A DESCRIPTION OF DISCRETE SERIES USING STEP ALGEBRAS

JOUKO MICKELSSON

1. Introduction.

In this paper we study the discrete series of an arbitrary complex finite dimensional Lie algebra \( \mathfrak{g} \) with respect to a reductive subalgebra \( \mathfrak{f} \) in \( \mathfrak{g} \), \( \text{rank } \mathfrak{f} = \text{rank } \mathfrak{g} \). Because our notion of discrete series differs slightly from the usual one even when \( \mathfrak{g} \) is semi-simple we shall introduce some notation in order to explain this difference.

So let \( \mathfrak{g} \) first be semi-simple. Let \( \mathfrak{h} \subseteq \mathfrak{f} \) be a Cartan subalgebra of \( \mathfrak{g} \), \( \Psi \) the system of roots for \( (\mathfrak{g}, \mathfrak{h}) \) and \( \Delta_k \subseteq \Psi \) a positive system for \( (\mathfrak{f}, \mathfrak{h}) \). Let \( \mathfrak{h}^* \) be the dual of \( \mathfrak{h} \) and let \( (\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C} \) be the dual of the Killing form of \( \mathfrak{f} \) restricted to \( \mathfrak{h} \). The set \( \Lambda \) of integral weights consists of those \( \lambda \in \mathfrak{h}^* \) for which

\[
\langle \lambda, \alpha \rangle = \frac{2\langle \lambda, \alpha \rangle}{(\alpha, \alpha)} \in \mathbb{Z} \quad \forall \alpha \in \Delta_k ,
\]

and the set \( \Lambda^+ \) of dominant integral elements is

\[
\Lambda^+ = \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}_+ \quad \forall \alpha \in \Delta_k \}.
\]

where \( \mathbb{Z}_+ = \{0, 1, 2, \ldots \} \). Next we set

\[
\Lambda^+_{\text{reg}} = \{ \lambda \in \Lambda^+ \mid \langle \lambda, \alpha \rangle \neq 0 \quad \forall \alpha \in \Psi \} .
\]

The elements of \( \Lambda^+_{\text{reg}} \) are called regular weights. If \( \lambda \in \Lambda^+_{\text{reg}} \) then one can define a positive system \( \Delta^\lambda \) for \( (\mathfrak{g}, \mathfrak{h}) \) by

\[
\Delta^\lambda = \{ \alpha \in \Psi \mid \langle \lambda, \alpha \rangle > 0 \} .
\]

Clearly \( \Delta_k \subseteq \Delta^\lambda \). The discrete representations \( D_{d^\lambda, \lambda - \delta_k + \delta_n} \) are parametrized by regular \( \lambda \), where

\[
\delta_k = \frac{1}{2} \sum_{\alpha \in \Delta_k} \alpha , \quad \delta_n = \frac{1}{2} \sum_{\alpha \in \Delta^\lambda \setminus \Delta_k} \alpha .
\]

The discrete representations have the following three properties:

(1) \[
D_{d^\lambda, \lambda - \delta_k + \delta_n} = \sum_{\mu} m_{\lambda}(\mu)X_{\mu}
\]

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where $X_{\mu}$ is the irreducible finite dimensional $\mathfrak{f}$-module with highest weight $\mu$ and the $m_\lambda(\mu)$'s are integers, $0 \leq m_\lambda(\mu) < \infty$.

(2) \[ m_\lambda(\lambda - \delta_k + \delta_n) = 1. \]

(3) If $m_\lambda(\mu) \neq 0$ then $\mu = \lambda - \delta_k + \delta_n + v$, where $v$ is a sum of elements in $\Lambda$.

For more information, see [1], [2] and [9].

In our approach we choose a basis $\{\alpha_1, \ldots, \alpha_l\}$ in $\Delta_k$ and define a lexicographical ordering in $\Lambda$ with respect to this basis (section 2). If $\Omega \subset \Lambda^+$ is any subset then there exists a minimal element in $\Omega$. Let $V$ be a $\mathfrak{f}$-finite $\mathfrak{g}$-module, that is, $V$ is a direct sum of the $\mathfrak{f}$-modules $X_\mu$,

\[ V = \sum_\mu \oplus n(\mu) X_\mu. \]

We set $V_\mu = n(\mu) X_\mu$. We say that $V_\lambda$ is a minimal component of $V$ if $V_\lambda \neq 0$ and $V_\mu = 0$ for $\mu < \lambda$. The ordering "<" is total when $\mathfrak{f}$ is semi-simple, therefore the minimal component is unique for semi-simple $\mathfrak{f}$ (and for any $\mathfrak{g}$). The motivation for our choice of the ordering "<" is the fact that for any $\mathfrak{f}$-finite $\mathfrak{g}$-module $V$ there exists a minimal component $V_\lambda$ and that it is compatible with the standard partial ordering on $\mathfrak{h}^*$ defined by the choice of $\Delta_k$: if $\lambda - \lambda'$ is a sum of elements of $\Delta_k$ then $\lambda > \lambda'$. Set

\[ \Delta = \{ \alpha \in \Psi \mid \alpha > 0 \}. \]

Then $\Delta$ is a positive system for $(\mathfrak{g}, \mathfrak{h})$ (when $\mathfrak{g}$ is semi-simple) and $\Delta_k \subset \Delta$. If $V_\lambda$ is a minimal component for $V$ then $V_{\lambda - \alpha} = 0$ for any $\alpha \in \Delta$. On the other hand, if $\Delta' \subset \Psi$ is an arbitrary positive system for $(\mathfrak{g}, \mathfrak{h})$ such that $\Delta_k \subset \Delta'$ (e.g. $\Delta' = \Delta'$ for a regular $\nu$) then there does not always exist $0 \neq V_\mu \subset V$ such that $V_{\mu - \alpha} = 0$ for all $\alpha \in \Delta'$.

For a certain subset $\Lambda_0^+$ of $\Lambda^+$ we show that for each $\lambda \in \Lambda_0^+$ there exists a unique equivalence class $[V]$ of irreducible $\mathfrak{f}$-finite $\mathfrak{g}$-modules $V$ with minimal component $V_\lambda$. For each $\lambda \in \Lambda_0^+$ there is an irreducible $\mathfrak{g}$-module $V_\lambda$ with the three properties

(1)' \[ V_\lambda = \sum_\mu \oplus n_\lambda(\mu) X_\mu, \quad 0 \leq n_\lambda(\mu) < \infty. \]

(2)' \[ n_\lambda(\lambda) = 1. \]

(3)' If $n_\lambda(\mu) \neq 0$ then $\mu = \lambda + v$, where $v$ is a sum of elements in $\Delta \setminus \Delta_k$.

