MATH. SCAND. 41 (1977), 331–346

ON THE LEVY–CHINTSCHIN FORMULA ON LOCALLY COMPACT MAXIMALLY ALMOST PERIODIC GROUPS

EBERHARD SIEBERT

In his paper [3] W. Hazod proved a Lévy–Chintschin formula for convolution semigroups on an arbitrary locally compact group G based on the space $\mathscr{D}(G)$ of infinitely differentiable functions with compact support. On the other side in [7] the author has given a Lévy–Chintschin formula on a (Lie projective) maximally almost periodic group (MAP group) G based on its coefficient algebra $\mathfrak{K}(G)$.

Both results depend heavily on the fundamental paper [5] by G. A. Hunt on convolution semigroups on Lie groups. But the proofs given in [3] and [7] are independent and in some sense parallel.

The main purpose of this paper is to point out how one can derive the Lévy–Chintschin representation on $\mathfrak{K}(G)$ (for a MAP group G) if the representation on $\mathscr{D}(G)$ is given. So we start with the Lévy–Chintschin formula on $\mathscr{D}(G)$ as stated in [8] and extend it to the space $\mathscr{E}(G)$ of bounded continuous functions on G that are locally in $\mathscr{D}(G)$. But $\mathfrak{K}(G)$ is a subspace of $\mathscr{E}(G)$. Thus by restriction we get a representation formula, but with an unusual integral term. Nevertheless by a simple substitution one reaches the desired formula (Theorem 3.2).

This approach has several advantages: It is rather direct, it demands no Lie group approximation and the restriction to Lie projective groups (as in [7]) can be dropped. Furthermore the relation between the two different representations (on $\mathscr{D}(G)$ and on $\mathfrak{K}(G)$ resp.) is clarified.

For the Lévy–Chintschin formula on a MAP group G we need a Lévy function γ for G. The existence of γ has been proved in [7], Satz III.5. Yet the demonstration given there is incomplete since it is based on the incorrect Lemma III.12. But if one replaces this lemma by lemma 3.2 of the present paper one can prove the existence of a Lévy function for G. We will not give this modified demonstration here since it is similar to the proof of Satz 4.2 in [8] and since a complete proof will be contained in the forthcoming book [4].

Received August 23, 1976.

Fimally we show how the Gaussian component in the Lévy–Chintschin formula for a convolution semigroup (on a MAP group) can be calculated from its generating functional. This result stresses again the distinguished role played by the Gaussian distributions.

Notations.

For any $n \in \mathsf{N}$ we denote by $\mathsf{M}(n, \mathsf{C})$ the space of all $n \times n$-matrices over C. E_n is the unit matrix in $\mathsf{M}(n, \mathsf{C})$, and for $M \in \mathsf{M}(n, \mathsf{C})$ we denote by $\tilde{M}$ the adjoint matrix (that is, $\tilde{M} = \bar{M}^T$) and by $\operatorname{Tr}(M)$ the trace of M. We equip $\mathsf{M}(n, \mathsf{C})$ with the spectral norm $\|.\|$, that is,

$$\|M\| = \max_{1 \leqq j \leqq n} \sqrt{\beta_j}$$

where $\beta_1, \ldots, \beta_n$ are the eigenvalues of $M\tilde{M}$. If M is diagonalizable or (equivalently) normal (that is, $M\tilde{M} = \tilde{M}M$) then $\|M\| = \max_{1 \leqq j \leqq n} |\alpha_j|$ where $\alpha_1, \ldots, \alpha_n$ are the eigenvalues of M.

Let E be a locally compact space. By $\mathscr{C}^b(E)$ we denote the space of all continuous bounded *complex valued* functions on E and by $\mathscr{K}(E)$ the subspace of functions with compact support. If $\mathscr{V}$ is a subspace of $\mathscr{C}^b(E)$ let

$$\mathscr{V}_+ := \{f \in \mathscr{V} : f \geqq 0\} .$$

A linear functional L on $\mathscr{V}$ is called *real* if $L(\bar{f}) = \overline{L(f)}$ for all $f \in \mathscr{V}$. The support of $f \in \mathscr{C}^b(E)$ is denoted by $\operatorname{supp}(f)$. For a subset A of E its indicator function is 1_A. $\mathscr{M}^b_+(E)$ is the cone of all positive bounded Radon measures on E equipped with the weak topology $\sigma(\mathscr{M}^b(E), \mathscr{C}^b(E))$. For $x \in E$ the Dirac measure in x is denoted by ε_x.

By G we always denote a locally compact group. Let $\mathfrak{V}(G)$ be the system of neighbourhoods of the unit e in G which are in addition Borel sets. G_0 is the connected component of e in G, and it is $G^\times = G \backslash \{e\}$. For $f \in \mathscr{C}^b(G)$ the function f^* is defined by $f^*(x) = f(x^{-1})$ (all $x \in G$), The function $u \in \mathscr{C}^b(G)$ is called a *local unit*, if there exists a $U \in \mathfrak{V}(G)$ such that $1_U \leqq u \leqq 1_G$. A convolution semigroup on G is a family $(\mu_t)_{t>0}$ in $\mathscr{M}^b_+(G)$ such that $\mu_t(G) = 1$,

$$\mu_s * \mu_t = \mu_{s+t} \qquad \text{(convolution)}$$

for all $s, t > 0$ and $\lim_{t \downarrow 0} \mu_t = \varepsilon_e$.

If G is a Lie group $\mathscr{L}(G)$ denotes its Lie algebra and $\exp_G$ the exponential mapping from $\mathscr{L}(G)$ into G. For $X \in \mathscr{L}(G)$ and for all differentiable functions $f \in \mathscr{C}^b(G)$ we define

$$(Xf)(e) = \lim_{t \to 0} \frac{1}{t} [f(\exp_G(tX)) - f(e)] .$$

1. Spaces of infinitely differentiable functions and linear functionals.

Let G be a locally compact group. Following F. Bruhat [2] we consider the following two subspaces of $\mathscr{C}^{\mathrm{b}}(G)$:

1. $\mathscr{D}(G)$ is the space of infinitely differentiable functions with compact support on G (see [2] or [8]).

