TRANSFORMS VANISHING AT INFINITY
IN A CERTAIN DIRECTION
AND SEMI-IDEEMPOTENT MEASURES

LOUIS PIGNO

Let $G$ be a compact abelian group with character group $\Gamma$. Throughout the paper, we shall assume there is a non-trivial group homomorphism

$$\varphi: \Gamma \to R$$

where $R$ is the additive group of reals. If $\varphi$ is an isomorphism, then the semigroup $\mathcal{P}$ is the set $\{\gamma \in \Gamma: \varphi(\gamma) \geq 0\}$. For convenience, assume $\Gamma$ is countable.

Let $M(G)$ be the usual convolution algebra of finite complex-valued Borel measures on $G$. The Fourier–Stieltjes transform of the measure $\mu \in M(G)$ is the function $\hat{\mu}$ defined on $\Gamma$ by

$$\hat{\mu}(\gamma) = \int_G \gamma(-x) \, d\mu(x).$$

We will also use $\hat{\cdot}$ to denote the Gelfand transform. Let $M_0(G)$ be the ideal of measures $\mu \in M(G)$ such that $\hat{\mu} \in C_0(\Gamma)$.

We designate by $M_\varphi(G)$ the set of those $\mu \in M(G)$ such that $\hat{\mu}$ vanishes at infinity in the direction of $\varphi$. By this is meant $\{\gamma_n\}_{n=1}^{\infty} \subset \Gamma$ with $\varphi(\gamma_n) \to \infty \Rightarrow \hat{\mu}(\gamma_n) \to 0$. Here, $\varphi(\gamma_n) \to \infty$ in the usual topology of $R$.

It is easy to check that $M_\varphi(G)$ is a closed ideal of $M(G)$ such that if $\tau \in M_\varphi(G)$ and $\zeta \ll \tau$ then $\zeta \in M_\varphi(G)$. Thus

$$M(G) = M_\varphi(G) \oplus M_\varphi^\perp(G)$$

where

$$M_\varphi^\perp(G) = \{\zeta \in M(G): \zeta \perp \tau \text{ for each } \tau \in M_\varphi(G)\}.$$

Let $\delta_0$ be the identity measure in $M(G)$ and for any set of non-zero integers $\{N_1, \ldots, N_m\}$ put $\delta_i = N_i \delta_0$, $i = 1, 2, \ldots, m$. We state our first result.

**Theorem 1.** Let $\mu \in M(G)$ with $\mu^* \prod_{i=1}^m (\mu - \delta_i) \in M_\varphi(G)$. Then $\mu = \mu_0 + \mu_\perp$ where $\mu_0 \in M_\varphi(G)$, $\mu_\perp \in M_\varphi^\perp(G)$ and $\hat{\mu}_\perp(\Gamma) \subset \mathbb{Z}$.

Received July 27, 1976; in revised form September 19, 1976.
Proof. We adapt the method of [5]: Suppose $\mu \in M(G)$ and
\[ \mu \ast \prod_{i=1}^{m} (\mu - \delta_i) \in M_\phi(G) . \]

Since $M_\phi(G)$ is an ideal (1) gives:
\[ \mu \ast \prod_{i=1}^{m} (\mu - \delta_i) \in M_\phi(G) . \]

Let $S$ be the structure semi-group for $M(G)$ and consider the image of $M(G)$ in $M(S)$ in the usual way; see [8]. For $\zeta \in M(G)$ the image of $\zeta$ is denoted by $(\zeta)_s$. Then (2) becomes:
\[ (\mu)_s \ast \prod_{i=1}^{m} (\mu - \delta_i)_s \in M_\phi(S) \]

where $M_\phi(S)$ is the image of $M_\phi(G)$.

We shall assume
\[ (\mu)_s \neq 0 . \]

Let $\hat{S}$ denote the semi-characters of $S$ and $\hat{\Gamma}$ the closure of $\Gamma$ in $\hat{S}$. Recall that $\hat{S}$ is the maximal ideal space of $M(G)$. Now (4) implies the existence of an infinite set $\{\gamma_n\}^\infty_1 \subset \Gamma$ and an $\epsilon > 0$ such that $|\hat{\mu}_\perp(\gamma_n)| \geq \epsilon$ and $\varphi(\gamma_n) \to \infty$.

Thus $\{\gamma_n\}^\infty_1$ has a cluster point $\beta_0 \in \hat{\Gamma}\setminus\Gamma$. Since conjugation is continuous and multiplication of semi-characters separately continuous we may infer that $|\beta_0|^2 \in \hat{\Gamma}\setminus\Gamma$.

Put $T(\beta_0) = \{ s \in S : \beta_0(s) = 0 \}$. Then define
\[ (\mu)_s = \mu_1 + \mu_2 \]

where
\[ \mu_1 = (\mu)_s|_{T(\beta_0)} \quad \text{and} \quad \mu_2 = (\mu)_s|_{S\setminus T(\beta_0)} . \]

Notice $\mu_2 \neq 0$ by (4).

