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ON THE FUNCTION $\sum_{n=0}^{\infty} \frac{\zeta(n+2)}{n!} x^n$

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The sequence

$$\zeta(2) = \sum \frac{1}{n^2},\ \zeta(3) = \sum \frac{1}{n^3},\ \zeta(4) = \sum \frac{1}{n^4}, \ldots$$

bears in some respects an analogy with the sequence of Bernoulli numbers $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}, \ldots$. For the Bernoulli numbers appear in expressions for the coefficients of polynomial solutions to the difference equation

$$f(x+1) - f(x) = x^k ,$$

$k = 0, 1, 2, \ldots$, whereas for $k = -m$, $m \geqq 2$, this equation has the solution

$$f(x) = -\sum_{n=0}^{\infty} \frac{1}{(x+n)^m}$$

whose Laurent expansion at the origin is

$$-\frac{1}{x^m} - \zeta(m) + m\zeta(m+1)x - \frac{m(m+1)}{2}\zeta(m+2)x^2 + \ldots .$$

If, following this analogy, we replace B_k *by* $\zeta(k+2)$ in the well-known formula

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k ,$$

we obtain the entire function

$$\mathfrak{G}(x) = \sum_{k=0}^{\infty} \frac{\zeta(k+2)}{k!} x^k . \tag{1}$$

This function has apparently not been studied, and in view of its connections with important functions in analysis and number theory, it seems worthwhile to give here a number of results concerning $\mathfrak{G}(x)$, together with a proof of the functional equation for Riemann's zeta function, to which the study of $\mathfrak{G}(x)$ leads naturally.

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To begin with, by substituting $\sum_{n=1}^{\infty} 1/n^{k+2}$ for $\zeta(k+2)$ in (1) and reversing the order of summation, one finds

$$\mathfrak{G}(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{x/n} . \tag{2}$$

This expression shows that $\mathfrak{G}(x)$ is almost periodic in every half-plane $\operatorname{Re} x \leqq \alpha$; in fact, if N is divisible by $2, 3, \ldots, K$ and $\operatorname{Re} x \leqq \alpha$, then

$$|\mathfrak{G}(x+2\pi iN) - \mathfrak{G}(x)| \leqq 2 \sum_{n=K+1}^{\infty} \frac{1}{n^2} e^{\alpha/n} = O\left(\frac{1}{K}\right) \quad \text{as } K \to \infty .$$

It follows that every root ϱ of the equation $\mathfrak{G}(x) = c$ gives rise to an infinite sequence of roots $\{\varrho_k\}_{k=-\infty}^{\infty}$, such that ϱ_k is very nearly equal to $\varrho + 2\pi iNk$, where N is the least common multiple of some sufficiently large initial segment of the positive integers. On the other hand, since $\mathfrak{G}$ is an entire function of order 1, and is clearly not of the form $e^{c_1 x} + c_2$ for c_1, c_2 constants, it follows by Hadamard's theory of entire functions of finite order that $\mathfrak{G}(x)$ takes on every complex value. These facts show in particular that the sum $\sum_{\varrho} 1/|\varrho|$, taken over all zeroes ϱ of $\mathfrak{G}(x)$, diverges. From Hadamard's factorization theorem we conclude that $\mathfrak{G}(x)$ has an everywhere convergent product expansion of the form

$$\alpha e^{\beta x} \prod_{\varrho} \left(1 - \frac{x}{\varrho}\right) e^{x/\varrho} . \tag{3}$$

The values of α, β and $\sum_{\varrho} 1/\varrho^k$ $(k \geqq 2)$ are easily computed in terms of $\zeta(2)$, $\zeta(3)$, $\zeta(4), \ldots$ using (1) and (3); one finds in particular

$$\alpha = \zeta(2), \qquad \beta = \frac{\zeta(3)}{\zeta(2)}, \qquad \sum_{\varrho} \frac{1}{\varrho^2} = \frac{\zeta(3)^2 - \zeta(2)\zeta(4)}{\zeta(2)^2} .$$

After these remarks I leave aside the interesting question of the distribution of zeroes of $\mathfrak{G}(x)$, as well as the connection between the values of $\mathfrak{G}(x)$ and the multiplicative structure of the integers, which is reflected for example in the formula:

$$\lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \mathfrak{G}(2\pi ink) = \sum_{d \mid n} \frac{1}{d^2} .$$