2. $\mathscr{E}(G)$ is the space of bounded infinitely differentiable functions on G, i.e.

$$\mathscr{E}(G) = \{f \in \mathscr{C}^{\mathrm{b}}(G) : f.g \in \mathscr{D}(G) \text{ for all } g \in \mathscr{D}(G)\} .$$

DEFINITION 1. Let $\mathscr{V} = \mathscr{D}(G)$ or let $\mathscr{V} = \mathscr{E}(G)$ and let L be a real linear functional on $\mathscr{V}$.

a) L is called *nearly positive* if for every $f \in \mathscr{V}_+$, $f(e) = 0$ we have $L(f) \geqq 0$.

b) L is called a *primitive form* if for all $f, g \in \mathscr{V}$ we have

$$L(f \cdot g^*) = L(f)g(e) - f(e)L(g) .$$

c) L is called a *quadratic form* if L is nearly positive and if for all $f, g \in \mathscr{V}$ we have

$$L(f \cdot g) + L(f \cdot g^*) = 2(L(f)g(e) + f(e)L(g)) .$$

d) L is called *concentrated in the origin* if for all $f \in \mathscr{V}$ vanishing in a neighbourhood of e we have $L(f) = 0$.

REMARKS 1. Let L be a linear functional on $\mathscr{V}_{\mathrm{R}} = \{f \in \mathscr{V} : f = \bar{f}\}$ of one the four types introduced in definition 1. Then the unique extension of L to a real linear functional on $\mathscr{V}$ is of the same type.

2. A real linear functional L on $\mathscr{V}$ is concentrated in the origin if and only if for any local unit $u \in \mathscr{V}$ we have $L(u.f) = L(f)$ for all $f \in \mathscr{V}$.

3. Any primitive and any quadratic form on $\mathscr{V}$ is concentrated in the origin (see [8, Lemma 3.1]).

LEMMA 1. (i) *Let L be a linear functional on $\mathscr{D}(G)$ concentrated in the origin. Then L can be extended uniquely to a linear functional $\hat{L}$ on $\mathscr{E}(G)$ concentrated in the origin.*

If $u \in \mathscr{D}(G)$ is a local unit then $\hat{L}(f) = L(u.f)$ for all $f \in \mathscr{E}(G)$.

(ii) *If L is a primitive (respectively quadratic) form on $\mathscr{D}(G)$ its extension $\hat{L}$ is a primitive (respectively quadratic) form on $\mathscr{E}(G)$.*

PROOF. (i) Let $f \in \mathscr{E}(G)$. If $u, v \in \mathscr{D}(G)$ are local units, we have by remark 2: $L(u.f) = L(u.v.f) = L(v.f)$. Thus by $\hat{L}(f) := L(u.f)$ (all $f \in \mathscr{E}(G)$) there is defined a real linear functional $\hat{L}$ on $\mathscr{E}(G)$ independent of the special choice of u. It is obvious that L is concentrated in the origin and that it is the unique extension of L with this property.

(ii) Let L be a primitive form on $\mathscr{D}(G)$. With u also $u.u^*$ is a local unit in $\mathscr{D}(G)$. By remark 3 it follows for $f, g \in \mathscr{E}(G)$:

$$\begin{aligned}\hat{L}(f.g^*) &= L((u.u^*)(f.g^*)) = L((u.f).(u.g)^*)\\ &= L(u.f)(u.g)(e) - (u.f)(e)L(u.g) = \hat{L}(f)g(e) - f(e)\hat{L}(g)\,.\end{aligned}$$

Analogously one proves the statement for a quadratic form.

Definition 2. Every linear mapping Γ from $\mathscr{D}(G)$ into itself with the following properties is called a *Lévy mapping* for G:

(LM 1) For every primitive form ψ on $\mathscr{D}(G)$ and for every $f \in \mathscr{D}(G)$ we have $\psi(f - \Gamma(f)) = 0$.

(LM 2) For every $f \in \mathscr{D}(G)$ we have $\Gamma(f)^* = -\Gamma(f)$.

(LM 3) For every $x \in G$ the functional $f \to (\Gamma(f))(x)$ is a primitive form on $\mathscr{D}(G)$.

Remark 4. On every locally compact group G there exists a Lévy mapping. [By [8, Satz 2] there exists a Lévy mapping Γ on $\mathscr{D}_{\mathbf{R}}(G)$. One extends Γ to $\mathscr{D}(G)$ by

$$\Gamma(f_1 + i.f_2) := \Gamma(f_1) + i.\Gamma(f_2)$$

for all $f_1, f_2 \in \mathscr{D}_{\mathbf{R}}(G)$. Taking into account remark 1 one sees immediately that this gives us a Lévy mapping defined on $\mathscr{D}(G)$.]

Lemma 2. *Let Γ be a Lévy mapping for G. Given a local unit $u \in \mathscr{D}(G)$ one defines a linear mapping $\hat{\Gamma}$ from $\mathscr{E}(G)$ into $\mathscr{D}(G)$ by $\hat{\Gamma}(f) = \Gamma(u.f)$ for all $f \in \mathscr{E}(G)$. This definition of $\hat{\Gamma}$ is independent of the special choice of u, and $\hat{\Gamma}$ enjoys the properties* (LM 1), (LM 2), (LM 3) (*with $\mathscr{D}(G)$ replaced by $\mathscr{E}(G)$*).

Proof. From (LM 3) and lemma 1 it follows that $\hat{\Gamma}$ is defined and independent of u. The verification of (LM 1), (LM 2), (LM 3) is immediate.

Definition 3. A positive Radon measure η on $G^\times$ is called a *Lévy measure* for G if $\int_{G^\times} f\,d\eta < +\infty$ for all $f \in \mathscr{D}_+(G)$, $f(e) = 0$ and if $\eta(\complement U) < +\infty$ for all $U \in \mathfrak{V}(G)$.

