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THE HILBERT SERIES OF SOME GRADED ALGEBRAS AND THE POINCARÉ SERIES OF SOME LOCAL RINGS

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1. Introduction.

Let R be a ring (not necessarily commutative) and let $X_1,\ldots,X_r$ be indeterminates. $R\langle X_1,\ldots,X_r\rangle$ denotes the non-commutative polynomial ring in $X_1,\ldots,X_r$ over R. Let k be a (commutative) field and let $A=k\langle X_1,\ldots,X_r\rangle/I$, where I is a finitely generated homogeneous ideal of $k\langle X_1,\ldots,X_r\rangle$. A_n denotes the homogeneous component of A of degree n. Let us consider the following two formal series:

$$H_k^A(Z) = \sum_{n\geqq 0} (\dim_k A_n)Z^n, \tag{1.1}$$

$$P^A(Z) = \sum_{n\geqq 0} (\dim_k(\mathrm{Tor}_n^A(k,k)))Z^n. \tag{1.2}$$

The former is called the *Hilbert series* of A and the latter is called the *Poincaré series* of A. It has been conjectured by Govorov [4] that $H_k^A(Z)$ is a rational function. It is well-known that the conjecture is true if A is commutative. However, the conjecture has been proved in few cases when A is non-commutative ([1], [3], [4]). We do not know how the study has been made on the rationality of $P^A(Z)$, but it seems probable that $P^A(Z)$ is rational if A is commutative. In connection with this, there is a conjecture that the Poincaré series of local rings are rational. Let R be a local ring and m its maximal ideal. The *Poincaré series* $P^R(Z)$ of R is defined as

$$P^R(Z) = \sum_{n\geqq 0} (\dim_k(\mathrm{Tor}_n^R(k,k)))Z^n, \tag{1.3}$$

where $k=R/m$. Many studies have been made on this conjecture in the last twenty years (see [5], [6] for example). Most of the results may be described as that the Poincaré series of a local ring close to a regular local ring in a sense is rational. Recently Fröberg [2] gave an interesting result that the Poincaré series of a quotient ring of a regular local ring modulo an ideal generated by any set of monomials of degree 2 in a regular system of parameters is rational.

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He constructed the minimal resolution of a local ring of the type sbove in a special way.

In this paper we will generalize Fröberg's method and calculate the Hilbert series of some type of graded algebras, which are quotient rings of $k\langle X_1,\ldots,X_r\rangle$ modulo ideals generated by some elements of degree 2, and the Poincaré series of some type of local rings, which are quotient rings of regular local rings modulo ideals generated by some monomials and binomials of degree 2 in regular systems of parameters. In section 2 we construct a complex which includes Fröberg's complex as a special case. In section 3 we show that the minimal resolution of a local ring of the type above is connected with our complex by means of the spectral sequence. In sections 4 and 5 we give some conditions under which the complex is acyclic, that is applied to calculating the Hilbert series and the Poincaré series in sections 6 and 7. As to the Hilbert series we reached the same conclusion as in [1] as a special case. All Poincaré series obtained in this paper are included in one of the following forms:

$$P^R(Z) = \frac{1}{1-rZ+uZ^2}, \tag{1.4}$$

$$P^R(Z) = \frac{1+Z}{1-(r-1)Z-uZ^2}, \tag{1.5}$$

where r is the embedding dimension of R and $0 \leqq u < r$.

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2. Construction of the complex.

We set a situation as follows: $\imath=\{1,\ldots,r\}$ is the set of r numbers. The product set $\imath\times\imath$ is a disjoint union of subsets $\alpha_1,\ldots,\alpha_u$ and β. For every $(i,j)\in\alpha_1\cup\ldots\cup\alpha_u$ an element c_{ij} of $k^*=k\setminus\{0\}$ is given.

We consider the following relations among $2r$ indeterminates X_i, Y_i $(i\in\imath)$:

$$\begin{cases} c_{ij}X_iX_j = c_{kl}X_kX_l & \text{for } (i,j),(k,l)\in\alpha_v,\ v=1,\ldots,u, \\ X_iX_j = 0 & \text{for } (i,j)\in\beta, \end{cases} \tag{2.1}$$

$$\sum_{(i,j)\in\alpha_v} c_{ij}^{-1}Y_iY_j = 0 \quad \text{for } v=1,\ldots,u. \tag{2.2}$$

Let A (respectively B) be the quotient ring $k\langle X_1,\ldots,X_r\rangle$ (respectively $k\langle Y_1,\ldots,Y_r\rangle$) modulo relation (2.1) (respectively (2.2)). Let C be the quotient ring of $k\langle X_1,\ldots,X_r\rangle\langle Y_1,\ldots,Y_r\rangle$ modulo relations (2.1) and (2.2). Clearly

$$C = A\otimes_k B. \tag{2.3}$$

The images of X_i and Y_i in A, B or C are denoted by x_i and y_i respectively. A (respectively B) is a graded k-algebra by the degree with respect to x_i (respectively y_i). C is a left graded A-module by the degree with respect to y_i (elements of A are considered to be of degree 0 in this case). A_n (respectively B_n, C_n) denotes the homogeneous component of degree n of A (respectively B, C). C_n is a free left A-module due to (2.3).