The set $\Lambda_0^+$ plays more or less the role of $\Lambda_{\text{reg}}^+$ in the earlier approach. We shall call here the set of modules $V_\lambda$ as the discrete series.

The method applied here is the same which was used in [8] for describing the irreducible $\mathfrak{gl}(2, \mathbb{C}) \oplus \mathfrak{gl}(2, \mathbb{C})$-finite $\mathfrak{gl}(4, \mathbb{C})$-modules. The irreducible modules with minimal $\mathfrak{f}$-type $\lambda$ are parametrized by the action of certain
algebra $D_\lambda$, associated to the step algebra $S(g, \mathfrak{f})$, on the minimal component. If $\lambda \in \Lambda_0^+$ then $D_\lambda \cong \mathbb{C}$. In an earlier paper we gave a sufficient condition for $\lambda \in \Lambda^+$ in order that a $g$-module with minimal component $V_\lambda$ belongs to the discrete series ([7, theorem 4.9]). However, the condition in [7] is unnecessarily severe.

In section 3 we first give a general but rather complicated description of the set $\Lambda_0^+$. The structure of $\Lambda_0^+$ is worked out more explicitly for the following three classes of pairs $(g, \mathfrak{f})$:

$$(\text{gl} (p+q, \mathbb{C}), \text{gl} (p, \mathbb{C}) \oplus \text{gl} (q, \mathbb{C})), \quad (C_{p+q}, C_p \oplus C_q)$$

and $$(D_{p+q}, D_p \oplus D_q)$$

where $C_{p}$ and $D_{p}$ are classical simple Lie algebras of rank $l$. In all cases we have studied so far it is found that

$$\Lambda_0^+ = \delta + \Lambda^+,$$

where $\delta = 1/N \sum_{\alpha \in \Delta \setminus \Delta_k} \alpha$ and $N$ is an integer depending on the pair $(g, \mathfrak{f})$.

To get a better idea of the methods used in the present work, the reader is recommended to look at the thesis of van den Hombergh, [4]. There all non-decomposable Harish-Chandra modules for certain real rank one pairs $(g, \mathfrak{f})$ were classified using the step algebra $S(g, \mathfrak{f})$.


Let $\mathfrak{f}$ be a reductive subalgebra in a complex finite dimensional Lie algebra $g$. Thus the adjoint representation $\text{ad} \mathfrak{f}$ of $\mathfrak{f}$ in $g$ is completely reducible and there is an $\text{ad} \mathfrak{f}$-invariant complement $p$ of $\mathfrak{f}$ in $g$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{f}$ and fix a positive system $\Delta_k$ for $(\mathfrak{f}, \mathfrak{h})$. Let $\{\alpha_1, \ldots, \alpha_l\}$ be a basis of $\Delta_k$. We define $\langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle$, $\Lambda$ and $\Lambda^+$ as in introduction. Next we define

$$\mathfrak{f}_s = [\mathfrak{f}, \mathfrak{f}], \quad \mathfrak{h}_s = \mathfrak{h} \cap \mathfrak{f}_s.$$ 

For $\lambda \in \mathfrak{h}^*$ we define $\lambda^s \in \mathfrak{h}_s^*$ as the restriction of $\lambda$ to the subspace $\mathfrak{h}_s \subset \mathfrak{h}$. If $\lambda \in \Lambda$ then

$$(1) \quad \lambda^s = \sum_{i=1}^{l} r_i \alpha_i^s$$

where each $r_i$ is real and rational. If $\lambda \in \Lambda^+$ then $r_i \geq 0, 1 \leq i \leq l$. If $\lambda, \lambda' \in \Lambda$ and $(\lambda - \lambda')^s \neq 0$ then we define $\lambda > \lambda'$ if the first non-zero number in the row $r_1 - r'_1, r_2 - r'_2, \ldots$ is positive. This ordering on $\Lambda$ is total if and only if $\mathfrak{f}$ is semisimple, $\mathfrak{h}_s = \mathfrak{h}$. We define $\lambda \gg \lambda'$ if $\lambda - \lambda'$ is a sum of the simple roots $\alpha_i$. Clearly $\lambda \gg \lambda'$ implies $\lambda > \lambda'$. The set $\{\lambda^s \mid \lambda \in \Lambda^+\}$ can be identified through (1) with a subset of $\mathbb{R}^l$ which is known to be nowhere dense (in the ordinary topology of
R^l) and is bounded below by the vector 0. It follows that any subset Ω⊂Λ^+ has a minimal element and that is unique if ℱ is semi-simple.

Let \{t_1, \ldots, t_n\} be a basis in p consisting of weight vectors,

\[ [h, t_i] = \mu_i(h)t_i, \quad h \in \mathfrak{h}, \quad 1 \leq i \leq n. \]

We can assume that \( \mu_1^l \geq \mu_2^l \geq \ldots \geq \mu_n^l \). The choice of \( \Lambda_k \) defines the splitting \( \mathfrak{f} = \mathfrak{f}_+ \oplus \mathfrak{h} \oplus \mathfrak{f}_- \). We denote by \( U(a) \) the enveloping algebra of a Lie algebra \( a \). We define

\[ S'(g, \mathfrak{f}) = \{ u \in U(g) \mid \mathfrak{f}_+ u \subset U(g)\mathfrak{f}_+ \} \]

and we set \( S(g, \mathfrak{f}) = S'(g, \mathfrak{f})/U(g)\mathfrak{f}_+ \), the step algebra of the pair \( (g, \mathfrak{f}) \). For each sequence \( (i) = (i_1, \ldots, i_n) \in \mathbb{Z}_+^n \) we put \( t(i) = t_{i_1}^l \ldots t_{i_n}^l \in U(g) \). Consider the subspace \( U_1 \subset U(g) \),

\[ U_1 = \sum_{(i)} t(i)U(\mathfrak{h}). \]

We can split

\[ U(g) = U_1 \oplus U(g)\mathfrak{f}_+ \oplus U(\mathfrak{f}_-)\mathfrak{f}_- U_1, \]

\[ U(g) = U_1 \oplus U(g)\mathfrak{f}_+ \oplus U_1 U(\mathfrak{f}_-)\mathfrak{f}_-. \]