Remark 5. A positive Radon measure η on $G^\times$ is a Lévy measure for G if and only if $\int_{G^\times} f\,d\eta < +\infty$ for all $f \in \mathscr{E}_+(G)$, $f(e) = 0$.

2. Extension of the Lévy–Chintschin formula from $\mathscr{D}(G)$ to $\mathscr{E}(G)$.

Let G be a locally compact group and $(\mu_t)_{t>0}$ a convolution semigroup on G. The generating functional $(A, \mathscr{A})$ of $(\mu_t)_{t>0}$ is defined by

$$\mathscr{A} = \left\{ f \in \mathscr{C}^{\mathrm{b}}(G) : \lim_{t\downarrow 0} \frac{1}{t} \int [f - f(e)] d\mu_t \text{ exists} \right\}$$

and

$$A(f) = \lim_{t\downarrow 0} \frac{1}{t} \int [f - f(e)]\, d\mu_t \quad \text{for } f \in \mathscr{A}\,.$$

$(A, \mathscr{A})$ has the following *properties*:

1. $\mathscr{D}(G)$ is contained in $\mathscr{A}$.
2. A is a real functional.
3. $\mathscr{A}$ contains the constants, and $A(1_G) = 0$.
4. A is nearly positive, that is, $A(f) \geqq 0$ for all $f \in \mathscr{A}_+$, $f(e) = 0$.
5. A is normed (on $\mathscr{D}(G)$), i.e.

$$\sup \{A(f) : f \in \mathscr{D}(G),\ 1_U \leqq f \leqq 1_G \text{ for some } U \in \mathfrak{B}(G)\} = 0\,.$$

6. A admits on $\mathscr{D}(G)$ a canonical decomposition (Lévy–Chintschin formula): There exist a primitive form A_1 on $\mathscr{D}(G)$, a quadratic form A_2 on $\mathscr{D}(G)$ and a Lévy measure η for G such that for all $f \in \mathscr{D}(G)$ we have (with a fixed Lévy mapping Γ for G):

$$A(f) = A_1(f) + A_2(f) + \int_{G^\times} [f - f(e) - \Gamma(f)]\, d\eta\,.$$

7. $\int_{G^\times} f\, d\eta = \lim_{t\downarrow 0} 1/t \int_G f\, d\mu_t$ for all $f \in \mathscr{K}(G^\times)$.

[In fact these properties have been established for real valued functions in [8]. But the extension to complex valued functions is immediate with the aid of property 2.]

We are now going to establish the decomposition described in 6. also for $A \mid \mathscr{E}(G)$. For this purpose let us define

$$\mathscr{C}_e^{\mathrm{b}}(G) = \{f \in \mathscr{C}^{\mathrm{b}}(G) : e \notin \operatorname{supp}(f)\}\,.$$

$\mathscr{C}_e^{\mathrm{b}}(G)$ may be considered as a subspace of $\mathscr{C}^{\mathrm{b}}(G^\times)$.

Lemma 1. *$\mathscr{C}_e^{\mathrm{b}}(G)$ is a subspace of $\mathscr{A}$, and $A(f) = \int_{G^\times} f\, d\eta$ for all $f \in \mathscr{C}_e^{\mathrm{b}}(G)$.*

Proof. Let $u \in \mathscr{D}(G)$ be a local unit. Then we get from properties 3 and 6 above:

$$\lim_{t\downarrow 0} \frac{1}{t} \int [1_G - u]\, d\mu_t = A(1_G - u) = -A(u) = \int [1_G - u]\, d\eta\,.$$

Furthermore for $f \in \mathscr{K}(G^\times)$ we have $f.[1_G - u] \in \mathscr{K}(G^\times)$, thus by property 7:

$$\lim_{t\downarrow 0}\frac{1}{t}\int f.[1_G-u]\,d\mu_t = \int f.[1_G-u]\,d\eta\,.$$

Together this gives us

$$\lim_{t\downarrow 0}\frac{1}{t}[1_G-u].\mu_t = [1_G-u].\eta$$

weakly in $\mathscr{M}^{\mathrm{b}}_{+}(G^{\times})$. Let $f\in\mathscr{C}^{\mathrm{b}}_{e}(G)$. Then u can be chosen such that $u.f=0$. It follows:

$$\begin{aligned}\int f\,d\eta &= \int f.[1_G-u]\,d\eta\\ &= \lim_{t\downarrow 0}\frac{1}{t}\int f.[1_G-u]\,d\mu_t = \lim_{t\downarrow 0}\frac{1}{t}\int f\,d\mu_t = A(f)\,.\end{aligned}$$

This proves the lemma.

PROPOSITION 1. (i) *$\mathscr{E}(G)$ is a subspace of $\mathscr{A}$.*

(ii) *Let $\hat{A}_1$, $\hat{A}_2$ and $\hat{\Gamma}$ be the canonical extensions of A_1, A_2 and Γ (from property* 6 *above) to $\mathscr{E}(G)$ (cf. lemma* 1.1 *and lemma* 1.2*). Then for every $f\in\mathscr{E}(G)$ the Lévy–Chintschin formula is valid*:

$$A(f) = \hat{A}_1(f)+\hat{A}_2(f)+\int_{G^{\times}}[f-f(e)-\hat{\Gamma}(f)]\,d\eta\,.$$

Here $\hat{A}_1$ is a primitive form and $\hat{A}_2$ a quadratic form on $\mathscr{E}(G)$.

PROOF. (i) Let $f\in\mathscr{E}(G)$, and let $u\in\mathscr{D}(G)$ be a local unit. Since $u.f\in\mathscr{D}(G)$ by the definition of $\mathscr{E}(G)$ we have $u.f\in\mathscr{A}$ by property 1. Since $(1_G-u).f\in\mathscr{C}^{\mathrm{b}}_{e}(G)$ we have $(1_G-u).f\in\mathscr{A}$ by lemma 1. Together this gives us $f\in\mathscr{A}$.