A differential mapping d of C is defined by the right multiplication by $\sum_{i=1}^r x_i y_i$, that is,

$$d(c) = c \sum_{i=1}^{r} x_i y_i, \quad c \in C . \tag{2.4}$$

d is a homomorphism of left graded A-modules of degree 1. The following is proved by an easy calculation.

Claim 1. $d \circ d = 0$.

Thus C is a free complex of left A-modules. Now we take the dual complex C^* of C:

$$C^* = \bigoplus_{n \geqq 0} C_n^*, \qquad C_n^* = \mathrm{Hom}_A(C_n, A) . \tag{2.5}$$

C^* is a free complex of right A-modules with the differential mapping d^* of degree -1. The nth component of d^* is denoted by d_n^*. From (2.4) we have

$$d_n^* f(m) = f\left(m \sum_{i=1}^{r} x_i y_i\right) = \sum_{i=1}^{r} x_i f(m y_i) , \tag{2.6}$$

where $f \in C_n^*$ and m is a monic monomial in y_i of degree $n-1$.

Since A is a graded algebra and C_n is a finitely generated left graded A-module by the degree with respect to x_i (monic monomials in y_i are considered to be of degree 0), C_n^* is a right graded A-module in the usual way. The homogeneous component of degree p of C_n^* is denoted by $C_{n,p}^*$:

$$C_{n,p}^* = \{ f \in C_n^* \mid f(m) \in A_p \text{ for any monic monomial } m \in C_n \text{ in } y_i \} .$$

We know from (2.6) that d_n^* is a homomorphism of right graded A-modules of degree 1. The pth component of d_n^* is denoted by $d_{n,p}^*$;

$$d_{n,p}^* : C_{n,p}^* \to C_{n-1,p+1}^* . \tag{2.7}$$

The nth homology group $H_n(C^*)$ of C^* is also a right graded A-module and its homogeneous component of degree p is denoted by $H_n(C^*)_p$. In view of (2.7) C^* is the direct product of the complexes

$$C^*(q) = \bigoplus_{q=n+p} C_{n,p}^* \quad (q=0,1,2,\ldots) .$$

Let $H_n(C^*(q))$ be the nth homology group of $C^*(q)$. Then we have

$$H_n(C^*)_p = H_n(C^*(n+p)) .$$

CLAIM 2.

$$H_0(C^*)_p = \begin{cases} k & \text{if } p=0, \\ 0 & \text{if } p\neq 0. \end{cases}$$

PROOF. Let $f \in C_1^*$ and $f(y_i)=a_i \in A$ $(i \in \iota)$. By (2.6)

$$d^*f(1) = \sum_{i=1}^{r} x_i f(y_i) = \sum_{i=1}^{r} x_i a_i .$$

Since we can choose a_i arbitrarily in A, the image of d^* is the right ideal of A generated by $x_1,\ldots,x_r$. Therefore

$$H_0(C^*) = A\Big/\left(\sum_{i=1}^{r} x_i A\right) = k .$$

Thus, C^* is a free complex of right A-modules over k. C^* is not always acyclic, but it is acyclic in some cases as we state later. In the rest of this section, we proceed our argment under the hypothesis that C^* is acyclic. Then the complex $C^*(q)$ is exact for $q>0$, hence its Euler-Poincaré characteristic is 0:

$$\begin{aligned}\chi(C^*(q)) &= \sum_{q=n+p} (-1)^n \dim_k C^*_{n,p} \\ &= \sum_{q=n+p} (-1)^n \operatorname{rank}_A C_n \cdot \dim_k A_p \\ &= \sum_{q=n+p} (-1)^n \dim_k B_n \cdot \dim_k A_p \\ &= 0 .\end{aligned}$$

When $q=0$, we have

$$\dim_k B_0 \cdot \dim_k \mathrm{A}_0 = 1 .$$

It follows from these that

$$H_k^B(-Z) \cdot H_k^A(Z) = 1 .$$

Moreover if A is commutative, we obtain from (2.6)

$$d_n^*(C_n^*) \subset C_{n-1}^* \mathfrak{M} ,$$

where $\mathfrak{M}=\oplus_{n>0}A_n$ is the irrelevant maximal ideal of A. In this case the complex C^* is called *minimal* and we have

$$\mathrm{Tor}_n^A(k,k) \cong C_n^* \otimes_A k .$$

Hence the nth Betti number $\dim_k(\mathrm{Tor}_n^A(k,k))$ of A is equal to

$$\dim_k(C_n^* \otimes_A k) = \mathrm{rank}_A C_n^* = \mathrm{rank}_A C_n = \dim_k B_n .$$

Therefore the Poincaré series $P^A(Z)$ of A coincides with the Hilbert series $H_k^B(Z)$ of B. Summarizing the preceding argument, we obtain

THEOREM 1. *If the complex C^* is acyclic, we have*

$$H_k^B(Z) = 1/H_k^A(-Z) , \tag{2.8}$$

and moreover if A is commutative,

$$P^A(Z) = H_k^B(Z) = 1/H_k^A(-Z) \tag{2.9}$$

and therefore $P^A(Z)=H_k^B(Z)$ is rational.