Let $\tau \in M_\phi(S)$ and consider
\[ |\tau| \hat{\gamma}(\beta_0)^2 = \int_{S\setminus T(\beta_0)} |\beta_0|^2(s) d|\tau|(s) . \]

Now for fixed $k$ we have $\lim_j \varphi(\gamma_j - \gamma_k) = \infty$, so since the Gelfand transform is continuous on $\hat{S}$ and multiplication of semi-characters separately continuous we may conclude that $|\tau| \hat{\gamma}(\beta_0)^2 = 0$. Thus, we gather from (5) that $M_\phi(S)$ is carried by $T(\beta_0)$. Recall for any $\omega_1, \omega_2 \in M(S)$ that
\[ \text{carrier} (\omega_1 \ast \omega_2) \subset \text{(carrier} \omega_1) \text{(carrier} \omega_2) . \]
Inasmuch as $T(\beta_0)$ is an ideal and $S \setminus T(\beta_0)$ is a semigroup we obtain via (6) that for any $\omega \in M(S)$ and $\tau \in M_\varphi(S)$ the condition:

$$
\mu_2 \ast \prod_{i=1}^m (\mu_2 - (\delta_i)_\omega) \perp \tau + \omega \ast \mu_1.
$$

We gather from (3) and (7) that

$$
\mu_2 \ast \prod_{i=1}^m (\mu_2 - (\delta_i)_\omega) = 0.
$$

Pulling back, we have

$$
\mu_\perp = \varrho_1 + \varrho_2, \quad \varrho_1 \perp \varrho_2
$$

where $(\varrho_i)_\omega = \mu_\eta, i = 1, 2$. As a consequence of (8)

$$
\varrho_2 \ast \prod_{i=1}^m (\varrho_2 - \delta_i) = 0, \quad (\varrho_2 \neq 0).
$$

Since $\varrho_i \in M_\varphi^+(G)$ ($i = 1, 2$) we see from (10) that $\|\mu - \varrho_2\| \leq \|\mu\| - 1$. So if $\varrho_1 \neq 0$ we apply this finite descent argument to $\mu - \varrho_2$ and therefore conclude that

$$
\hat{\mu}_{\perp}(\Gamma) \subset \mathbb{Z}.
$$

This completes the proof.

Theorem 1 has an application to semi-idempotent measures which we now give. A subset $E$ of $\Gamma$ is said to be a Sidon set if $f \in L^\infty(G)$ with $\text{supp} \hat{f} \subset E \Rightarrow \sum |\hat{f}(\gamma)| < \infty$. For any subset $A$ of $\Gamma$ put

$$
F(A) = \{\mu \in M(G) : \hat{\mu} \text{ is integer-valued on } A\}
$$

and

$$
I(A) = \{\mu \in M(G) : \hat{\mu} = 0 \text{ or } 1 \text{ on } A\}.
$$

Assume $\varphi$ is a non-trivial isomorphism of $\Gamma$ into $\mathbb{R}$. The following result is an analogue of a result announced by I. Kessler [1]. See also Y. Meyer [3, pp. 206–211].

**Theorem 2.** Let $E$ be a Sidon subset of $\Gamma$. Suppose $\mu \in F(\Gamma \setminus \mathcal{P} \cup E)$. Then there is a $v \in F(\Gamma)$ such that $\hat{\mu} = \hat{v}$ off $-\mathcal{P} \cup E$. In particular, if $\mu \in I(\Gamma \setminus \mathcal{P} \cup E)$ then $v \in I(\Gamma)$.

**Proof.** Suppose $\mu \in F(\Gamma \setminus \mathcal{P} \cup E)$ and let $N_i$ ($i = 1, 2, \ldots, m$) be the distinct non-zero integer-values of $\hat{\mu}$ off $-\mathcal{P} \cup E$. It is apparent that
\[(1) \quad \text{supp} \left\{ \mu \ast \prod_{i=1}^{m} (\mu - \delta_i) \right\} \subset -\mathcal{P} \cup E. \]

where $\delta_i = N_i \delta_0$. By Theorem 2 of [7, p. 368] we see that (1) gives:

\[(2) \quad \left\{ \mu \ast \prod_{i=1}^{m} (\mu - \delta_i) \right\} \in C_0(\mathcal{P}). \]

As a consequence of (2) we gather that

\[(3) \quad \mu \ast \prod_{i=1}^{m} (\mu - \delta_i) \in M_\varphi(G). \]