(ii) With the aid of lemma 1 and property 6 we obtain:

$$\begin{aligned}A(f) &= A(u.f)+A((1_G-u).f) = A_1(u.f)+A_2(u.f)+\\ &\quad+\int_{G^{\times}}[u.f-f(e)-\Gamma(u.f)]\,d\eta+\int_{G^{\times}}(1_G-u).f\,d\eta\\ &= \hat{A}_1(f)+\hat{A}_2(f)+\int_{G^{\times}}[f-f(e)-\hat{\Gamma}(f)]\,d\eta\,.\end{aligned}$$

The last statement follows from lemma 1.1.

3. Lévy–Chintschin formula on maximally almost periodic groups.

Let G be a locally compact group. By $\mathrm{Rep}_n\,(G)$ we denote the space of all n-dimensional continuous unitary representations of G, equipped with the compact-open topology ($n \in \mathbf{N}$). Rep (G), the topological sum of the $\mathrm{Rep}_n(G)$, $n \in \mathbf{N}$, is locally compact. The dimension of $D \in \mathrm{Rep}(G)$ is denoted by $n(D)$. If Rep (G) separates the points of G then G is called maximally almost periodic (MAP).

The linear span $\mathfrak{R}(G)$ of the coefficients of all $D \in \mathrm{Rep}(G)$ is an algebra over $\mathbf{C}$, the coefficient algebra of G. A linear functional N on $\mathfrak{R}(G)$ can thus also be considered as matrix valued mapping on Rep (G), and N is called continuous if it is continuous on Rep (G) (see [7, I, § 1]).

In the sequel we always assume that G is a MAP *group.*

REMARKS 1. Let G be a Lie projective group. Then for any compact subset C in Rep (G) there exist a compact normal subgroup K in G such that G/K is a Lie group and a compact subset C in Rep (G/K) such that $C = \{\dot{D} \circ p : \dot{D} \in \dot{C}\}$ (where p denotes the canonical mapping from G onto G/K).

2. $\mathfrak{R}(G)$ is a subspace of $\mathscr{E}(G)$ (see [2]).

DEFINITION 1. Let L be a real linear functional on $\mathfrak{R}(G)$.

a) L is called a *primitive form* if for all $f, g \in \mathfrak{R}(G)$ we have

$$L(f.g^*) = L(f)g(e) - f(e)L(g) .$$

b) L is called a *quadratic form* if $L(D)$ is a positive semidefinite Hermitian matrix for any $D \in \mathrm{Rep}(G)$, and if for all $f, g \in \mathfrak{R}(G)$ we have

$$L(f.g) + L(f.g^*) = 2.(L(f)g(e) + f(e)L(g)) .$$

REMARK 3. If L is a primitive (respectively quadratic) form on $\mathscr{E}(G)$ then $(-L) \mid \mathfrak{R}(G)$ is a primitive (respectively quadratic) form on $\mathfrak{R}(G)$. [We only have to show: If L is a quadratic form on $\mathscr{E}(G)$ then for any $D \in \mathrm{Rep}(G)$ the matrix $-L(D)$ is positive semidefinite. Let $D = (d_{ij})_{1 \leqq i,j \leqq n}$, $(a_1, \ldots, a_n) \in \mathbf{C}^n$ and

$$f := \sum_{i,j=1}^{n} \bar{a}_i a_j d_{ij} .$$

Then $f^* = \bar{f}$ and $|f| \leqq f(e)$ (since f is a positive definite function), thus

$$f + f^* \leqq 2f(e) = (f + f^*)(e) .$$

Since L is nearly positive and since $L(f^*) = L(f)$ as well as $L(1_G) = 0$ we get $(-L)(f) \geqq 0$.]

DEFINITION 2. Every mapping γ from $G \times \mathrm{Rep}(G)$ into $\bigcup_{n \geqq 1} \mathbf{M}(n, \mathbf{C})$ with the following five properties is called a *Lévy function* for G:

(LF 1) γ is a continuous mapping from $G \times \mathrm{Rep}_n(G)$ into $\mathsf{M}(n, \mathsf{C})$ for any $n \in \mathsf{N}$.

(LF 2) For every compact subset C in Rep (G) we have

$$\sup\{\|\gamma(x, D)\| \ : \ x \in G, \ D \in C\} < +\infty .$$

(LF 3) For every compact subset C in Rep (G) we have

$$\lim_{x \to e} \sup_{D \in C} \|\gamma(x, D)\| = 0 .$$

(LF 4) For every $x \in G$ there exists a one parameter subgroup $(x_t)_{t \in \mathsf{R}}$ in G such that $D(x_t) = \exp t\gamma(x, D)$ for all $D \in$ Rep (G) and $t \in \mathsf{R}$.

(LF 5) For every compact subset C in Rep (G) there exists a $U \in \mathfrak{V}(C)$ such that $D(x) = \exp \gamma(x, D)$ for all $D \in C$, $x \in U$.

REMARK 4. For any $x \in G$ one can extend $\gamma(x, .)$ uniquely to a real linear functional on $\mathfrak{R}(G)$ (by [7, Lemma 1.1]) also denoted by $\gamma(x, .)$. By (LF 4) each $\gamma(x, .)$ is then a primitive form on $\mathfrak{R}(G)$.

The following result is fundamental for the Lévy–Chintschin formula on MAP groups.

THEOREM 1. *On any* (*locally compact*) MAP *group there exist Lévy functions.*

SKETCH OF THE PROOF. 1. Let G_1 be an open Lie projective subgroup of G and suppose that there exists a Lévy function γ_1 for G_1. For any $D \in$ Rep (G) we have $D \mid G_1 \in$ Rep (G_1). Let us define

$$\gamma(x, D) := 0 \qquad \text{for } x \in \complement G_1 ,$$

$$\gamma(x, D) := \gamma_1(x, D \mid G_1) \qquad \text{for } x \in G_1 .$$

Then it is easy to see that γ is a Lévy function for G.

2. Thus without loss of generality we may assume that G itself is a Lie projective group. In this case we have proved the existence of a Lévy function for G in [7, Satz III.5]. But on the one side we constructed there a Lévy function with the somewhat artificial domain $G \times$ Rep (G_0), and on the other side the proof was based on the incorrect Lemma III.12 in [7]. These difficulties can now be overcome with the aid of the following two lemmas.