The function $\Pi(s)\zeta(1-s)$ is easily expressed as an integral transform of $\mathfrak{G}(x)$. Starting from Euler's integral for the factorial function,

$$\Pi(s) = \int_0^{\infty} e^{-x} x^s \, dx \qquad (\operatorname{Re} s = \sigma > -1) , \tag{4}$$

we make the substitution $x = u/n$, multiply both sides by n^{s-1}, and sum over $n = 1, 2, \ldots$ to find

$$\Pi(s)\zeta(1-s) = \int_0^\infty \{u\mathfrak{G}(-u)\}u^{s-1}\,du\,, \tag{5}$$

valid for $-1<\sigma<0$. This formula is analogous to the classical expression

$$\frac{1}{s}\Pi(s)\zeta(s) = \int_0^\infty \left\{\frac{1}{e^u-1}\right\}u^{s-1}\,du \qquad (\sigma>1) \tag{6}$$

which one finds by making instead the substitution $x=nu$ in (4). Comparison of (5) and (6) leads to the functional equation for $\zeta(s)$, as will be shown below.

First it is useful to have an estimate for $\mathfrak{G}(-u)$ when u is a large positive number. This is provided by the Euler–Maclaurin formula, applied to the function

$$f(t) = \frac{1}{t^2}e^{-u/t}:$$

$$\sum_{n=2}^\infty \frac{1}{n^2}e^{-u/n} = \int_1^\infty \frac{1}{t^2}e^{-u/t}\,dt + \sum_{r=1}^q (-1)^r\frac{B_r}{r!}\left[\left(\frac{d}{dt}\right)^{r-1}\frac{1}{t^2}e^{-u/t}\right]_1^\infty + R_q \tag{7}$$

where

$$R_q = \frac{(-1)^{q-1}}{q}\int_1^\infty B_q(t-[t])\left\{\left(\frac{d}{dt}\right)^q\frac{1}{t^2}e^{-u/t}\right\}dt\,.$$

One sees easily by induction that $(d/dt)^r\ 1/t^2\ e^{-u/t}$ is of the form

$$\frac{1}{t^{r+2}}P_r\left(\frac{u}{t}\right)e^{-u/t}$$

where P_r is a polynomial of degree r. It follows immediately from (7) that

$$\mathfrak{G}(-u) = \frac{1}{u}+R_q+O(u^{q-1}e^{-u}) \qquad \text{as } u\to\infty\,.$$

Furthermore

$$\begin{aligned}|R_q| &\leqq c\int_1^\infty \frac{1}{t^{q+2}}\left|P_q\left(\frac{u}{t}\right)\right|e^{-u/t}\,dt\\ &= \frac{c}{u^{q+1}}\int_0^u y^q|P_q(y)|e^{-y}\,dy\\ &= O\left(\frac{1}{u^{q+1}}\right)\end{aligned}$$

since the second integral converges for $u=\infty$. Therefore

$$\mathfrak{G}(-u) = \frac{1}{u}+o\left(\frac{1}{u^q}\right) \qquad \text{as } u\to+\infty \tag{8}$$

for every q.

Using this estimate it is easy to derive from (5) a partial fractions expansion for $\Pi(s)\zeta(1-s)$. We write (5) as

$$(9)\qquad \Pi(s)\zeta(1-s) = \int_0^1 \{u\mathfrak{G}(-u)\}u^{s-1}\,du - \frac{1}{s} + \int_1^\infty \{u\mathfrak{G}(-u)-1\}u^{s-1}\,du$$

and evaluate the first integral by termwise integration of

$$\mathfrak{G}(-u)u^{s-1} = \sum_{k=0}^{\infty} \frac{\zeta(k+2)}{k!}(-1)^k u^{s-k-1}$$

to obtain

$$\Pi(s)\zeta(1-s) = -\frac{1}{s} + \sum_{k=0}^{\infty} \frac{\zeta(k+2)}{k!}\frac{(-1)^k}{s+k+1} + \int_1^\infty \{u\mathfrak{G}(-u)-1\}u^{s-1}\,du\,,$$

where by (8) the integral on the right is an entire function of s.