Let \( P' \) denote the projection onto the first summand in the first formula, and \( Q' \) the corresponding projection in the second formula. We define projections \( P, Q : U(g)/U(g)\mathfrak{f}_+ \to U_1 \) by

\[ P(u + U(g)\mathfrak{f}_+) = P'(u) \quad \text{and} \quad Q(u + U(g)\mathfrak{f}_+) = Q'(u). \]

For each \( i \in \{1, 2, \ldots, n\} \) there exists \( s_i \in S(g, \mathfrak{f}) \) of the form

\[ s_i = t_i p_i + \sum_{\mu_j \geq \mu_i} u_j t_j p_j, \]

where \( u_j \in U(\mathfrak{f}_-) \), \( p_j \in U(\mathfrak{h}) \) and \( p_i \in U(\mathfrak{h}) \) has the following property:

\[ p_i(\lambda) \neq 0 \quad \text{if} \quad \lambda + \mu_i + \delta_k \in \Lambda^+, \]

where \( \delta_k = \frac{1}{2} \sum \delta_k \); see proposition 1.1.8, page 18 in [4] and [3, proposition 1]. If \( p \in U(\mathfrak{h}) \) we denote by \( p(\lambda) \) the value of a polynomial on \( \mathfrak{h}^* \) obtained via the replacement \( h \mapsto \lambda(h) \).

For each \( \lambda \in \mathfrak{h}^* \) we define the left ideal

\[ I_\lambda = U(g)\{ h - \lambda(h) \cdot 1 \mid h \in \mathfrak{h} \} \]

and for each \( \lambda \in \Lambda^+ \) let \( J_\lambda \) be the left ideal in \( U(\mathfrak{f}) \) which annihilates the vector with highest weight in the finite dimensional \( \mathfrak{f} \)-module \( X_\lambda \). Let \( \pi_\lambda : U(g) \to U(g)/I_\lambda \) be the projection and set \( P_\lambda = \pi_\lambda \circ P, \quad Q_\lambda = \pi_\lambda \circ Q \). Then
if $\lambda + \mu_i + \delta_k \in \Lambda^+$. We say that $s \in S(g, \mathfrak{l})$ has weight $\mu$ if $s = s' + U(g)\mathfrak{l}_+$, where $s' \in S'(g, \mathfrak{l})$ such that $[h, s'] = \mu(h)s'$ for all $h \in \mathfrak{h}$. The step $s_i$ has the weight $\mu_i$ ([4, proposition I.1.8]).

**Lemma 2.1.** Suppose $s \in S(g, \mathfrak{l})$ has weight $\mu$, and $\lambda \in \Lambda^+$. If $\lambda + \mu \notin \Lambda^+$, then $s \in U(g)J_{\lambda}/U(g)\mathfrak{l}_+$.

**Proof.** Consider the $g$-module $V = U(g)/U(g)J_{\lambda}$. It contains a finite dimensional $\mathfrak{l}$-module with highest weight $\lambda$ and highest vector $x = 1 + U(g)J_{\lambda}$. Clearly $V = U(g)x$. From [6, proposition 4.2], it follows that $V$ is $\mathfrak{l}$-finite. The elements of $S(g, \mathfrak{l})$ act in a natural way on $\mathfrak{l}_+$-extreme vectors in $V$. Consider the vector $y = sx$. Then

$$\mathfrak{l}_+y = 0, \quad hy = (\lambda + \mu(h))y \quad \forall h \in \mathfrak{h}.$$ 

Therefore $y = sx = 0$ if $\lambda + \mu \notin \Lambda^+$. But the annihilator of $x$ is $U(g)J_{\lambda}$ and the assertion follows.

We say that the pair $(g, \mathfrak{l})$ is of type (A) if

$$|\langle \mu_i, \alpha_j \rangle| \leq 1 \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq l.$$ 

Here again $\langle \mu, \alpha \rangle = 2(\mu, \alpha)/(\alpha, \alpha)$.

**Lemma 2.2.** Let $(g, \mathfrak{l})$ be of type (A) and let $\lambda \in \Lambda^+$ such that $\lambda + \mu_{i_0} \in \Lambda^+$ for some $1 \leq i_0 \leq n$. Then for $t_j$ such that $\mu_j = \mu_{i_0}$ there exists

$$r_j = \sum_{\mu_i = \mu_{i_0}} a_i s_i \in S(g, \mathfrak{l}) \quad (a_i \in \mathbb{C})$$

such that $Q_\lambda(r_j) = t_j$.

**Proof.** (1) Let $N_\lambda$ be the Verma module for $\mathfrak{l}$ with highest weight $\lambda$. First we show that $N_{\lambda - v + \mu} \subset N_\lambda$ when $\mu = \mu_{i_0}$ and $v = \mu$, $v \in \{\mu_1, \ldots, \mu_n\}$. Namely, if $N_{\lambda - v + \mu} \subset N_\lambda$ then $\lambda - v + \mu + \delta_k = w(\lambda + \delta_k)$ for some $w$ in the Weyl group of $\mathfrak{l}$. Now $\lambda \in \Lambda^+$ so $w(\lambda + \delta_k) \notin \Lambda^+$ for $w \neq 1$ and $\langle w(\lambda + \delta_k), \alpha_i \rangle < 0$ for some $1 \leq i \leq l$. But $\langle \lambda + \mu, \alpha_i \rangle \geq 0$ so

$$0 > \langle w(\lambda + \delta_k), \alpha_i \rangle = \langle \lambda + \mu, \alpha_i \rangle + \langle \delta_k, \alpha_i \rangle - \langle v, \alpha_i \rangle$$

$$\geq 1 - \langle v, \alpha_i \rangle \geq 0 ,$$

a contradiction.

(2) Let $S_\mu \subset S(g, \mathfrak{l})$ be the subspace spanned by the vectors $s_i$ with $\mu_i = \mu_{i_0} = \mu$. 