LEMMA 1. *Let G be a* (MAP) *Lie group. Then there exists a $D \in$* Rep (G) *such that the kernel of D is a discrete subgroup in G. If $\{X_1, \ldots, X_k\}$ is a base for $\mathscr{L}(G)$ then the vectors $(X_1 D)(e), \ldots, (X_k D)(e)$ are linearly independent* (*over* C).

LEMMA 2. *Let G be an arbitrary* MAP *group and let γ be a Lévy function for G. Let K be a compact normal subgroup in G such that G/K is a Lie group, $\{X_1,\dots,X_k\}$ a base for $\mathscr{L}(G/K)$ and $\{x_1,\dots,x_k\}$ a system of canonical coordinates for G/K in $\mathscr{D}(G/K)$ adapted to $\{X_1,\dots,X_k\}$* (cf. [7, III, § 1)]. *Finally let p denote the canonical mapping from G to G/K.*

Then there exist functions $y_1,\dots,y_k \in \mathscr{C}^b(G)$ such that

$$\gamma(x,\dot{D}\circ p) = \sum_{i=1}^{k} y_i(x)(X_i\dot{D})(\dot{e})$$

for all $D \in$ Rep (G/K) and $x \in G$. Furthermore there exists a neighbourhood $U \in \mathfrak{B}(G)$ such that $y_i(x)=x_i(p(x))$ for all $x \in U$.

Now the existence of a Lévy function for G can be established by an appropriate modification of the proof for the existence of a Lévy mapping (in [8, Satz 4.2]). A complete proof will be contained in the forthcoming book [4].

COROLLARY. *Let γ be a Lévy function for G and C a compact subset in* Rep (G). *Then there exist a neighbourhood $U \in \mathfrak{B}(G)$ and a function $f \in \mathscr{D}_+(G), f(e)=0$ such that*

$$\|\gamma(x,D)^2\| \leqq f(x) \quad \textit{for all } x \in U \textit{ and } D \in C\,.$$

PROOF. Let G_1 be an open Lie projective subgroup of G and γ_1 a Lévy function for G_1. Since $D \to D\,|\,G_1$ is a continuous mapping there exists by (LF 3) and (LF 5) a neighbourhood $U \in \mathfrak{B}(G)$, $U \subseteqq G_1$, such that

$$\gamma(x,D) = \log D(x) = \gamma_1(x,D\,|\,G_1) \quad \text{for all } x \in U,\ D \in C\,.$$

Thus without loss of generality we can assume that G is a Lie projective group.

Let K, p and $\dot{C}$ be as in remark 1 (corresponding to C). Moreover we use the notations introduced in lemma 2. Then by this very lemma there exists a $U \in \mathfrak{B}(G)$ such that

$$\|\gamma(x,\dot{D}\circ p)^2\| \leqq k\left[\max_{1\leqq i,j\leqq k} \|(X_i\dot{D})(\dot{e})\|\,\|(X_j\dot{D})(\dot{e})\|\right]\sum_{i=1}^{k}(x_i\circ p)^2(x)$$

for all $x \in U$, $D=\dot{D}\circ p \in C$.

But $\dot{D} \to (X_i\dot{D})(\dot{e})$ is continuous on Rep (G/K). [If $(\dot{x}_t)_{t\in R}$ is the one parameter subgroup in G/K corresponding to X_i then $\dot{D}(\dot{x}_t)=\exp t(X_i\dot{D})(\dot{e})$. This proves the statement (cf. [7, Lemma II.5]).] Therefore the exists a constant $c>0$ such that

$$\|(X_i\dot{D})(\dot{e})\| \leqq c \quad \text{for all } \dot{D} \in \dot{C} \ \ (1\leqq i\leqq k)\,.$$

Since $g=\sum_{i=1}^{k}(x_i\circ p)^2 \in \mathscr{D}_+(G)$, $g(e)=0$, the assertion follows with $f=(k.c.c).g$.

Finally we need a family of special functions: For any $D \in \operatorname{Rep}(G)$ let f_D be the function defined by

$$f_D(x) = \operatorname{Re}[\operatorname{Tr}(E_{n(D)}-D(x))] \qquad (\text{all } x \in G).$$

Obviously $f_D \in \mathfrak{K}_+(G)$, $f_D^*=f_D$, $f_D \leqq 2n(D)$ and

$$\ker(D) = \{x \in G : f_D(x)=0\}.$$

LEMMA 3. (i) *For any* $D \in \operatorname{Rep}(G)$ *we have*

$$\|\tfrac{1}{2}(D(x)+\widetilde{D(x)})-E_{n(D)}\| \leqq f_D(x) \qquad \textit{for all } x \in G.$$

(ii) *There exists a constant* $c>0$ *such that for any* $D \in \operatorname{Rep}(G)$ *we have*

$$\frac{1}{n^2} f_D(x^n) \leqq c.f_D(x) \qquad \textit{for all } n \in \mathbf{N} \textit{ and } x \in G.$$

Let γ *be a Lévy function for* G *and* C *a compact subset in* $\operatorname{Rep}(G)$.