REMARK 1. (2.9) is called Fröberg's formula. Since it does not always hold, C^* is not always acyclic (see Fröberg [2]).

3. Minimal resolutions of local rings.

Let R be a regular local ring of dimension r. Let m be the maximal ideal of R and let $x_1,\dots,x_r$ be a regular system of parameters of R. Let $k=R/m$ and assume $k\subset R$. Let I be an ideal of R generated by some monomials and binomials of degree 2 in $x_1,\dots,x_r$ with coefficients in k. Then $R'=R/I$ is the quotient ring of R modulo some relations in $x_1,\dots,x_r$ of type (2.1) in section 2. Let B (respectively C') be the quotient ring of $k\langle Y_1,\dots,Y_r\rangle$ (respectively $R'\langle Y_1,\dots,Y_r\rangle$) modulo the relations in $Y_1,\dots,Y_r$ of type (2.2) corresponding to the relations in $x_1,\dots,x_r$. Then $C'\cong R'\otimes_k B$. A differential mapping d of C' is defined in the same way as that of C is defined in section 2. Then C' is a free complex of R'-modules and the dual complex

$$C'^* = \bigoplus_{n\geqq 0} \mathrm{Hom}_{R'}(C'_n, R')$$

of C' is a free complex of R'-modules over k.

Let A be a quotient ring of $k\langle X_1,\dots,X_r\rangle$ modulo the same relations in X_i as in x_i above and let $C=A\otimes_k B$. It is not difficult to observe

$$A \cong \mathrm{Gr}(R') = \bigoplus_{n\geqq 0} (m'^n/m'^{n+1}) , \tag{3.1}$$

where m' is the maximal ideal of R'. We concider the m'-adic filtration on C'^*. There is the spectral sequence associated with the filtered complex C'^* (see Serre [7, Chapitre II]). We can show through a simple calculation using (3.1) that $E^n_{1,p} \cong C^*_{n,p}$, this is expressed as

$$C^*_{n,p} \Rightarrow H_n(C'^*)\,. \tag{3.2}$$

Therefore, if C^* is acyclic, so is C'^* and C'^* becomes the minimal resolution of k. Hence the nth Betti number of R' is equal to

$$\dim_k (C'^*_n \otimes_{R'} k) = \operatorname{rank}_{R'} C'^*_n = \operatorname{rank}_{R'} C'_n = \dim_k B_n\,.$$

Thus we obtain

THEOREM 2. *If C^* is acyclic, then C'^* is the minimal resolution of k and*

$$P^{R'}(Z) = H^B_k(Z) = 1/H^A_k(-Z) = 1/H^{\mathrm{Gr}\,(R')}_k(-Z)\,, \tag{3.3}$$

and therefore $P^{R'}(Z)$ is rational.

4. General analysis of the complex C^*.

Let A, B and C be the same as in section 2. Let $M = \oplus_{p \geqq 0} M_p$ be a finitely generated left graded module over the graded algebra A. We define a complex $C^*(M)$ of graded k-modules by

$$C^*(M) = \bigoplus_{n \geqq 0} C^*_n(M), \qquad C^*_n(M) = \operatorname{Hom}_A(C_n, M)\,.$$

The differential mapping of $C^*(M)$ is induced from the differential mapping d of C and is denoted by $d^*(M)$ or simply by d^* if M is fixed. $C^*_n(M)_p$ denotes the homogeneous component of $C^*_n(M)$ of degree p:

$$C^*_n(M)_p = \{f \in C^*_n(M) \mid f(m) \in M_p \text{ for any monic monomial } m \in C_n \text{ in } y_i\}.$$

The nth component $d^*_n(M)$ of $d^*(M)$ is a homomorphism of graded k-modules of degree -1. The nth homology group $H_n(C^*(M))$ of $C^*(M)$ is also a graded k-module and its homogeneous component of degree p is denoted by $H_n(C^*(M))_p$.

CLAIM 3. $H_0(C^*(M)) = M/\mathfrak{M}M$.

PROOF. Similar to the proof of Claim 2 in section 2.