If \( s \in S_\mu \), we can write

\[
(*) \quad s = \sum_{\mu_i = \mu} a_{i}s_i = \sum_{\mu_i = \mu} a_{i}t_{i}p_{i} + \sum_{\mu_i \gg \mu} v_{i}t_{i}p_{i}
\]

where \( p_i \in U(h) \) such that \( p_i(\lambda) \neq 0 \) for \( \mu_i = \mu \) and \( v_i \in U(f_-)f_- \). Thus

\[
Q_\lambda(S_\mu) \subset p_\mu,
\]

where \( p_\mu \) consists of vectors with weight \( \mu \) in \( p \). Because \( \dim S_\mu = \dim p_\mu \), we only have to show that the mapping \( Q_\lambda : S_\mu \rightarrow p_\mu \) is injective. If the second term in (*) is in \( I_\lambda \) then

\[
Q_\lambda(s) = \sum_{\mu_i = \mu} a_{i}t_{i}p_{i}(\lambda)
\]

and \( Q_\lambda(s) = 0 \) implies that all \( a_i = 0 \), and thus \( s = 0 \). In other cases we write

\[
s = \sum_{\mu_i = \mu} t_{i}p'_{i} + \sum_{\mu_i \gg \mu} t_{i}v'_{i}p'_{i} + t_{j_0}v_{j_0}p_{j_0}
\]

where again \( p'_i \in U(h) \), \( v'_i \in U(f_-)f_- \) and \( \mu_{j_0} \) is a minimal weight such that \( v_{j_0}p_{j_0} \notin I_\lambda \). If now

\[
Q_\lambda(s) = \sum_{\mu_i = \mu} t_{i}p'_i(\lambda) = 0
\]

then \( s \in U(g)I_\lambda \) by [7, lemma 4.4], or [4, proposition II.2.12]. From \( f_+s \subset U(g)f_+ \) it follows that

\[
f_+v_{j_0} \subset U(f)f_+ + I_\lambda.
\]

But \( \text{ad} h(v_{j_0}) = (\mu - \mu_{j_0})(h) \) for \( h \in h \) and this implies \( N_{\lambda - \mu_{j_0} + \mu} \subset N_\lambda \), a contradiction with (1). Thus \( Q_\lambda(s) \neq 0 \) for \( s \neq 0 \), \( s \in S_\mu \).

**Lemma 2.3.** Let \( s \in S(g,f) \) be of weight \( \mu \) and let \( \lambda \in \Lambda \) such that \( \lambda + \mu \in \Lambda^+ \) and \( P_\lambda(s) = 0 \). Then \( s \in I_\lambda/U(g)f_+ \).

**Proof.** First we write

\[
s = \sum_{\mu(i) = \mu} t(i)p(i) + \sum_{\mu(i) \gg \mu} v(i)t(i)p(i)
\]

where \( p(i) \in U(h) \), \( v(i) \in U(f_-)f_- \) and \( \mu(i) = i_1\mu_1 + \ldots + i_n\mu_n \). Then \( p(i)(\lambda) = 0 \) for \( \mu(i) = \mu \). If \( s \notin I_\lambda \) then we can choose a minimal \( \mu(i_0) \gg \mu \) such that \( p(i_0)(\lambda) \neq 0 \). From \( f_+s \subset U(g)f_+ \) it follows that

\[
f_+v(i_0) \subset U(f)f_+ + I_{\lambda + \mu(i_0)}.
\]

Thus \( v(i_0) \) is \( f_+ \)-extreme with weight \( \lambda + \mu \) in the Verma module

\[
N_{\lambda + \mu(i_0)} = U(f)/(U(f)f_+ + I_{\lambda + \mu(i_0)}).
\]
It follows that

\[ N_{\lambda + \mu} \subset N_{\lambda + \mu(i_0)} . \]

There is an element \( w \) in the Weyl group of \( \mathfrak{f} \) such that \( \lambda + \mu + \delta_k = w(\lambda + \mu(i_0) + \delta_k) \) so

\[ \lambda + \mu(i_0) + \delta_k = w^{-1}(\lambda + \mu + \delta_k) \ll \lambda + \mu + \delta_k \]

because of \( \lambda + \mu \in \Lambda^+ \). But this inequality is in contradiction with \( \mu(i_0) \gg \mu \).

Let \( S_0(\mathfrak{g}, \mathfrak{f}) \) be the subalgebra of \( S(\mathfrak{g}, \mathfrak{f}) \) which is generated by the \( s_i \)'s and \( U(\mathfrak{h}) \). We define \( S^k \subset S_0(\mathfrak{g}, \mathfrak{f}) \) as the subspace of elements which are at most of degree \( k \) in the variables \( s_i \). We set

\[ S^k(\mu) = \{ s \in S^k \mid s \text{ is of weight } \mu \} , \]

\[ S_+ (\mu) = S^1(\mu) + \left\{ s \in S^2(\mu) \mid s = \sum_{\mu_i \leq \mu_j} s_i s_j p_{ij}; \ p_{ij} \in U(\mathfrak{h}) \right\} , \]

\[ S_- (\mu) = S^1(\mu) + \left\{ s \in S^2(\mu) \mid s = \sum_{\mu_i \leq \mu_j} s_i s_j p_{ij}; \ p_{ij} \in U(\mathfrak{h}) \right\} , \]

\[ S_\pm (\lambda, \mu) = (S_\pm (\mu) + U(\mathfrak{g})J_\lambda)/U(\mathfrak{g})J_\lambda, \ \lambda \in \Lambda^+ . \]

**Lemma 2.4.** Let \( (\mathfrak{g}, \mathfrak{f}) \) be of type \((A)\), \( \lambda \in \Lambda^+ \), and \( \mu \in \Lambda \) such that \( \lambda + \mu \in \Lambda^+ \). Then \( S^2(\mu) \subset S_+ (\mu) + I_\lambda \).

**Proof.** Because of lemma 2.3 it is sufficient to show that for any \( t_i t_j \) with \( \mu_i + \mu_j = \mu \) there is \( s_{ij} \in S_+ (\mu) \) such that \( P_\lambda(s_{ij}) = t_i t_j \). For any \( t_i \) with \( \mu_i = \mu \) there is \( s_i \in S^1(\mu) \) such that \( P_\lambda(s_i) = t_i p_i(\lambda) \) where \( p_i(\lambda) \neq 0 \). Thus we can forget the first order terms. Now \( t_i t_j = t_j t_i \mod \mathfrak{g} \) so we can assume for example that \( \mu_i \leq \mu_j \).