(iii) *There exist a neighbourhood* $U \in \mathfrak{V}(G)$ *and constants* $c_1, c_2>0$ *such that*

$$c_1\|\gamma(x,D)^2\| \leqq f_D(x) \leqq c_2\|\gamma(x,D)^2\| \qquad \textit{for all } x \in U,\ D \in C.$$

(iv) *There exist a neighbourhood* $U \in \mathfrak{V}(G)$ *and a constant* $c_3>0$ *such that*

$$\|D(x)-E_{n(D)}-\gamma(x,D)\| \leqq c_3.f_D(x) \qquad \textit{for all } x \in U,\ D \in C.$$

PROOF. Let $D \in \operatorname{Rep}_m(G)$ and let $\alpha_1(x),\dots,\alpha_m(x)$ be the eigenvalues of $D(x)$. Since $D(x)$ is a unitary matrix we have $\alpha_j(x)=\exp i\vartheta_j(x)$ with $\vartheta_j(x) \in [-\pi,\pi]$. Thus

$$f_D(x) = \sum_{1\leqq j\leqq m}(1-\cos\vartheta_j(x)).$$

(i) Since $D(x)$ is diagonalizable we get

$$\|\tfrac{1}{2}(D(x)+\widetilde{D(x)})-E_m\| = \max_{1\leqq j\leqq m}|\cos\vartheta_j(x)-1| \leqq f_D(x).$$

(ii) There exists a constant $c>0$ such that

$$\frac{1}{n^2}(1-\cos(n\vartheta)) \leqq c.(1-\cos\vartheta) \qquad \text{for all } n \in \mathbf{N} \text{ and } \vartheta \in [-\pi,\pi]$$

(see [1, p. 183]). Since $\alpha_j(x^n)=\alpha_j(x)^n$ we get

$$\frac{1}{n^2}(1-\cos\vartheta_j(x^n)) = \frac{1}{n^2}\big(1-\cos(n\vartheta_j(x))\big) \leqq c\big(1-\cos\vartheta_j(x)\big)$$

$(1 \leqq j \leqq m)$ and thus

$$\frac{1}{n^2} f_D(x^n) \leqq c \cdot f_D(x) \quad \text{for all } n \in \mathbb{N} \ (x \in G)\,.$$

(iii) Without loss of generality we may assume $C \subseteq \mathrm{Rep}_m(G)$ for some $m \in \mathbb{N}$. Let $D \in C$ be arbitrary but fixed. By (LF 3) and (LF 5) there exists a neighbourhood $U \in \mathfrak{B}(G)$ such that

$$D(x) = \exp \gamma(x, D), \qquad \|D(x) - E_m\| < 1$$

and

$$\gamma(x, D) = \log D(x) = -\sum_{k \geqq 1} \frac{1}{k!}(D(x) - E_m)^k \quad \text{for all } x \in U\,.$$

Therefore $\gamma(x, D)$ is diagonalizable simultaneously with $D(x)$, and $\vartheta_1(x), \ldots, \vartheta_m(x)$ are the eigenvalues of $\gamma(x, D)$. This gives us

$$\|\gamma(x, D)^2\| = \|\gamma(x, D)\|^2 = \max_{1 \leqq j \leqq m} \vartheta_j(x)^2\,.$$

By minorizing U we can assume in addition that

$$|\vartheta_j(x)| \leqq \frac{1}{2\sqrt{e}} \qquad \text{for all } x \in U,\ 1 \leqq j \leqq m$$

(cf. (LF 3)).

For $|\vartheta| \leqq 1/2\sqrt{e}$ the elementary inequalities

$$\frac{\vartheta^2}{4} \leqq 1 - \cos\vartheta \leqq \frac{\vartheta^2}{2}$$

hold. From this we get for all $x \in U$:

$$\begin{aligned} \tfrac{1}{4}\|\gamma(x, D)^2\| &= \tfrac{1}{4} \max_{1 \leqq j \leqq m} \vartheta_j(x)^2 \leqq \sum_{j=1}^{m} (1 - \cos\vartheta_j(x)) \\ &= f_D(x) \leqq \tfrac{1}{2} \sum_{j=1}^{m} \vartheta_j(x)^2 \leqq \frac{m}{2} \cdot \max_{1 \leqq j \leqq m} \vartheta_j(x)^2 = \frac{m}{2} \|\gamma(x, D)^2\|\,. \end{aligned}$$

(iv) Choose U as in (iii). Then for $x \in U$ we have $\|\gamma(x, D)\| \leqq 1$ and

$$\begin{aligned} \|D(x) - E_m - \gamma(x, D)\| &= \left\| \sum_{k \geqq 2} \frac{1}{k!} \gamma(x, D)^k \right\| \\ &= \left\| \gamma(x, D)^2 \sum_{k \geqq 2} \frac{1}{k!} \gamma(x, D)^{k-2} \right\| \\ &\leqq e \cdot \|\gamma(x, D)\|^2\,. \end{aligned}$$

The assertion follows by (iii).

For every compact subset C of Rep (G) we define the function F_C by $F_C = \sup_{D \in C} f_D$. F_C is a positive bounded lower semicontinuous function on G.

LEMMA 4. *Let η be a Lévy measure for G (definition 1.3). Then for any compact subset C in* Rep (G) *the function F_C is η-integrable.*

PROOF. By lemma 3, (iii) and the corollary to theorem 1 there exist a neighbourhood $U \in \mathfrak{B}(G)$ and a function $f \in \mathscr{D}_+(G)$, $f(e)=0$, such that $F_C(x) \leqq f(x)$ for all $x \in U$. Since on the other side F_C is bounded the assertion follows.

DEFINITION 3. A positive Radon measure η on $G^\times$ is called a *weak Lévy measure* for G if

$$\int_{G^\times} F_C \, d\eta < +\infty$$

for all compact subsets C in Rep (G) and if $\eta(\complement U) < +\infty$ for all $U \in \mathfrak{B}(G)$.

REMARKS. 5. By lemma 4 any Lévy measure is a weak Lévy measure.

6. On a Lie projective group G the concepts of Lévy measure and weak Lévy measure coincide; moreover a positive Radon measure η on $G^\times$ such that $\eta(\complement U) < +\infty$ for all $U \in \mathfrak{B}(G)$ is a Lévy measure if and only if $\int f_D \, d\eta < +\infty$ for all $D \in$ Rep (G). [If G is a Lie group and φ a Hunt function for G (see [7, III, § 1]) there exist $D \in$ Rep (G), $U \in \mathfrak{B}(G)$ and $c > 0$ such that $\varphi(x) \leqq c . f_D(x)$ for all $x \in U$: This follows with the aid of lemma 1 (cf. part 2 in the proof of [6, Lemma III.9]).]

