ON THE FUNCTION $\sum_{n=0}^{\infty} \frac{\zeta(n+2)}{n!} x^n$

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The sequence

$$
\zeta(2) = \sum \frac{1}{n^2}, \quad \zeta(3) = \sum \frac{1}{n^3}, \quad \zeta(4) = \sum \frac{1}{n^4}, \ldots
$$

bears in some respects an analogy with the sequence of Bernoulli numbers $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}, \ldots$. For the Bernoulli numbers appear in expressions for the coefficients of polynomial solutions to the difference equation

$$
f(x+1) - f(x) = x^k,
$$

$k = 0, 1, 2, \ldots$, whereas for $k = -m$, $m \geq 2$, this equation has the solution

$$
f(x) = -\sum_{n=0}^{\infty} \frac{1}{(x+n)^m}
$$

whose Laurent expansion at the origin is

$$
-\frac{1}{x^m} - \zeta(m) + m\zeta(m+1)x - \frac{m(m+1)}{2}\zeta(m+2)x^2 + \ldots.
$$

If, following this analogy, we replace $B_k$ by $\zeta(k+2)$ in the well-known formula

$$
\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k,
$$

we obtain the entire function

$$
(1) \quad \mathcal{G}(x) = \sum_{k=0}^{\infty} \frac{\zeta(k+2)}{k!} x^k.
$$

This function has apparently not been studied, and in view of its connections with important functions in analysis and number theory, it seems worthwhile to give here a number of results concerning $\mathcal{G}(x)$, together with a proof of the functional equation for Riemann's zeta function, to which the study of $\mathcal{G}(x)$ leads naturally.

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To begin with, by substituting $\sum_{n=1}^{\infty} 1/n^{k+2}$ for $\zeta(k+2)$ in (1) and reversing
the order of summation, one finds
\begin{equation}
\mathcal{G}(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} e^{x/n}.
\end{equation}

This expression shows that $\mathcal{G}(x)$ is almost periodic in every half-plane $\text{Re } x \leq \alpha$; in fact, if $N$ is divisible by $2, 3, \ldots, K$ and $\text{Re } x \leq \alpha$, then
\begin{equation}
|\mathcal{G}(x + 2\pi i N) - \mathcal{G}(x)| \leq 2 \sum_{n=K+1}^{\infty} \frac{1}{n^x} e^{x/n} = O\left(\frac{1}{K}\right) \quad \text{as } K \to \infty.
\end{equation}

It follows that every root $\rho$ of the equation $\mathcal{G}(x) = c$ gives rise to an infinite
sequence of roots $\{\rho_k\}_{k=-\infty}^{\infty}$, such that $\rho_k$ is very nearly equal to $\rho + 2\pi i k$, where $N$ is the least common multiple of some sufficiently large initial segment of the positive integers. On the other hand, since $\mathcal{G}$ is an entire function of order 1, and is clearly not of the form $e^{\rho_x} + c_2$ for $c_1, c_2$ constants, it follows by Hadamard's theory of entire functions of finite order that $\mathcal{G}(x)$ takes on every complex value. These facts show in particular that the sum $\sum_{q} 1/|q|$, taken over all zeroes $q$ of $\mathcal{G}(x)$, diverges. From Hadamard's factorization theorem we conclude that $\mathcal{G}(x)$ has an everywhere convergent product expansion of the form
\begin{equation}
\alpha e^{\beta x} \prod_{q} \left(1 - \frac{x}{q}\right) e^{x/q}.
\end{equation}

The values of $\alpha, \beta$ and $\sum_{q} 1/|q|^k$ ($k \geq 2$) are easily computed in terms of $\zeta(2), \zeta(3), \zeta(4), \ldots$ using (1) and (3); one finds in particular
\begin{equation}
\alpha = \zeta(2), \quad \beta = \frac{\zeta(3)}{\zeta(2)}, \quad \sum_{q} \frac{1}{\rho^2} = \frac{\zeta(3)^2 - \zeta(2)\zeta(4)}{\zeta(2)^2}.
\end{equation}

After these remarks I leave aside the interesting question of the distribution of
zeroes of $\mathcal{G}(x)$, as well as the connection between the values of $\mathcal{G}(x)$ and the
multiplicative structure of the integers, which is reflected for example in the
formula:
\begin{equation}
\lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \mathcal{G}(2\pi i k) = \sum_{d|n} \frac{1}{d^2}.
\end{equation}