We prove the existence of \( s_{ij} \) by induction on \( j \). The assertion is true for \( j = 1 \) because \( s_1 = t_1 \) and

\[ P_\lambda(s_i s_1) = P_{\lambda + \mu_1}(s_i) t_1 = t_i t_1 p_i(\lambda + \mu_1) . \]

Now \( \lambda + \mu_1 + \mu_1 + \delta_k = \lambda + \mu + \delta_k \in \Lambda^+ \) so \( p_i(\lambda + \mu_1 + \delta_k) \neq 0 \) and we can set

\[ s_{i1} = (p_i(\lambda + \mu_1))^{-1} s_i s_1 . \]

Suppose that the assertion is true for \( j = k \). But

\[ P_\lambda(s_i s_{k+1}) = t_i t_{k+1} p_i(\lambda + \mu_{k+1}) p_{k+1}(\lambda) + \sum_{\mu_i + \mu_j = \mu \mu_i \gg \mu_{k+1}} t_i t_j a_{rs} \]

where \( a_{rs} \in \mathbb{C} \). By the induction hypothesis, there is \( s \in S_+ (\mu) \) such that \( P_\lambda(s) \) is
equal to the last term in the above formula. If \( a = p_i(\lambda + \mu_{k+1})p_{k+1}(\lambda) \neq 0 \) we can define
\[
s_{ik+1} = a^{-1}(s_is_{k+1} - s) .
\]
The first factor \( p_i(\lambda + \mu_{k+1}) \neq 0 \) for the same reason as before. As for the second,
\[
\langle \lambda + \mu_{k+1} + \delta_k, \alpha_m \rangle = \langle \lambda, \alpha_m \rangle + \langle \mu_{k+1}, \alpha_m \rangle + 1 \geq \langle \lambda, \alpha_m \rangle
\]
for a pair \((g, l)\) of type \((A)\), so \( \lambda + \mu_{k+1} + \delta_k \in \Lambda^+ \) and therefore \( p_{k+1}(\lambda) \neq 0 \).

**Lemma 2.5.** Let \((g, l)\) be of type \((A)\). Let \( \lambda \in \Lambda^+ \), \( \mu \in \Lambda \) such that \( \lambda + \mu \in \Lambda^+ \). Let \( n_+(\lambda, \mu) \) (respectively \( n_-(\lambda, \mu) \)) be the number of pairs \((t_i, t_j)\) such that \( \mu = \mu_i + \mu_j \), \( \mu_i \leq \mu_j \) and \( \lambda + \mu_i \in \Lambda^+ \) (respectively \( \lambda + \mu_i \in \Lambda^+ \)). If \( n_+(\lambda, \mu) \leq n_-(\lambda, \mu) \) then \( S_+(\lambda, \mu) = S_-(\lambda, \mu) \).

**Proof.** From lemma 2.4 it follows that \( S_-(\lambda, \mu) \subset S_+(\lambda, \mu) \). All we need to show is \( \dim S_-(\lambda, \mu) \geq \dim S_+(\lambda, \mu) \).

From lemma 2.1 follows immediately the inequality
\[
\dim S_+(\lambda, \mu) \leq n_+(\lambda, \mu) + \dim \left( (S^1(\mu) + U(\mu)J_\lambda)/U(\mu)J_\lambda \right) .
\]
For each pair \((t_i, t_j)\) such that \( \mu_i + \mu_j = \mu \), \( \mu_i \leq \mu_j \) and \( \lambda + \mu_i \in \Lambda^+ \) we choose \( r_i, r_j \in S^1 \) such that \( Q_\lambda(r_i) = t_i \) and \( Q_{\lambda+\mu}(r_j) = t_j \) (see lemma 2.2). To show that
\[
\dim S_-(\lambda, \mu) \geq n_-(\lambda, \mu) + \dim \left( (S^1(\mu) + U(\mu)J_\lambda)/U(\mu)J_\lambda \right) .
\]
we prove that the elements \( r_j^ir_i \) are linearly independent in \( S_-(\lambda, \mu) \). Suppose that
\[
s = \sum a_{ij}r_j^ir_i \in U(\mu)J_\lambda \quad (a_{ij} \in \mathbb{C}) .
\]
Then \( Q_\lambda(s) = 0 \). Let \( a_{i0} \neq 0 \) but \( a_{ij} = 0 \) when \( i < i_0 \). Then
\[
Q_\lambda(s) = a_{i0}t_{i0}t_i + \sum_{\mu_j \leq \mu_{i0} \atop \mu_i > \mu_{i0}} b_{ij}t_{fi} \neq 0 ,
\]
a contradiction. Thus all \( a_{ij} = 0 \) and the \( r_j^ir_i \)'s are linearly independent in \( S_-(\lambda, \mu) \).

3. **Discrete series.**

We denote
\[
\Lambda^+_0 = \{ \lambda \in \Lambda^+ \mid n_+(\lambda, \mu_i + \mu_j) \leq n_-(\lambda, \mu_i + \mu_j) \quad \forall i, j \text{ such that } \mu_i < 0 < \mu_j \text{ and } \lambda + \mu_i + \mu_j \in \Lambda^+ \}. 
\]
We say that \( A_0^+ \) is stable if
\[
(A_0^+ + \mu_k) \cap A^+ \subset A_0^+ \quad \forall \mu_k > 0.
\]

As we shall see later, in many interesting cases \( A_0^+ \) is in fact stable.

A \( g \)-module \( V \) is said to be \( \mathfrak{l} \)-finite if it is a sum of finite dimensional \( \mathfrak{l} \)-modules. If \( \lambda \in A^+ \) then \( V_\lambda \) denotes the sum of all \( \mathfrak{l} \)-submodules in \( V \) with highest weight \( \lambda \). We set
\[
V_\lambda^+ = \{ x \in V_\lambda \mid \mathfrak{f}_+ x = 0 \}.
\]

Let \( D \) be the centralizer of \( h \) in \( S_0(\mathfrak{g}, \mathfrak{l}) \), i.e. the subalgebra of elements with weight zero. We set
\[
A_{\beta, \alpha} = \{ u \in U(\mathfrak{g}) \mid uV_\alpha^+ \subset V_\beta^+ \text{ for any } \mathfrak{g}-\text{module } V \},
\]
\[
M_\alpha = \sum_{\beta < \alpha} A_{\beta, \alpha},
\]
\[
D_\alpha = D/D \cap U(\mathfrak{g})M_\alpha,
\]
\[
R_+ = \{ \mu_i \mid \mu_i > 0 \}, \quad R_- = \{ \mu_i \mid \mu_i < 0 \}.
\]

If \( V \) is a \( g \)-module such that \( V_\alpha = 0 \) for \( \alpha < \lambda \), then \( V_\lambda^+ \) is in a natural way a \( D_\lambda \)-module. In [8, theorem 1], it was shown that the mapping \( V \mapsto V_\lambda^+ \) determines a bijection from the set of equivalence classes of irreducible \( \mathfrak{l} \)-finite \( g \)-modules for which \( V_\lambda \neq 0 \) and \( V_\alpha = 0 \) if \( \alpha < \lambda \), onto the set of equivalence classes of irreducible \( D_\lambda \)-modules.