PROPOSITION 1. *Let γ be a Lévy function for G and η a weak Lévy measure for G. Then for every $f \in \mathfrak{K}(G)$ the integral*

$$\psi_\eta(f) := -\int_{G^\times} [f(x) - f(e) - \gamma(x, f)] \eta(dx)$$

exists. ψ_η is a real linear functional on $\mathfrak{K}(G)$ and is continuous (considered as a mapping on Rep (G)*).*

PROOF. The existence of $\psi_\eta(D)$ for $D \in$ Rep (G) follows from lemma 3, (iv) and (LF 2).

Let $D_0 \in$ Rep (G) and $\varepsilon > 0$ be given. We choose a compact neighbourhood C of D_0. By lemma 3, (iv) and by $\int F_C \, d\eta < +\infty$ there exists a neighbourhood $U \in \mathfrak{B}(G)$ such that

$$\int_{U\setminus\{e\}} \|D(x)-E_{n(D)}-\gamma(x,D)\|\eta(dx) \leqq \frac{\varepsilon}{4}$$

for all $D \in C$. By $\eta(\complement U) < +\infty$ and by (LF 1) there exists a neighbourhood C_0 of D_0, $C_0 \subseteq C$, such that

$$\left\|\int_{\complement U} [(D_0(x)-D(x))-(\gamma(x,D_0)-\gamma(x,D))]\eta(dx)\right\| \leqq \frac{\varepsilon}{2}$$

for all $D \in C_0$. Together we get $\|\psi_\eta(D_0)-\psi_\eta(D)\| \leqq \varepsilon$ for all $D \in C_0$.

Now we are in the position to prove the Lévy–Chintschin formula on MAP groups.

THEOREM 2. *Let G be a (locally compact)* MAP *group, γ a Lévy function for G and $(\mu_t)_{t>0}$ a convolution semigroup on G.*

(i) *For any $f \in \mathfrak{R}(G)$ the limit*

$$\psi(f) = \lim_{t\downarrow 0}\frac{1}{t}\int [f(e)-f]\,d\mu_t$$

exists. ψ is a continuous real linear functional on $\mathfrak{R}(G)$.

(ii) *There exist a continuous primitive form ψ_1 on $\mathfrak{R}(G)$, a continuous quadratic form ψ_2 on $\mathfrak{R}(G)$ and a weak Lévy measure η for G such that the following decomposition of ψ is valid:*

(LC) $$\psi = \psi_1+\psi_2+\psi_\eta$$

or more explicitly

$$\psi(f) = \psi_1(f)+\psi_2(f)-\int_{G^\times} [f(x)-f(e)-\gamma(x,f)]\,\eta(dx)$$

for all $f \in \mathfrak{R}(G)$.

PROOF. (i) The existence of ψ follows from proposition 2.1. The continuity of ψ has been shown in [7, Lemma II.5].

(ii) If A is the generating functional of $(\mu_t)_{t>0}$ then $A(f) = -\psi(f)$ for $f \in \mathfrak{R}(G)$. With the notation of proposition 2.1 we thus get

$$\psi(f) = -\hat{A}_1(f)-\hat{A}_2(f)-\int_{G^\times} [f-f(e)-\hat{\Gamma}(f)]\,d\eta\,.$$

By lemma 1.1 and remark 3.3

$$\psi_1' := -\hat{A}_1 \,|\, \mathfrak{R}(G)$$

is a primitive form and

$$\psi_2 := -\hat{A}_2 | \mathfrak{K}(G)$$

is a quadratic form on $\mathfrak{K}(G)$. By proposition 1

$$\psi_1''(f) = -\psi_\eta(f) - \int_{G^\times} [f - f(e) - \hat{\Gamma}(f)]\, d\eta = \int_{G^\times} [\hat{\Gamma}(f) - \gamma(.\,, f)]\, d\eta$$

exists for all $f \in \mathfrak{K}(G)$. By (LM 3) (cf. lemma 1.2) and remark 4 ψ_1'' is a primitive form on $\mathfrak{K}(G)$. Thus

$$\psi = \psi_1' + \psi_1'' + \psi_2 + \psi_\eta = \psi_1 + \psi_2 + \psi_\eta$$

where $\psi_1 := \psi_1' + \psi_1''$ is again a primitive form on $\mathfrak{K}(G)$.

By proposition 1 and by part (i) of this theorem $\psi_0 := \psi_1 + \psi_2 = \psi - \psi_\eta$ is continuous. For $D \in \mathrm{Rep}\,(G)$ we have

$$\psi_1(\tilde{D}) = -\psi_1(D) \quad \text{and} \quad \psi_2(\tilde{D}) = \psi_2(D)\,.$$

Thus we get

$$\psi_2(D) = \tfrac{1}{2}(\psi_0(D) + \psi_0(\tilde{D})) = \tfrac{1}{2}(\psi_0(D) + \widetilde{\psi_0}(D))\,.$$

This shows the continuity of ψ_2 and of $\psi_1 = \psi_0 - \psi_2$.

The linear functional ψ introduced in theorem 2 is called the *negative definite form* associated with the convolution semigroup $(\mu_t)_{t>0}$,

REMARK 7. It follows immediately from proposition 1 that conversely by (LC) there is always given a continuous negative definite form ψ on $\mathfrak{K}(G)$. Thus if G is a B-group, especially if G is a Moore group, then there exists a unique convolution semigroup on G with the associated negative definite form ψ (cf. [7, Satz III.6]).

Finally we prove a formula for the quadratic form ψ_2 in (LC) similar to the equation (10) in [1, Theorem 18.19].

By $\mathfrak{K}_n(G)$ we denote the space of all mappings $M = (m_{ij})_{1 \leq i, j \leq n}$ from G into $\mathsf{M}(n, \mathsf{C})$ whose coefficients m_{ij} belong to $\mathfrak{K}(G)$. Obviously $\mathrm{Rep}_n\,(G)$ is contained in $\mathfrak{K}_n(G)$, and $\mathfrak{K}_1(G) = \mathfrak{K}(G)$. If $M \in \mathfrak{K}_n(G)$ also $M^* = (m_{ij}^*)_{1 \leq i, j \leq n}$ is in $\mathfrak{K}_n(G)$. If $M_1, M_2 \in \mathfrak{K}_n(G)$ their matrix product $M_1 M_2$ is also in $\mathfrak{K}_n(G)$.