The function $\Pi(s)\zeta(1 - s)$ is easily expressed as an integral transform of $\mathcal{G}(x)$. Starting from Euler's integral for the factorial function,
\begin{equation}
\Pi(s) = \int_{0}^{\infty} e^{-x^s} dx \quad (\text{Re } s = \sigma > -1),
\end{equation}

we make the substitution $x = u/n$, multiply both sides by $n^{s-1}$, and sum over $n=1, 2, \ldots$ to find
(5) \[ \Pi(s)\zeta(1-s) = \int_0^\infty \{ u\Phi(-u) \} u^{s-1} \, du, \]
valid for \(-1 < \sigma < 0\). This formula is analogous to the classical expression

(6) \[ \frac{1}{s} \Pi(s)\zeta(s) = \int_0^\infty \left\{ \frac{1}{e^u-1} \right\} u^{s-1} \, du \quad (\sigma > 1) \]

which one finds by making instead the substitution \( x = nu \) in (4). Comparison of (5) and (6) leads to the functional equation for \( \zeta(s) \), as will be shown below.

First it is useful to have an estimate for \( \Phi(-u) \) when \( u \) is a large positive number. This is provided by the Euler–Maclaurin formula, applied to the function

\[ f(t) = \frac{1}{t^2} e^{-u/t}. \]

(7) \[ \sum_{n=2}^{\infty} \frac{1}{n^2} e^{-u/n} = \int_1^\infty \frac{1}{t^2} e^{-u/t} \, dt + \sum_{r=1}^q (-1)^r \frac{B_r}{r!} \left[ \left( \frac{d}{dt} \right)^{r-1} \frac{1}{t^2} e^{-u/t} \right]_1^\infty + R_q \]

where

\[ R_q = \frac{(-1)^{q-1}}{q} \int_1^\infty B_q(t-[t]) \left\{ \frac{d}{dt} \right\}^q \frac{1}{t^2} e^{-u/t} \, dt. \]

One sees easily by induction that \( (d/dt)^r 1/t^2 e^{-u/t} \) is of the form

\[ \frac{1}{t^{r+2}} P_r \left( \frac{u}{t} \right) e^{-u/t} \]

where \( P_r \) is a polynomial of degree \( r \). It follows immediately from (7) that

\[ \Phi(-u) = \frac{1}{u} + R_q + O(u^{q-1} e^{-u}) \quad \text{as } u \to \infty. \]

Furthermore

\[ |R_q| \leq c \int_1^\infty \frac{1}{t^{q+2}} \left| P_q \left( \frac{u}{t} \right) \right| e^{-u/t} \, dt \]

\[ = \frac{c}{u^{q+1}} \int_0^u y^q |P_q(y)| e^{-y} \, dy \]

\[ = O\left( \frac{1}{u^{q+1}} \right) \]

since the second integral converges for \( u = \infty \). Therefore

(8) \[ \Phi(-u) = \frac{1}{u} + o\left( \frac{1}{u^q} \right) \quad \text{as } u \to +\infty \]

for every \( q \).
Using this estimate it is easy to derive from (5) a partial fractions expansion for $\Pi(s)\zeta(1-s)$. We write (5) as

$$\Pi(s)\zeta(1-s) = \int_0^1 \{u\mathcal{G}(-u)\} u^{s-1} \, du - \frac{1}{s} + \int_1^\infty \{u\mathcal{G}(-u) - 1\} u^{s-1} \, du$$

and evaluate the first integral by termwise integration of

$$\mathcal{G}(-u)u^{s-1} = \sum_{k=0}^\infty \frac{\zeta(k+2)}{k!} (\frac{-1}{s})^k u^{s-k-1}$$

to obtain

$$\Pi(s)\zeta(1-s) = -\frac{1}{s} + \sum_{k=0}^\infty \frac{\zeta(k+2)}{k!} \frac{(1)}{s^{k+1}} + \int_1^\infty \{u\mathcal{G}(-u) - 1\} u^{s-1} \, du,$$

where by (8) the integral on the right is an entire function of $s$.