For an integer $q \geqq 0$ we define a complex $C^*(M)(q)$ *by*

$$C^*(M)(q) = \bigoplus_{q=n+p} C^*_n(M)_p\,.$$

The Euler-Poincaré characteristic of $C^*(M)(q)$ is denoted by $r_M(q)$ and we define $R_M(Z)=\sum_{q\geqq 0} r_M(q)Z^q$. Then

$$r_M(q) = \sum_{q=n+p} (-1)^n \dim_k C_n^*(M)_p$$

$$= \sum_{q=n+p} (-1)^n \dim_k \mathrm{B}_n \cdot \dim_k M_p .$$

Hence we get

$$H_k^B(-Z)\cdot H_k^M(Z) = R_M(Z) . \tag{4.1}$$

In particular,

$$H_k^B(-Z)\cdot H_k^A(Z) = R_A(Z) . \tag{4.2}$$

Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of left graded A-modules. We clearly have

$$H_k^{M_1}(Z)+H_k^{M_3}(Z) = H_k^{M_2}(Z) .$$

Hence by (4.1) we obtain

$$R_{M_1}(Z)+R_{M_3}(Z) = R_{M_2}(Z) . \tag{4.3}$$

Moreover, since C is a free complex, we have an exact sequence of the complexes:

$$0 \to C^*(M_1) \to C^*(M_2) \to C^*(M_3) \to 0 .$$

This derives the long exact sequence:

$$\begin{aligned} \ldots \to H_n(C^*(M_1))_p &\to H_n(C^*(M_2))_p \to H_n(C^*(M_3))_p \\ \to H_{n-1}(C^*(M_1))_{p+1} &\to H_{n-1}(C^*(M_2))_{p+1} \to \cdots . \end{aligned} \tag{4.4}$$

Hereafter we use the following notations: For $I=(i_1,\ldots,i_n)\in \iota^n$, x_I (respectively y_I) denotes the monomial $x_{i_1}\ldots x_{i_n}$ (respectively $y_{i_1}\ldots y_{i_n}$). Moreover, for $J=(j_1,\ldots,j_m)\in \iota^m$, IJ means $(i_1,\ldots,i_n,j_1,\ldots,j_m)\in \iota^{n+m}$. In particular, for $i\in \iota$, Ii (rexpectively iI) means $(i_1,\ldots,i_n,i)$ (respectively $(i,i_1,\ldots,i_n))\in \iota^{n+1}$.

Let ι' be a subset of ι. Let $\mathscr{I}$ (respectively $\mathscr{J}$) be the left ideal of A generated by x_i, $i\in \iota'$ (respectively $i\in \iota\setminus \iota'$) and let $\mathscr{I}'$ (respectively $\mathscr{J}'$) be the left ideal of B generated by y_i, $i\in \iota'$ (respectively $i \in \iota\setminus\iota'$). Let $\bar{x}_i$ denote the image of x_i in $A/\mathscr{J}$.

Claim 4. $H_n(C^*(A/\mathscr{J}))_0 \cong \mathrm{Hom}_k(B_n/\mathscr{I}'_n, k)$.

PROOF. The claim is true in case $n=0$ by Claim 3. Let $n>0$ and $f \in C_n^*(A/\mathscr{I})_0$. Let $f(y_I)=a_I \in k$ for $I \in \iota^n$ and assume that $d^*f=0$. Then for all $I' \in \iota^{n-1}$ we have

$$d^*f(y_{I'}) = \sum_{i \in \iota'} a_{I'i}\bar{x}_i = 0 ,$$

this shows $a_{I'i}=0$ for all $I' \in \iota^{n-1}$ and $i \in \iota'$. Therefore, f is naturally identified with an element of $\mathrm{Hom}_k((B/\mathscr{I}')_n, k)$, this proves the claim.

COROLLARY 1. $H_n(C^*)_0 = 0$ *for* $n > 0$.

COROLLARY 2. $H_n(C^*(k)) \cong B_n$.

Next we consider the following situation:

(4.5) For any $v=1,\dots,u$ either $\alpha_v \subset \iota \times \iota'$ or $\alpha_v \cap (\iota \times \iota') = \varnothing$.

Renumbering α_v, we may assume that $\alpha_1,\dots,\alpha_{u'}$ are all α_v such that $\alpha_v \subset \iota \times \iota'$.

CLAIM 5. Under the situtation above

$$H_n(C^*(A/\mathscr{I}))_1 = 0 .$$

PROOF. The claim is true in case $n=0$ by Claim 3. Let $n>0$ and $f \in C_n^*(A/\mathscr{I})_1$. Let $f(y_I)=\sum_{j \in \iota'} a_I^j \bar{x}_j$ for $I \in \iota^n$, where $a_I^j \in k$. Assume that $d^*f=0$. Then for all $I' \in \iota^{n-1}$,

$$\begin{aligned} d^*f(y_{I'}) &= \sum_{i \in \iota} x_i f(y_{I'}y_i) = \sum_{i \in \iota}\sum_{j \in \iota'} a_{I'i}^j \bar{x}_i \bar{x}_j \\ &= \sum_{v=1}^{u'} \sum_{(i,j) \in \alpha_v} a_{I'i}^j c_{ij}^{-1}(c_{ij}\bar{x}_i\bar{x}_j) = 0 . \end{aligned}$$

It follows from this that

$$\sum_{(i,j) \in \alpha_v} a_{I'i}^j c_{ij}^{-1} = 0 \tag{4.6}$$

for all $I' \in \iota^{n-1}$ and $v=1,\dots,u'$. We define $\bar{f} \in C_{n+1}^*(A/\mathscr{I})_0$ by

$$\bar{f}(y_{Ij}) = \begin{cases} a_I^j & \text{if } j \in \iota', \\ 0 & \text{if } j \notin \iota', \end{cases}$$

for $Ij \in \iota^{n+1}$. Then $\bar{f}$ is well-defined by (4.6) and $d^*\bar{f}=f$.