**Theorem 3.1.** Let \( (\mathfrak{g}, \mathfrak{l}) \) be a pair of type (A) such that \( \mu_i^\pm \neq 0 \forall i \in \{1, 2, \ldots, n\} \) (if \( \mathfrak{l} \) is semi-simple the last condition is equivalent with \( \text{rank } \mathfrak{l} = \text{rank } \mathfrak{g} \)). In addition, we assume that \( A_0^+ \) is stable. Then for each \( \lambda \in A_0^+ \) there is one and only one equivalence class \( [V] \) of irreducible \( \mathfrak{l} \)-finite \( g \)-modules such that \( V_\lambda \neq 0 \) but \( V_\alpha = 0 \) for \( \alpha < \lambda \). Furthermore, \( \dim V_\lambda^+ = 1 \), and \( \dim V_\alpha^+ \leq \) the number of sequences \( \{\mu_{i_1}, \ldots, \mu_{i_p}\} \) of elements in \( R_+ \) such that \( \mu_{i_1} + \ldots + \mu_{i_p} + \lambda = \alpha \).

**Proof.** We shall show that \( D_\lambda \cong \mathbb{C} \) from which the first assertion follows immediately using the remark above. A general element in \( D \) is a linear combination of vectors of type \( s = s_{i_1} \ldots s_{i_p} u \), when \( u \in U(\mathfrak{h}) \) and
\[
(*) \quad \mu_{i_1} + \ldots + \mu_{i_p} = 0.
\]

From \( \mu_i^\pm \neq 0 \) follows that either \( \mu_{i} > 0 \) or \( \mu_{i} < 0 \). Now \( I_\lambda \subset M^{\lambda, \lambda} \), thus \( u \) is a complex number modulo \( M_\lambda \). We shall show by induction on \( p \) that each \( s_{i_1} \ldots s_{i_p} u \) is a complex number modulo \( M_\lambda \). We saw already that this is the case for \( p = 0 \). Suppose that it is true for \( p = k \) and let us consider the case \( p = k + 1 \). If \( \mu_{i_p} < 0 \) then \( s_{i_p} \in M_\lambda \) and \( s \in D \cap U(\mathfrak{g})M_\lambda \). Suppose then that \( \mu_{i_p} > 0 \). By
(*) there is a last index $i_m$ for which $\mu_{i_m} < 0$. Let $v = \lambda + \mu_{i_p} + \ldots + \mu_{i_m+2}$. We may assume that $v \in \Lambda^+$; otherwise

$$s_{i_{m+2}} \ldots s_{i_p} \in U(g) J_\lambda \subset U(g) M_\lambda$$

and therefore $s \in D \cap U(g) M_\lambda$. Then

$$s_{i_m} s_{i_{m+1}} = \sum_{\mu_j' \geq \mu_j} a_{ij} s_i s_j + r$$

where $a_{ij} \in \mathbb{C}$ and $r \in S^1 + U(g) J_v$ (lemma 2.5) and $a_{ij}$ can be different from zero only when $\mu_i + \mu_j = \mu_{i_m} + \mu_{i_{m+1}}$. If $v \in U(g) J_v$ then $vs_{i_{m+2}} \ldots s_{i_p} \in U(g) J_\lambda$, thus

$$s = \sum_{\mu_{j} \geq \mu_{j}'} a_{ij} s_i \ldots s_{i_{m-1}} s_j s_{i_{m+2}} \ldots s_{i_p} + r'$$

where $r' \in S^{p-1} + U(g) M_\lambda$. Consider a typical term

$$s' = s_{i_1} \ldots s_{i_{m-1}} s_i s_j s_{i_{m+2}} \ldots s_{i_p}.$$

If $\mu_j > 0$ then $\mu_i' \geq \mu_j'$ implies $\mu_i > 0$ and we have reduced the number of factors $s_h$ with negative weight $\mu_i$ by one (compared with $s$). If $\mu_j < 0$ then we can consider $s_j s_{i_{m+2}}$ instead of $s_{i_m} s_{i_{m+1}}$ and continue as above. Noting that a $s_h$ with $\mu_i < 0$ on the right gives zero modulo $M_\lambda$, we can finally write

$$s = q_1 + q_2$$

where $q_2 \in S^{p-1} + M_\lambda$ and $q_1$ is a linear combination of monomials of degree $p$, each of them containing one less factors $s_h$ with negative weight $\mu_i$ than the original monomial $s$. We can make a second induction on the number of negative factors and we arrive at $s = w + q$, where $q \in S^{p-1} + M_\lambda$ and $w$ contains no negative factors. Because $w$ is of weight zero it contains no positive factors either, and therefore $w \in U(l)$, which implies $w \in \mathbb{C} \cdot 1 + I_\lambda$. By the first induction, $q \in \mathbb{C} \cdot 1 + D \cap U(g) M_\lambda$, and thus $s \in \mathbb{C} \cdot 1$ modulo $M_\lambda$. We have now shown that $D_\lambda \cong \mathbb{C}$. From this and the fact that $V_\lambda^+$ is an irreducible $D_\lambda$-module it follows that $\dim V_\lambda^+ = 1$.

Let $0 \neq x \in V_\lambda^+$ and $y \in V_\lambda^+$. Then by [4, corollary II.1.5 p. 29], there exists $s \in S_0(g, l)$ such that $y = sx$. Using the same technique as above we can eliminate all factors $s_i$ with $\mu_i < 0$ from $s$. Thus

$$s \equiv \sum_{p=0}^{k} \sum_{\mu_i + \ldots + \mu_p = \alpha - \lambda} a(i_1, \ldots, i_p) s_{i_1} \ldots s_{i_p} \mod M_\lambda,$$

where $a(i_1, \ldots, i_p) \in \mathbb{C}$. This proves the last assertion.
Next we shall describe explicitly the set $A_0^+$ for three classes of classical reductive Lie algebras with respect to a reductive subalgebra of equal rank. Looking at the root space structure of the Lie algebras $A_t, C_t$ and $D_t$ (see e.g. [5]) it is easily seen that these pairs are of type (A). In each case we shall see that $A_0^+$ is stable so that theorem 3.1 applies.

a) $(g, l) = (gl (p + q, \mathbb{C}), gl (p, \mathbb{C}) \oplus gl (q, \mathbb{C}))$.