The functions $\mathcal{G}(x)$ and $1/(e^x - 1)$, already related by an analogy and by their appearance in (5) and (6), can be brought into even closer connection. I take as point of departure the integral formula

$$\mathcal{G}(x) = \int_{c+i\infty}^{c-i\infty} \frac{e^{xz}}{z^2} \frac{dz}{e^{2\pi iz} - 1} \quad (0 < c < 1)$$

which is easily verified using the calculus of residues. This integral can be reduced to a Fourier integral, as follows. We deform the path of integration into the path $\Gamma$ consisting of the segment $(i\infty, i\delta]$, the semicircle of radius $\delta$ around 0 in the half-plane $\text{Re} z \geq 0$ and the segment $[-i\delta, -i\infty)$, where $0 < \delta < 1$, and add the equation

$$0 = \int_{\Gamma} \frac{e^{x/z}}{z^2} \left\{-\frac{1}{2\pi i z} + \frac{1}{2}\right\} dz$$

to obtain

$$\mathcal{G}(x) = \int_{\Gamma} \frac{e^{x/z}}{z^2} \left\{\frac{1}{e^{2\pi i z} - 1} - \frac{1}{2\pi i z} + \frac{1}{2}\right\} dz$$

in which the function in brackets has a zero of order one at the origin. Letting $z = 1/iy$ in the last integral, one finds

$$\mathcal{G}(x) = -i \int_{\Gamma'} \left\{\frac{1}{e^{2\pi iy} - 1} - \frac{y}{2\pi} + \frac{1}{2}\right\} e^{ixy} \, dy$$

where $\Gamma'$ is the closed path consisting of the segment $[-1/\delta, 1/\delta]$, together with the semicircle $c$ of radius $1/\delta$ around the origin lying in the half-plane $\text{Im} y \leq 0$, this path being traversed in the positive sense. Now when $x$ is a real number
< 0, the integrand is \( O(|y|^{-1} e^{i|y| \text{Im}(y)}) \) on the circle \( c \), from which it follows easily that the integral over \( c \) tends to zero as the radius increases, so that, letting \( \delta \to 0 \),

\[
\mathcal{G}(x) = \int_{-\infty}^{\infty} i \left\{ \frac{1}{e^{2\pi i/y} - 1} - \frac{y}{2\pi i} + \frac{1}{2} \right\} e^{ixy} dy
\]

\[
= \int_{-\infty}^{\infty} i \varphi \left( \frac{2\pi}{y} \right) e^{ixy} dy
\]

where

\[
\varphi(z) = \frac{1}{e^2 - 1} - \frac{1}{z} + \frac{1}{2}.
\]

Since \( \varphi \) is an odd function, (10) is equivalent to

\[
\frac{\text{sgn}(x)}{2} \mathcal{G}(-|x|) = \int_0^\infty \varphi \left( \frac{2\pi}{y} \right) \sin xy dy.
\]

This result can be verified, at least formally, by substituting for \( \varphi(2\pi/y) \) the partial fraction expansion

\[
\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{y}{1+k^2 y^2}
\]

and integrating termwise using the equation

\[
\frac{1}{\pi} \int_0^\infty \frac{y \sin xy}{\beta^2 + y^2} dy = \frac{\text{sgn}(x)}{2} e^{-\beta |x|}.
\]

In order to invert the relation (11), I first render the integral absolutely convergent by an integration by parts:

\[
\int_0^\infty \varphi \left( \frac{2\pi}{y} \right) \sin xy dy = \lim_{n \to \infty} \int_{1/n}^n \varphi \left( \frac{2\pi}{y} \right) d \left( -\frac{1}{x} \cos xy \right)
\]

\[
= \lim_{n \to \infty} \left[ -\varphi \left( \frac{2\pi}{n} \right) \frac{1}{x} \cos nx + \varphi(2\pi n) \frac{1}{x} \cos x + \frac{1}{x} \int_{1/n}^n \frac{d}{dy} \varphi \left( \frac{2\pi}{y} \right) \cos xy dy \right]
\]

\[
= \frac{1}{2x} + \frac{1}{x} \int_0^\infty \left\{ \frac{d}{dy} \varphi \left( \frac{2\pi}{y} \right) \right\} \cos xy dy.
\]