COROLLARY. $H_n(C^*)_1 = 0$ (*without situation* (4.5)).

CLAIM 6. *Under situation* (4.5) *the following two statements are equivalent*:

(4.7) $$H_n(C^*(A/\mathscr{I}))_p = 0 \quad \text{for } n>0 \text{ and } p \geqq 2\,,$$

(4.8) $$H_n(C^*(\mathscr{I}))_p = 0 \quad \text{for } n>0 \text{ and } p \geqq 2\,.$$

PROOF. The proof is immediate from the long exact sequence (4.4) induced from the exact sequence:

$$0 \to \mathscr{I} \to A/\mathscr{I} \to k \to 0\,.$$

If the statements in Claim 6 are satisfied (we will prove this in some cases in the following section), we have by Claim 4 and Claim 5

$$\begin{aligned} r_{A/\mathscr{I}}(q) &= (-1)^q \dim_k \left(H_q(C^*(A/\mathscr{I}))_0\right) \\ &= (-1)^q \dim_k \left(B_q/\mathscr{I}'_q\right)\,. \end{aligned}$$

Hence by (4.1)

(4.9) $$H_k^B(Z) \cdot H_k^{A/\mathscr{I}}(-Z) = H_k^{B/\mathscr{I}'}(Z)\,.$$

5. Several cases where C^* is acyclic.

We postulate situation (4.5). We consider the following condition:

(5.1) There exist $i, j \in \imath'$ such that (i,j), $(j,i) \in \beta$ and

$$\alpha_v \cap (\imath' \times i) \neq \varnothing, \quad \alpha_v \cap (\imath' \times j) \neq \varnothing \quad \text{for all } v=1,\dots,u'\,.$$

Let $\imath' = \{1,\dots,r'\}$ $(r' \leqq r)$ and put $d_n = \dim_k (A_n/\mathscr{I}_n)$.

CLAIM 7. *On condition* (5.1) *we have*

(5.2) $$d_0 = 1,\ d_1 = r',\ d_2 = u' \text{ and } d_n = 0 \text{ for } n \geqq 3\,.$$

THEOREM 3. *On condition* (5.1)

$$H_n(C^*(A/\mathscr{I}))_p = 0 \quad \text{for } p>0\,.$$

PROOF. It suffices to prove that $H_n(C^*(A/\mathscr{I}))_2 = 0$ for $n>0$. Due to the assumption we can choose elements i_v, j_v in $\imath'$ for each $v=1,\dots,u'$ such that (i_v, i), $(j_v, j) \in \alpha_v$. Let $f \in C_n^*(A/\mathscr{I})_2$ and $f(y_I) = \sum_{v=1}^{u'} a_I^v \cdot \bar{x}_{i_v} \cdot \bar{x}_i$ $(a_I^v \in k)$ for $I \in \imath^n$. Since $A_3/\mathscr{I}_3 = 0$, we see $d^*f = 0$. Hence we must find $\bar{f} \in C_{n+1}^*(A/\mathscr{I})_1$ such that $d^*\bar{f} = f$. We define $\bar{f}$ as follows:

For $I \in \imath^n$ and $v = 1,\dots,u'$,

$$\bar{f}(y_{Ii_v}) = a_I^v \cdot \bar{x}_i\,,$$

for $I' \in \imath^{n-2s-1}(0 \leqq s \leqq [(n-2)/2])$ and $w = 1, \ldots, u'$,

$$\bar{f}(y_{I' j_w j \underbrace{ij \ldots ij}_{ij \text{ repeats } s \text{ times}}}) = -\sum_{\substack{(k_1, i_{v(1)}) \in \alpha_w,\ (l_1, j_{w(1)}) \in \alpha_{v(1)} \\ (k_2, i_{v(2)}) \in \alpha_{w(1)},\ (l_2, j_{w(2)}) \in \alpha_{v(2)} \\ \cdots\cdots \\ (k_s, i_{v(s)}) \in \alpha_{w(s-1)},\ (l_s, j_{w(s)}) \in \alpha_{v(s)} \\ (k_{s+1}, i_{v(s+1)}) \in \alpha_{w(s)}}} c^* \cdot a^{v(s+1)}_{I' k_1 l_1 \ldots k_s l_s k_{s+1}} \cdot \bar{x}_i ,$$

where

$$c^* = \frac{c_{i_{v(1)} i} \cdots c_{i_{v(s)} i} c_{j_w j} c_{j_{w(1)} j} \cdots c_{j_{w(s)} j}}{c_{k_1 i_{v(1)}} \cdots c_{k_{s+1} i_{v(s+1)}} c_{l_1 j_{w(1)}} \cdots c_{l_s j_{w(s)}}} ,$$