The Lie algebra $g = gl (p + q, \mathbb{C})$ consists of complex $(p + q) \times (p + q)$-matrices. We define $e_{ij}$ as the matrix for which

$$(e_{ij})_{kl} = \delta_{ik}\delta_{jl},$$

where $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ii} = 1$. We define $gl (p, \mathbb{C})$ as the subalgebra generated by the elements $e_{ij}, 1 \leq i, j \leq p$. The subalgebra $gl (q, \mathbb{C})$ is spanned by the elements $e_{ij}, p + 1 \leq i, j \leq p + q$. We set $l = gl (p, \mathbb{C}) \oplus gl (q, \mathbb{C})$. A Cartan subalgebra $h \subset l$ of $g$ is spanned by the diagonal matrices $e_{ii}, 1 \leq i \leq p + q$. The semi-simple part $l_s$ consists of trace zero matrices, $l_s = sl (p, \mathbb{C}) \oplus sl (q, \mathbb{C})$ and $h_s = h \cap l_s$. A positive system $\Delta_k$ for $(l, h)$ is defined by setting

$$l_+ = \sum_{1 \leq i < j \leq p} C \cdot e_{ij} + \sum_{p + 1 \leq i < j \leq p + q} C \cdot e_{ij}.$$ 

Then $l_-$ is obtained by transposing the matrices in $l_+$. The simple roots $\alpha_1, \ldots, \alpha_{p+q-2}$ correspond to the vectors $e_{12}, e_{23}, \ldots, e_{p-1,p}, e_{p+1,p+2}, \ldots, e_{p+q-1,p+q}$. If $\lambda \in h^*$, we denote $\lambda_i = \lambda (e_{ii})$. The set of weights $\Lambda$ consists of those $\lambda \in h^*$ for which the numbers $\lambda_i - \lambda_j$ ($1 \leq i, j \leq p$) and $\lambda_k - \lambda_l$ ($p + 1 \leq k, l \leq p + q$) are all real integers. The dominant integral weights are given by

$$A^+ = \{ \lambda \in \Lambda \mid \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p; \lambda_{p+1} \geq \lambda_{p+2} \geq \ldots \geq \lambda_{p+q} \}.$$ 

An $ad l$-invariant complement $g$ of $l$ in $g$ is spanned by the vectors

$$(e_{ij}, e_{ji}; 1 \leq i \leq p, p + 1 \leq j \leq p + q).$$

We define

$$\lambda^j = \frac{1}{j} \sum_{k=1}^{j} \lambda_k - \frac{1}{p-j} \sum_{k=j+1}^{p} \lambda_k \quad \text{when} \ 1 \leq j \leq p - 1;$$

$$\lambda^j = \frac{1}{j-p+1} \sum_{k=p+1}^{j+1} \lambda_k - \frac{1}{q-j+p-1} \sum_{k=j+2}^{p+q} \lambda_k \quad \text{when} \ p \leq j \leq p + q - 2.$$ 

Then

$$\lambda^s = \sum_{j=1}^{p+q-2} \lambda^s_j x_j^s,$$
where $\lambda^s = \lambda \mid_{\mathfrak{h}}$. As before, $\lambda > \lambda'$ if the first non-zero number in the sequence $\lambda^1 - \lambda'^1, \lambda^2 - \lambda'^2, \ldots$ is positive.

We denote the root corresponding to $e_{ij}$ by $\alpha_{ij}$. Then

$$ R_- = \{ \alpha_{j1} \mid p + 1 \leq j \leq p + q \} \cup \{ \alpha_{ij} \mid 2 \leq i \leq p, p + 1 \leq j \leq p + q \}, $$

$$ R_+ = \{ \alpha_{1j} \mid p + 1 \leq j \leq p + q \} \cup \{ \alpha_{ji} \mid 2 \leq i \leq p, p + 1 \leq j \leq p + q \}. $$

**Theorem 3.2.** For the pair $(g, \mathfrak{l})$ defined above, $A_0^+ = \{ \lambda \in \Lambda^+ \mid \lambda_1 - \lambda_2 > 0 \}$. This set is stable.

**Proof.** We have to show that for each $\nu = \alpha_- + \alpha_+$, where $\alpha_- \in R_-$ and $\alpha_+ \in R_+$, $n_-(\nu, \nu) \geq n_+(\nu, \nu)$ for all $\lambda \in \Lambda^+$ such that $\lambda_1 > \lambda_2$. As an example we shall consider the case $\nu = 0$. The other cases are treated in similar manner and are left to the reader. When $\nu = 0$ the number $n_-(\lambda, \nu)$ (respectively $n_+(\lambda, \nu)$) will be equal to the number $n_-(\lambda)$ (respectively $n_+(\lambda)$) of the roots $\alpha_{ij} \in R_-$ (respectively $\alpha_{ij} \in R_+$) such that $\lambda + \alpha_{ij} \in \Lambda^+$. If we denote $\lambda' = \lambda + \alpha_{ij}$, then $\lambda'_k = \lambda_k$ for $k \neq i, j$, $\lambda'_i = \lambda_i + 1$ and $\lambda'_j = \lambda_j - 1$. Thus $\lambda' \in \Lambda^+$ iff $\lambda_{i-1} > \lambda_i$ and $\lambda_j > \lambda_{j+1}$. Let $n_1(\lambda)$ be the number of indices $2 \leq i \leq p - 1$ for which $\lambda_i > \lambda_{i+1}$ and let $n_2(\lambda)$ be the number of indices $p + 1 \leq j \leq p + q - 1$ for which $\lambda_j > \lambda_{j+1}$. It is easily seen that

$$ n_+(\lambda) = (n_1(\lambda) + 2)(n_2(\lambda) + 1) \quad \text{for all } \lambda \in \Lambda^+; $$

$$ n_-(\lambda) = (n_1(\lambda) + 2)(n_2(\lambda) + 1), \quad \lambda \in \Lambda^+, \lambda_1 > \lambda_2; $$

$$ n_-(\lambda) = n_1(\lambda) \cdot (n_2(\lambda) + 1), \quad \lambda \in \Lambda^+, \lambda_1 = \lambda_2. $$

Therefore $n_-(\lambda) \geq n_+(\lambda)$ iff $\lambda_1 > \lambda_2$. The stability of $A^+_0$ follows from the fact that $\lambda'_1 - \lambda'_2 = \lambda_1 - \lambda_2$ or $\lambda'_1 - \lambda'_2 = \lambda_1 - \lambda_2 + 1$ for any $\alpha_{ij} \in R_+$.

b) $(g, \mathfrak{l}) = (C_{p+q}, C_p \oplus C_q)$.