By (11), it follows that

\[
\frac{1}{2} \left( \frac{1}{2}(\mathcal{G}(-|x|) - 1) = \int_0^\infty \left\{ \frac{d}{dy} \varphi \left( \frac{2\pi}{y} \right) \right\} \cos xy dy.
\]
Since
\[ \frac{d}{dy} \varphi \left( \frac{2\pi}{y} \right) = O \left( \frac{1}{y^2} \right) \quad \text{as} \quad y \to \infty , \]
the integral on the right converges absolutely, so that by Fourier's theorem
\[ (12) \quad \frac{d}{dy} \varphi \left( \frac{2\pi}{y} \right) = \frac{1}{\pi} \int_0^\infty \left\{ x \Theta(-x) - 1 \right\} \cos xy \, dx . \]
This integral, which converges rapidly in virtue of the estimate (8), is also the derivative of
\[ \frac{1}{\pi} \int_0^\infty \left\{ x \Theta(-x) - 1 \right\} \frac{1}{x} \sin xy \, dx = \frac{1}{\pi} \int_0^\infty \Theta(-x) \sin xy \, dx - \frac{1}{\pi} \cdot \text{sgn}(y) \]
which has the value \(-\frac{1}{2}\) for \(y = \infty\). Since \(\varphi(2\pi/\infty) = 0\), (12) implies
\[ (13) \quad \frac{1}{\pi} \int_0^\infty \left\{ x \Theta(-x) - 1 \right\} \frac{1}{x} \sin xy \, dx = \varphi \left( \frac{2\pi}{y} \right) - \frac{1}{2} = \frac{1}{e^{2\pi/y} - 1} - \frac{y}{2\pi} . \]
Finally I show how the functional equation for Riemann's zeta function
\[ (14) \quad \zeta(s) = 2^s \pi^{s-1} \Pi(-s) \sin \left( \frac{\pi s}{2} \right) \zeta(1-s) \]
follows from (5), (6), (13) and
\[ (15) \quad \frac{1}{s} \Pi(-s) \sin \left( \frac{\pi s}{2} \right) = \int_0^\infty \frac{\sin y}{y^{s+1}} \, dy \quad (-1 < \sigma < 1) . \]
First, in order to obtain integrals with a common domain of absolute convergence, I use \(1/s = \int_0^1 u^{s-1} \, du \) (\(\sigma > 0\)) to rewrite (9) as
\[ (16) \quad \Pi(s) \zeta(s) = \int_0^\infty \left\{ u \Theta(-u) - 1 \right\} u^s \frac{du}{u} \quad (\sigma > 0) . \]
A similar process, applied to (6) instead of (5), gives
\[ (17) \quad \frac{1}{s} \Pi(s) \zeta(s) = \int_0^\infty \left\{ \frac{1}{e^u - 1} - \frac{1}{u} \right\} u^s \frac{du}{u} \quad (0 < \sigma < 1) , \]
and the substitution \(y = u^{-1}\) takes (15) into
\[ (18) \quad \frac{1}{s} \Pi(-s) \sin \left( \frac{\pi s}{2} \right) = \int_0^\infty \sin \left( \frac{1}{u} \right) u^s \frac{du}{u} \quad (0 < \sigma < 1) . \]
The integrals in (16), (17), (18) all converge absolutely for $0 < \sigma < 1$. Next, let $\mathcal{F}$ denote the family of all integrable functions $f$ defined on $(0, \infty)$ and satisfying
\[\int_0^\infty |f(u)|u^\sigma \frac{du}{u} < \infty, \quad 0 < \sigma < 1,\]
and let $\hat{f}$ and $f \ast g$ be defined for arbitrary $f, g \in \mathcal{F}$ by
\[\hat{f}(s) = \int_0^\infty f(u)u^s \frac{du}{u} \quad (0 < \sigma < 1)\]
and
\[(f \ast g)(u) = \int_0^\infty f(t)g\left(\frac{u}{t}\right)\frac{dt}{t} \quad (0 < u < \infty).\]

Then it is easily seen that: (i) $f \ast g \in \mathcal{F}$ and $(f \ast g) = \hat{f} \cdot \hat{g}$, and (ii) if $h \in \mathcal{F}$ and $k(u) = \alpha h(\beta u)$ for constants $\alpha, \beta, \beta > 0$, then $\hat{k}(s) = \alpha \beta^{-1} \hat{h}(s)$. Taking now
\[f(u) = u \zeta(-u) - 1, \quad g(u) = \sin\left(\frac{1}{u}\right), \quad h(u) = \frac{1}{e^u - 1} - \frac{1}{u},\]
(13), (16), (17) and (18) imply:
\[(f \ast g)(u) = \pi h(2\pi u),\]
\[\hat{f}(s) = \Pi(s)\zeta(1 - s), \quad \hat{g}(s) = \frac{1}{s} \Pi(-s)\sin\left(\frac{\pi s}{2}\right), \quad \hat{h}(s) = \frac{1}{s} \Pi(s)\zeta(s),\]
from which the functional equation follows immediately in view of properties (i) and (ii) above.