for $I'' \in \imath^{n-2s}(1 \leqq s \leqq [(n-1)/2])$ and $v = 1, \ldots, u'$,

$$\bar{f}(y_{I'' i_v \underbrace{ij \ldots ij}_{s}}) = \sum_{\substack{(l_1, j_{w(1)}) \in \alpha_v,\ (k_1, i_{v(1)}) \in \alpha_{w(1)} \\ (l_2, j_{w(2)}) \in \alpha_{v(1)},\ (k_2, i_{v(2)}) \in \alpha_{w(2)} \\ \cdots\cdots \\ (l_s, j_{w(s)}) \in \alpha_{v(s-1)},\ (k_s, i_{v(s)}) \in \alpha_{w(s)}}} c^{**} \cdot a^{v(s)}_{I'' l_1 k_1 \ldots l_s k_s} \cdot \bar{x}_i ,$$

where

$$c^{**} = \frac{c_{i_v i} c_{i_{v(1)} i} \cdots c_{i_{v(s-1)} i} c_{j_{w(1)} j} \cdots c_{j_{w(s)} j}}{c_{k_1 i_{v(1)}} \cdots c_{k_s i_{v(s)}} c_{l_1 j_{w(1)}} \cdots c_{l_s j_{w(s)}}} ,$$

and for other $J \in \imath^{n+1}$, $\bar{f}(y_J) = 0$.

It is not difficult to observe that $\bar{f}$ is well-defined and $d^* \bar{f} = f$, we leave it to the reader.

REMARK 2. As a special case of condition (5.1) we have

(5.1′) there exists $i \in \imath'$ such that $(i, i) \in \beta$ and $\alpha_v \cap (\imath' \times i) \neq \varnothing$ for all $v = 1, \ldots, u'$.

Next we assume in (4.5) that $u' = 1$, that is, there is only one α_v $(= \alpha_1)$ such that $\alpha_v \subset \imath \times \imath'$. Moreover we assume that $\alpha_1 \cap (\imath' \times \imath') \neq \varnothing$. We say in this case that the relation (2.1) is *simple* on $\imath'$. Let again $\imath' = \{1, \ldots, r'\}$, $r' \leqq r$ and put $d_n = \dim_k (A_n / \mathscr{I}_n)$.

CLAIM 8. *If the relation is simple on* $\imath'$, *then*

(5.2) $$d_0 = d_2 = 1 \text{ and } d_1 = r' ,$$

(5.3) *either* (i) $d_n = 1$ *for all* $n \geqq 2$, *or* (ii) *there is* $n_0 \geqq 2$ *such that* $d_n = 1$ *for* $2 \leqq n \leqq n_0$ *and* $d_n = 0$ *for* $n > n_0$.

In (ii) above we refer to n_0 as the *order* of the relation on ι'. In case (i) we say the order is infinite. When the relation is simple on ι', condition (5.1) turns to

(5.4) there exist $i,j,k,l \in \iota'$ such that $(i,j), (j,i) \in \beta$ and $(k,i), (l,j) \in \alpha_1$,

and condition (5.1′) turns to

(5.4′) there exists $i,j \in \iota'$ such that $(i,i) \in \beta$ and $(j,i) \in \alpha_1$.

In this case the order is 2 by Claim 7 and $H_n(C^*(A/\mathscr{J}))_p = 0$ for $p>0$ by Theorem 3. In the proof of the following theorem $x_I \equiv x_J$ means $x_I = cx_J$ for some $c \in k^*$.

THEOREM 4. *If the relation is simple and of infinite order on ι', then*

$$H_n(C^*(A/\mathscr{J}))_p = 0 \textit{ for } p>0 .$$

PROOF. It suffices to prove that $H_n(C^*(A/\mathscr{J}))_p = 0$ for $n>0$, $p \geqq 2$. Let $(i,j) \in \alpha_1 \cap (\iota' \times \iota')$ and let $I \in \iota \times \iota \times \iota'$ such that $x_I \neq 0$. Then there is $k \in \iota$ such that $x_I \equiv x_{kij}$. Since $(k,i) \in \alpha_1$, it follows that $x_{kij} \equiv x_{ijj} \neq 0$, this implies $(j,j) \in \alpha_1$. Now assume $(1,1) \in \alpha_1$. Then $\bar{x}_I \equiv \bar{x}_1^p$ for all $I \in \iota^p$ such that $x_I \neq 0$. Let $f \in C_n^*(A/\mathscr{J})_p$ and $f(y_I) = a_I \bar{x}_1^p$ ($a_I \in k$) for $I \in \iota^n$. Assume $d^*f = 0$, that is, for all $I' \in \iota^{n-1}$

$$d^*f(y_{I'}) = \sum_{i \in \iota} x_i f(y_{I'i}) = \sum_{i \in \iota} a_{I'i} \bar{x}_i \bar{x}_1^p = \sum_{(i,1) \in \alpha_1} a_{I'i} c_{i1}^{-1} c_{11} \bar{x}_1^{p+1} = 0 .$$

Thus we conclude that

$$\sum_{(i,1) \in \alpha_1} a_{I'i} c_{i1}^{-1} = 0 \qquad \text{for all } I' \in \iota^{n-1} .$$

We define $\bar{f} \in C_{n+1}^*(A/\mathscr{J})_{p-1}$ as follows: For $J \in \iota^{n+1}$

$$\bar{f}(y_J) = \begin{cases} a_I \bar{x}_1^{p-1} & \text{if } J = I1 \text{ for } I \in \iota^n, \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to observe that $\bar{f}$ is well-defined and $d^*\bar{f} = f$. This completes the proof.