The functions $\mathfrak{G}(x)$ and $1/(e^x-1)$, already related by an analogy and by their appearence in (5) and (6), can be brought into even closer connection. I take as point of departure the integral formula

$$\mathfrak{G}(x) = \int_{c+i\infty}^{c-i\infty} \frac{e^{x/z}}{z^2}\frac{dz}{e^{2\pi i z}-1} \qquad (0<c<1)$$

which is easily verified using the calculus of residues. This integral can be reduced to a Fourier integral, as follows. We deform the path of integration into the path Γ consisting of the segment $(i\infty, i\delta]$, the semicircle of radius δ around 0 in the half-plane $\operatorname{Re} z \geqq 0$ and the segment $[-i\delta, -i\infty)$, where $0<\delta<1$, and add the equation

$$0 = \int_\Gamma \frac{e^{x/z}}{z^2}\left\{-\frac{1}{2\pi i z}+\frac{1}{2}\right\}dz$$

to obtain

$$\mathfrak{G}(x) = \int_\Gamma \frac{e^{x/z}}{z^2}\left\{\frac{1}{e^{2\pi i z}-1}-\frac{1}{2\pi i z}+\frac{1}{2}\right\}dz$$

in which the function in brackets has a zero of order one at the origin. Letting $z=1/iy$ in the last integral, one finds

$$\mathfrak{G}(x) = -i\int_{\Gamma'}\left\{\frac{1}{e^{2\pi/y}-1}-\frac{y}{2\pi}+\frac{1}{2}\right\}e^{ixy}\,dy$$

where Γ' is the closed path consisting of the segment $[-1/\delta, 1/\delta]$, together with the semicircle c of radius $1/\delta$ around the origin lying in the half-plane $\operatorname{Im} y \leqq 0$, this path being traversed in the positive sense. Now when x is a real number

<0, the integrand is $O(|y|^{-1}e^{|x|\,\mathrm{Im}(y)})$ on the circle c, from which it follows easily that the integral over c tends to zero as the radius increases, so that, letting $\delta \to 0$,

$$\mathfrak{G}(x) = \int_{-\infty}^{\infty} i\left\{\frac{1}{e^{2\pi/y}-1} - \frac{y}{2\pi} + \frac{1}{2}\right\} e^{ixy}\, dy \tag{10}$$

$$= \int_{-\infty}^{\infty} i\varphi\left(\frac{2\pi}{y}\right) e^{ixy}\, dy$$

where

$$\varphi(z) = \frac{1}{e^z - 1} - \frac{1}{z} + \frac{1}{2}.$$

Since φ is an odd function, (10) is equivalent to

$$\frac{\operatorname{sgn}(x)}{2}\mathfrak{G}(-|x|) = \int_0^{\infty} \varphi\left(\frac{2\pi}{y}\right) \sin xy\, dy. \tag{11}$$

This result can be verified, at least formally, by substituting for $\varphi(2\pi/y)$ the partial fraction expansion

$$\frac{1}{\pi}\sum_{k=1}^{\infty} \frac{y}{1+k^2y^2}$$

and integrating termwise using the equation

$$\frac{1}{\pi}\int_0^{\infty} \frac{y \sin xy}{\beta^2 + y^2}\, dy = \frac{\operatorname{sgn}(x)}{2} e^{-\beta|x|}.$$

In order to invert the relation (11), I first render the integral absolutely convergent by an integration by parts:

$$\int_0^{\infty} \varphi\left(\frac{2\pi}{y}\right) \sin xy\, dy = \lim_{n\to\infty} \int_{1/n}^{n} \varphi\left(\frac{2\pi}{y}\right) d\left(-\frac{1}{x}\cos xy\right)$$

$$= \lim_{n\to\infty}\left[-\varphi\left(\frac{2\pi}{n}\right)\frac{1}{x}\cos nx + \varphi(2\pi n)\frac{1}{x}\cos\frac{x}{n} + \frac{1}{x}\int_{1/n}^{n}\left\{\frac{d}{dy}\varphi\left(\frac{2\pi}{y}\right)\right\}\cos xy\, dy\right]$$

$$= \frac{1}{2x} + \frac{1}{x}\int_0^{\infty}\left\{\frac{d}{dy}\varphi\left(\frac{2\pi}{y}\right)\right\}\cos xy\, dy.$$

By (11), it follows that

$$\frac{1}{2}(|x|\cdot\mathfrak{G}(-|x|) - 1) = \int_0^{\infty}\left\{\frac{d}{dy}\varphi\left(\frac{2\pi}{y}\right)\right\}\cos xy\, dy.$$