Now (3) in combination with Theorem 1 permits the conclusion:

\[(4) \quad \mu_\perp \ast \prod_{i=1}^{m} (\mu_\perp - \delta_i) = 0. \]

where $\mu_\perp \in M_\varphi^\perp(G)$. Since $\mu_\perp \in F(\Gamma)$ it is evident that $\mu_0 \in F(\Gamma \setminus -\mathcal{P} \cup E)$. Consider

\[\mathcal{F} = \{ \gamma \notin -\mathcal{P} \cup E : |\hat{\mu}_0(\gamma)| \geq 1 \}. \]

We claim $\mathcal{F}$ is a finite subset of $\mathcal{P}$. To establish our claim, we shall assume $\mathcal{F}$ is infinite and force a contradiction:

Suppose $\mathcal{F}$ is infinite. Clearly $0 \leq \varphi(\mathcal{F}) \leq M$ for some $M \in \mathbb{R}^+$ since $\mu_0 \in M_\varphi(G)$. Let $r_0$ be the largest accumulation point of the set $\varphi(\mathcal{F})$ and let $\gamma_j \in \mathcal{F}$ be a sequence of distinct elements such that $\varphi(\gamma_j) \to r_0$. Then without loss of generality,

\[(5) \quad \tilde{\gamma}_j \mu_0 \to v \quad \text{weak-}^* \]

where $v$ is singular with respect to Haar measure on $G$. As a consequence of $\gamma_j \in \mathcal{F}$, (5) gives:

\[(6) \quad \hat{v}(0) \neq 0. \]

Now by Theorem 1.4 of [2, p. 8]

\[(7) \quad \lim_{j} (E - \gamma_j) \text{ is a finite subset of } \Gamma. \]

Thus, except for a possible finite set of positive $\gamma$'s,

\[(8) \quad \lim_{j} \hat{\mu}(\gamma + \gamma_j) = \hat{v}(\gamma) = 0. \]

because $\gamma + \gamma_j$ eventually does not belong to $\mathcal{F}$. Appeal to Theorem 1 of [7] yields $\hat{v}(0) = 0$ and this contradicts (6).

Thus, $\mathcal{F}$ is a finite set so there is a trigonometric polynomial $p$ on $G$ such that $\hat{\rho} = \hat{\mu}_0$ off $-\mathcal{P} \cup E$ and $\hat{\rho} = 0$ on $-\mathcal{P} \cup E$. Well, for the $v$ of our Theorem, take $v = \mu_\perp + p$. This concludes the proof.
The assumption that $\Gamma$ be countable in our paper is of course inessential. The assumption that $\varphi$ is a non-trivial isomorphism in Theorem 2 is equivalent to $\Gamma$ having an archimedean order.

Let $G$ be a non-discrete LCA group. The method of proof of Theorem 1 yields the following theorem.

**Theorem 3.** If

$$\mu \ast \prod_{i=1}^{m} (\mu - \delta_i) \in M_0(G)$$

then $\mu$ has a decomposition $\mu = \mu_0 + \mu_\perp$ where $\mu_0 \in M_0(G)$, $\mu_\perp \in M_\perp(G)$ and $\hat{\mu}_\perp(\Gamma) \subseteq \mathbb{Z}$.

For discrete $\Gamma$ we call $\mathcal{R} \subseteq \Gamma$ a weak Rajchman set if $\text{supp} \hat{\mu} \subseteq \mathcal{R}$ implies $\mu \in M_0(G)$. For examples of Rajchman sets, the reader is referred to [6]. An easy consequence of Theorem 3 is:

If $\mu \in F(\Gamma \setminus \mathcal{R})$ then there is a $v \in F(\Gamma)$ such that $\hat{\mu} = \hat{v}$ off $\mathcal{R}$. In particular, if $\mu \in I(\Gamma \setminus \mathcal{R})$ then $v \in I(\Gamma)$.

We remark that it is possible to prove a result which encompasses Theorem 1. For $\Gamma$ discrete suppose $\Phi$ is any family of non-trivial homomorphisms from $\Gamma$ into $\mathbb{R}$. We designate by $M_\Phi(G)$ the set of those $\mu \in M(G)$ with the following property: $\{\gamma_n\}_1^\infty \subseteq \Gamma$ with $\varphi(\gamma_n) \to \infty$ for all $\varphi \in \Phi$ implies $\hat{\mu}(\gamma_n) \to 0$. The proof of Theorem 1 can be adapted to obtain our final theorem.

**Theorem 4.** If $\mu \ast \prod_{i=1}^{m} (\mu - \delta_i) \in M_\Phi(G)$ then $\mu$ has a decomposition $\mu = \mu_0 + \mu_\perp$ where $\mu_0 \in M_\Phi(G)$, $\mu_\perp \in M_\perp(G)$ and $\hat{\mu}_\perp(\Gamma) \subseteq \mathbb{Z}$.

The author takes pleasure in thanking Professor Y. Domar of Uppsala University for helpful correspondence. The results of this paper were announced in [4].

**References**


KANSAS STATE UNIVERSITY
MANHATTAN
KANSAS 66506
U.S.A.