THEOREM 5. *Let ι be a disjoint union of subsets $\iota_1, \ldots, \iota_t$. Assume that for every $v = 1, \ldots, u$ there is s $(1 \leqq s \leqq t)$ such that $\alpha_v \subset \iota \times \iota_s$. If for each $s = 1, \ldots, t$ the relation either satisfies condition* (5.1) *with $\iota' = \iota_s$ or is simple and of infinite order on ι_s, then C^* is acyclic.*

PROOF. Immediate from Claim 6, Theorem 3, Theorem 4 and the long exact sequence (4.4) induced from the exact sequence:

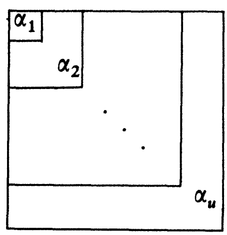
$$0 \to \mathscr{I}_1 \oplus \ldots \oplus \mathscr{I}_t \to A \to k \to 0,$$

where $\mathscr{I}_s$ is the ideal of A generated by x_i, $i \in \iota_s$.

REMARK 3. There is another extreme case; every α_v consists of either one element or two elements (i,j) and (j,i). In this case C^* is isomorphic to Fröberg's complex [2]. He proved the acyclicity of the complex in this case under a little more assumption. There are other cases where C^* is acyclic (see the following example), but we do not know the general condition for that.

EXAMPLE 1. We consider the following situation: ι is a disjoint union of non-empty subsets $\iota_1,\ldots,\iota_u$. α_v is defined inductively by $\alpha_1 = \iota_1 \times \iota_1$ and

$$\alpha_v = \left\{ \bigcup_{v',v'' \leqq v} (\iota_{v'} \times \iota_{v''}) \right\} \setminus \alpha_{v-1} \qquad (v=2,\ldots,u).$$



The relations in X_i, Y_i are

$$\text{(5.5)} \qquad \begin{cases} X_iX_j = X_kX_l & \text{for } (i,j), (k,l) \in \alpha_v, \\ \displaystyle\sum_{(i,j)\in\alpha_v} Y_iY_j = 0, & \end{cases} \qquad (v = 1,\ldots,u).$$

In this case C^* is acyclic. In fact, let $f \in C^*_{n,p}$ $(n,p>0)$ and $d^*f=0$. For each v we choose an element i_v in ι_v, so that for any $K \in \iota^p$ there is v such that $x_K = x^p_{i_v}$. Let

$$f(y_I) = \sum_{v=1}^{u} a^v_I x^p_{i_v},\ a^v_I \in k \text{ for } I \in \iota^n.$$

Then for $I' \in \iota^{n-1}$

$$\begin{aligned} d^*f(y_{I'}) &= \sum_{i\in\iota,\,1\leqq v\leqq u} a^v_{I'i} x_i x^p_{i_v} \\ &= \sum_{v=1}^{u} \sum_{(i,i_{v'})\in\alpha_v} a^{v'}_{I'i} x^{p+1}_{i_v} = 0. \end{aligned}$$

Hence we find that for all $v=1,\dots,u$

$$\sum_{(i,i_{v'})\in\alpha_v} a^{v'}_{I'i} = 0 .$$

Define $\bar{f} \in C^*_{n+1,p-1}$ by

$$\bar{f}(y_J) = \begin{cases} a^v_I x^{p-1}_{i_1} & \text{if } J = Ii_v \text{ for some } I \in \iota^n \text{ and } v, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\bar{f}$ is well-defined and $d^*\bar{f}=f$, this proves the acyclicity of C^*.

6. Determination of the Hilbert series.

Let assume the condition of Theorem 5. Moreover we suppose the following: For $s \leqq t'$ $(t' \leqq t)$ the relation is simple and of infinite order on ι_s, and for $s > t'$ the relation satisfies condition (5.1) with $\iota' = \iota_s$, and the number of α_v contained in $\iota \times \iota_s$ is u_s. In this case the Hilbert series of A is given by

$$H^A_k(Z) = 1 + rZ + u^*Z^2 + \frac{t'Z^2}{(1-Z)} = \frac{1+(r-1)Z-(r-t'-u^*)Z^2-u^*Z^3}{1-Z},$$

where $u^* = u_{t'+1} + \dots + u_t$. Therefore, by Theorem 1 and Theorem 5 we have