Let ψ be a linear functional on $\mathfrak{K}(G)$. Then ψ can be extended uniquely to a linear mapping from $\mathfrak{K}_n(G)$ into $\mathsf{M}(n, \mathsf{C})$ by $\psi(M) = (\psi(m_{ij}))_{1 \leq i, j \leq n}$.

LEMMA 5. (i) *If ψ is a primitive form on $\mathfrak{K}(G)$ then $\psi(D^n) = n\psi(D)$ for all $n \in \mathsf{N}$ ($D \in \mathrm{Rep}\,(G)$).*

(ii) *If ψ is a quadratic form on $\mathfrak{K}(G)$ then $\psi(D^n) = n^2\psi(D)$ for all $n \in \mathsf{N}$ ($D \in \mathrm{Rep}\,(G)$).*

PROOF. (i) If ψ is a primitive form on $\mathfrak{K}(G)$ then

$$\psi(f.g) = \psi(f)g(e)+f(e)\psi(g) \quad \text{for all } f,g \in \mathfrak{K}(G) .$$

It follows

$$\psi(M_1M_2) = \psi(M_1)M_2(e)+M_1(e)\psi(M_2) \quad \text{for all } M_1,M_2 \in \mathfrak{K}_m(G) .$$

Thus we get for $D \in \mathrm{Rep}_m(G)$ by induction on n:

$$\psi(D^{n+1}) = \psi(D^n)+\psi(D) = n\psi(D)+\psi(D) = (n+1)\psi(D) .$$

(ii) If ψ is a quadratic form on $\mathfrak{K}(G)$ then it follows from the defining functional equation (definition 1) for all $M_1,M_2 \in \mathfrak{K}_m(G)$:

$$\psi(M_1M_2)+\psi(M_1M_2^*) = 2[\psi(M_1)M_2(e)+M_1(e)\psi(M_2)] .$$

Since $DD^*=E_m$ and $D^*=\tilde{D}$ for $D \in \mathrm{Rep}_m(G)$ we get by induction on n:

$$\begin{aligned}\psi(D^{n+1})+(n-1)^2\psi(D) &= \psi(D^{n+1})+\psi(D^{n-1}) = \psi(D^nD)+\psi(D^nD^*)\\ &= 2[\psi(D^n)+\psi(D)] = 2[n^2\psi(D)+\psi(D)] ,\end{aligned}$$

and thus $\psi(D^{n+1})=(n+1)^2\psi(D)$.

PROPOSITION 2. *Let ψ be the negative definite form associated with the convolution semigroup $(\mu_t)_{t>0}$ on G and $\psi=\psi_1+\psi_2+\psi_\eta$ the decomposition* (LC). *Then for any $D \in \mathrm{Rep}(G)$ we have*

$$\psi_2(D) = \lim_{n\geqq 1}\frac{1}{n^2}(\psi(D^n)+\widetilde{\psi(D^n)}) .$$

Especially ψ_2 is uniquely determined by ψ and thus by $(\mu_t)_{t>0}$.

PROOF. In view of lemma 5 we may assume without loss of generality $\psi=\psi_\eta$. Furthermore we have

$$\gamma(x,(D^*)^n) = -\gamma(x,D^n)$$

(remark 4). Therefore by lemma 3, (i) and (ii):

$$\begin{aligned}\left\|\frac{1}{n^2}(\psi(D^n)+\widetilde{\psi(D^n)})\right\| &= \frac{1}{n^2}\left\|\int_{G^\times}[\tfrac{1}{2}(D(x)^n+\widetilde{D(x)^n})-E_{n(D)}]\eta(dx)\right\|\\ &= \frac{1}{n^2}\left\|\int_{G^\times}[\tfrac{1}{2}(D(x^n)+\widetilde{D(x^n)})-E_{n(D)}]\eta(dx)\right\|\\ &\leqq \int_{G^\times}\frac{1}{n^2}f_D(x^n)\eta(dx) \leqq c.\int_{G^\times}f_D(x)\eta(dx)\end{aligned}$$

for all $n \in \mathbf{N}$. By Lebesgue's theorem we thus have

$$\lim_{n \geq 1} \frac{1}{n^2} (\psi(D^n) + \widetilde{\psi(D^n)}) = 0 .$$

REFERENCES

1. Chr. Berg and G. Forst, *Potential theory on locally compact abelian groups* (Ergebnisse Math. 87), Springer-Verlag, Berlin - Heidelberg - New York, 1975.
2. F. Bruhat, *Distributions sur un groupe localement compact et applications à l'étude des representation des groupes p-adiques*, Bull. Soc. Math. France 89 (1961), 43–76.
3. W. Hazod, *Über die Lévy–Hincin-Formel auf lokalkompakten topologischen Gruppen*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 25 (1973), 301–322.
4. H. Heyer, *Probability measures on locally compact groups*, In preparation for Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin - Heidelberg - New York.
5. G. A. Hunt, *Semigroups of measures on Lie groups*, Trans. Amer. Math. Soc. 81 (1956), 264–293.
6. E. Siebert, *Wahrscheinlichkeitsmaße auf lokalkompakten maximal fastperiodischen Gruppen*, Dissertation, Tübingen, 1972 (unpublished).
7. E. Siebert, *Stetige Halbgruppen von Wahrscheinlichkeitsmaßen auf lokalkompakten maximal fastperiodischen Gruppen*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 25 (1973), 269–300.
8. E. Siebert, *Über die Erzeugung von Faltungshalbgruppen auf beliebigen lokalkompakten Gruppen*, Math. Z. 131 (1973), 313–333.

MATHEMATISCHES INSTITUT
UNIVERSITÄT TÜBINGEN
AUF DER MORGENSTELLE 10
D-7400 TÜBINGEN
WEST-GERMANY