Since

$$\frac{d}{dy}\varphi\left(\frac{2\pi}{y}\right) = O\left(\frac{1}{y^2}\right) \quad \text{as } y \to \infty\,,$$

the integral on the right converges absolutely, so that by Fourier's theorem

$$\frac{d}{dy}\varphi\left(\frac{2\pi}{y}\right) = \frac{1}{\pi}\int_0^\infty \{x\mathfrak{G}(-x)-1\}\cos xy\,dx\,. \tag{12}$$

This integral, which converges rapidly in virtue of the estimate (8), is also the derivative of

$$\frac{1}{\pi}\int_0^\infty \{x\mathfrak{G}(-x)-1\}\frac{1}{x}\sin xy\,dx = \frac{1}{\pi}\int_0^\infty \mathfrak{G}(-x)\sin xy\,dx - \frac{1}{2}\cdot\operatorname{sgn}(y)$$

which has the value $-\frac{1}{2}$ for $y=\infty$. Since $\varphi(2\pi/\infty)=0$, (12) implies

$$\frac{1}{\pi}\int_0^\infty \{x\mathfrak{G}(-x)-1\}\frac{1}{x}\sin xy\,dx = \varphi\left(\frac{2\pi}{y}\right)-\frac{1}{2} = \frac{1}{e^{2\pi/y}-1}-\frac{y}{2\pi}\,. \tag{13}$$

Finally I show how the functional equation for Riemann's zeta function

$$\zeta(s) = 2^s\pi^{s-1}\Pi(-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s) \tag{14}$$

follows from (5), (6), (13) and

$$\frac{1}{s}\Pi(-s)\sin\left(\frac{\pi s}{2}\right) = \int_0^\infty \frac{\sin y}{y^{s+1}}dy \quad (-1<\sigma<1)\,. \tag{15}$$

First, in order to obtain integrals with a common domain of absolute convergence, I use $1/s=\int_0^1 u^{s-1}\,du$ $(\sigma>0)$ to rewrite (9) as

$$\Pi(s)\zeta(s) = \int_0^\infty \{u\mathfrak{G}(-u)-1\}u^s\frac{du}{u} \quad (\sigma>0)\,. \tag{16}$$

A similar process, applied to (6) instead of (5), gives

$$\frac{1}{s}\Pi(s)\zeta(s) = \int_0^\infty \left\{\frac{1}{e^u-1}-\frac{1}{u}\right\}u^s\frac{du}{u} \quad (0<\sigma<1)\,, \tag{17}$$

and the substitution $y=u^{-1}$ takes (15) into

$$\frac{1}{s}\Pi(-s)\sin\left(\frac{\pi s}{2}\right) = \int_0^\infty \sin\left(\frac{1}{u}\right)u^s\frac{du}{u} \quad (0<\sigma<1)\,. \tag{18}$$

The integrals in (16), (17), (18) all converge absolutely for $0<\sigma<1$. Next, let $\mathscr{F}$ denote the family of all integrable functions f defined on $(0,\infty)$ and satisfying

$$\int_0^\infty |f(u)|u^\sigma \frac{du}{u} < \infty, \qquad 0<\sigma<1 ,$$

and let $\hat{f}$ and $f*g$ be defined for arbitrary $f,g \in \mathscr{F}$ by

$$\hat{f}(s) = \int_0^\infty f(u)u^s \frac{du}{u} \qquad (0<\sigma<1)$$

and

$$(f*g)(u) = \int_0^\infty f(t)g\left(\frac{u}{t}\right)\frac{dt}{t} \qquad (0<u<\infty) .$$

Then it is easily seen that: (i) $f*g \in \mathscr{F}$ and $(f*g)\hat{} = \hat{f}\cdot\hat{g}$, and (ii) if $h \in \mathscr{F}$ and $k(u)=\alpha h(\beta u)$ for constants α, β, $\beta>0$, then $\hat{k}(s)=\alpha\beta^{-1}\hat{h}(s)$. Taking now

$$f(u) = u\mathfrak{G}(-u)-1, \qquad g(u) = \sin\left(\frac{1}{u}\right), \qquad h(u) = \frac{1}{e^u-1}-\frac{1}{u},$$

(13), (16), (17) and (18) imply:

$$(f*g)(u) = \pi h(2\pi u) ,$$

$$\hat{f}(s) = \Pi(s)\zeta(1-s), \qquad \hat{g}(s) = \frac{1}{s}\Pi(-s)\sin\left(\frac{\pi s}{2}\right), \qquad \hat{h}(s) = \frac{1}{s}\Pi(s)\zeta(s) ,$$

from which the functional equation follows immediately in view of properties (i) and (ii) above.

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