RESULT 1. *In the situation above*

$$H^B_k(Z) = \frac{1+Z}{1-(r-1)Z-(r-t'-u^*)Z^2+u^*Z^3} \tag{6.1}$$

Next we consider the more special case where $u=1$, that is, B is the quotient ring of $k\langle Y_1,\dots,Y_r\rangle$ modulo the following single relation:

$$m = \sum_{(i,j)\in\alpha_1} c_{ij} Y_i Y_j = 0 . \tag{6.2}$$

To calculate the Hilbert series we may assume k is algebraically closed by Backelin [1, Lemma 4]. Through a suitable change of a coordinate system, relation (6.2) turns to a relation of type (5.4′) unless $m = (\sum_{i\in\iota} c_i Y_i)^2$ for some $c_i \in k$ ([1, Lemma 3]). If $m = (\sum_{i\in\iota} c_i Y_i)^2$, the relation in X_i corresponding to (6.2) is simple and of infinite order on ι in terms of section 5. Hence we come to the same conclusion as [1, Corollary of Theorem 2].

RESULT 2. *Let B be a quotient ring of $k\langle Y_1,\dots,Y_r\rangle$ by an ideal generated by a homogeneous element m of degree 2. If m is a square of a linear combination of Y_i over the algebraic closure of k, then*

$$H_k^B(Z) = \frac{1+Z}{1-(r-1)(Z+Z^2)}, \tag{6.3}$$

and otherwise

$$H_k^B(Z) = \frac{1}{1-rZ+Z^2}. \tag{6.4}$$

7. Determination of the Poincaré series.

Let R be an equi-characteristic regular local ring of dimension r and let R' be a quotient ring of R by an ideal I of R. Since the Poincaré series of R' is invariant under the completion of R', we do not lose the generality even if we assume that R is the formal power series ring $k[[X_1,\dots,X_r]]$. Now we will give the Poincaré series of $R'=R/I$ when I is generated by some types of monomials and binomials in $X_1,\dots,X_r$ of degree 2. First we obtain by (6.3) in Result 2.

RESULT 3. *Let ι' be a non-empty subset of $\iota=\{1,\dots,r\}$ and let*

$$I = (X_iX_j-X_kX_l, X_mX_n;\ i,j,k,l\in\iota',\ m\notin\iota',\ n\in\iota).$$

Then

$$P^{R'}(Z) = \frac{1+Z}{1-(r-1)(Z+Z^2)}. \tag{7.1}$$

A subset α of $\iota\times\iota$ is called *symmetric* if $(i,j)\in\alpha$ implies $(j,i)\in\alpha$. If α is symmetric and $\alpha\neq\iota'\times\iota'$ for any $\iota'\subset\iota$, we can find $i,j,k,l\in\iota$ such that (i,j), $(j,i)\notin\alpha$ and (k,i), $(l,j)\in\alpha$. Thus by (6.4) in Result 2 we have

RESULT 4. *Let α be a symmetric subset of $\iota\times\iota$ and assume $\alpha\neq\iota'\times\iota'$ for any $\iota'\subset\iota$. Suppose $c_{ij}=c_{ji}\in k^*$ for all $(i,j)\in\alpha$. Let*

$$I = (c_{ij}X_iX_j-c_{kl}X_kX_l,\ X_mX_n\ ;\ (i,j),(k,l)\in\alpha,\ (m,n)\notin\alpha).$$

Then

$$P^{R'}(Z) = \frac{1}{1-rZ+Z^2}. \tag{7.2}$$

More generally Theorem 3 gives

RESULT 5. *Let $\iota\times\iota$ be a disjoint union of symmetric subsets $\alpha_1,\dots,\alpha_u$ and β.*

Suppose $c_{ij}=c_{ji}\in k^*$ *for all* $(i,j)\in\alpha_v$, $v=1,\ldots,u$. *Assume that there are couples* $(i_0,j_0)\in\beta$ *and* $(k_v,i_0),(l_v,j_0)\in\alpha_v$, $v=1,\ldots,u$. *Let*

$$I = (c_{ij}X_iX_j-c_{kl}X_kX_l,\ X_mX_n\ ;\ \ (i,j),(k,l)\in\alpha_v,\ v=1,\ldots,u,\ (m,n)\in\beta)\ .$$

Then

$$P^{R'}(Z) = \frac{1}{1-rZ+uZ^2}\ . \tag{7.3}$$

EXAMPLE 2. We give two local rings of embedding dimension 3 of the type in Result 5, whose Poincaré series is $1/(1-3Z+2Z^2)$.

$$I = (X_1^2, X_2X_3, X_1X_2-X_3^2, X_1X_3-X_2^2)\ . \tag{7.4}$$

$$I = (X_1^2, X_1X_2-X_3^2, X_1X_3-X_2^2, X_1X_3-X_2X_3)\ . \tag{7.5}$$

Lastly Example 1 in section 5 gives

RESULT 6. *In the same situation as in Example* 1, *let*

$$I = (X_iX_j-X_kX_l\ ;\ \ (i,j),(k,l)\notin\alpha_v,\ v=1,\ldots,u)\ .$$

Then

$$P^{R'}(Z) = \frac{1+Z}{1-(r-1)Z-(r-u)Z^2}\ . \tag{7.6}$$

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