Let $\gamma$ be the $(2p + 2q) \times (2p + 2q)$-matrix defined by

$$ \gamma_{ij} = \begin{cases} 0 & \text{if } i \neq -j \\ 1 & \text{if } i = -j < 0; \quad i, j = \pm 1, \pm 2, \ldots, \pm (p + q) \\ -1 & \text{if } i = -j > 0 \end{cases} $$

Then the classical Lie algebra $g = C_{p+q}$ consists of complex $(2p + 2q) \times (2p + 2q)$-matrices $a$ such that $a^t \gamma + \gamma a = 0$ ($a^t$ is the transpose of $a$). A basis

$$ \{ f_{ij} \mid i, j = \pm 1, \ldots, \pm (p + q); |i| \leq |j| \} $$

for $g$ can be chosen in such a way that

$$ [f_{ij}, f_{kl}] = \gamma_{ik} f_{jl} + \gamma_{il} f_{jk} + \gamma_{jk} f_{il} + \gamma_{jl} f_{ik}. $$
where we have defined the auxiliary vectors \( f_{ij} = f_{ji} \) for \( |i| > |j| \). A subalgebra \( C_p \) is spanned by the vectors \( f_{ij} \) where \( |i|, |j| \leq p \) and there is a subalgebra \( C_q \) spanned by the elements \( f_{k|l|} \) for \( |k|, |l| \geq p + 1 \). We define \( \mathfrak{f} = C_p \oplus C_q \). A Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) in \( \mathfrak{f} \) is spanned by the vectors \( h_i = f_{i, -i}, \quad i = 1, 2, \ldots, p + q \).

We denote the root corresponding to \( f_{ij} \) (\( i \neq -j \)) by \( \alpha_{ij} \). We set

\[
\alpha_1 = \alpha_{1-2}, \quad \alpha_2 = \alpha_{2-3}, \ldots, \quad \alpha_{p-1} = \alpha_{p-1, -p}, \quad \alpha_p = \alpha_{pp},
\]

\[
\alpha_{p+1} = \alpha_{p+1, -(p+2)}, \ldots, \quad \alpha_{p+q-1} = \alpha_{p+q-1, -(p+q)}, \quad \alpha_{p+q} = \alpha_{p+q, p+q}.
\]

Thus \( \{\alpha_1, \ldots, \alpha_{p+q}\} \) is a set of simple roots for \( (\mathfrak{f}, \mathfrak{h}) \) and

\[
\Lambda_k = \{\alpha_{ij} \mid |i| \leq |j|, \quad i > 0; \quad |i|, |j| \leq p \quad \text{or} \quad |i|, |j| \geq p + 1\}.
\]

Now

\[
\Lambda = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \quad \forall i\}.
\]

We define \( \lambda_i = \lambda(h_i) \). Then

\[
\Lambda^+ = \{\lambda \in \Lambda \mid \lambda_1 \geq \ldots \geq \lambda_p \geq 0; \quad \lambda_{p+1} \geq \ldots \geq \lambda_{p+q} \geq 0\}.
\]

Next we set

\[
\lambda_i = \sum_{k=1}^{i} \lambda_k \quad \text{for} \quad 1 \leq i \leq p - 1; \quad \lambda^p = \frac{1}{2}(\lambda_p + \lambda_{p-1});
\]

\[
\lambda_i = \sum_{k=p+1}^{i} \lambda_k \quad \text{for} \quad p + 1 \leq i \leq p + q - 1;
\]

\[
\lambda^{p+q} = \frac{1}{2}(\lambda_{p+q} + \lambda_{p+q-1}).
\]

Then

\[
\lambda = \sum_{i=1}^{p+q} \lambda^i \alpha_i, \quad \lambda \in \Lambda,
\]

and \( \lambda > \lambda' \) if \( \lambda \neq \lambda' \) and the first non-zero number in the sequence \( \lambda_1 - \lambda_1', \lambda_2 - \lambda_2', \ldots \) is positive. Here

\[
R_+ = \{\alpha_{ij} \mid 1 \leq i \leq p, \quad |j| \geq p + 1\},
\]

\[
R_- = \{\alpha_{ij} \mid -p \leq i \leq -1, \quad |j| \geq p + 1\}.
\]

Let \( \varphi: R_+ \to R_- \) be the bijection defined by \( \varphi(\alpha_{1j}) = \alpha_{-p,j}, \varphi(\alpha_{ij}) = \alpha_{-(i-1)}, j \) for \( 2 \leq i \leq p, \quad |j| \geq p + 1 \). If \( \lambda \in \Lambda^+ \), \( \lambda_p > 0 \), then it is easily seen that \( \lambda + \alpha_{ij} \in \Lambda^+ \) iff \( \lambda + \varphi(\alpha_{ij}) \in \Lambda^+ \) for any \( \alpha_{ij} \in R_+ \). Let \( v = \alpha_{ij} + \alpha_{+} \) where \( \alpha_{-} \in R_- \) and \( \alpha_{+} \in R_+ \). If \( v = 0 \), \( \alpha_{ij} \in R_+ \), then \( \alpha_{kl} + \alpha_{ij} = v \) if \( k = -i \) and \( l = -j \). In that case, for any \( (\alpha_{-i}, -p \alpha_{ij}) \) such that \( \lambda + \alpha_{ij} \in \Lambda^+ \), there is \( (-\varphi(\alpha_{ij}), \varphi(\alpha_{ij})) \) with \( \lambda + \varphi(\alpha_{ij}) \in \Lambda^+ \), where \( \lambda \in \Lambda^+ \), \( \lambda_p > 0 \). Thus \( n_-(\lambda, v) = n_+(\lambda, v) \) when \( v = 0 \), \( \lambda_p > 0 \). The case \( v = \alpha_{kl} + \alpha_{ij} \) (\( l \neq -j \)) is treated in the same way. If \( v = \alpha_{kl} + \alpha_{ij} \) where
If \( k 
eq -i \) and \( \beta, \gamma \in \Lambda \) then \( \beta + \alpha_{ij} \in \Lambda^+ \) iff \( \beta + \alpha_{kj} \in \Lambda^+ \). It follows that for each pair \((\alpha_{kl}, \alpha_{ij})\) such that \( \beta = \alpha_{kl} + \alpha_{ij} \) and \( \beta + \alpha_{ij} \in \Lambda^+ \) there corresponds a pair \((\alpha_{kj}, \alpha_{ij})\) such that \( \beta = \alpha_{kj} + \alpha_{ij} \) and \( \beta + \alpha_{kj} \in \Lambda^+ \). We have now shown that \( n_-(\beta, \gamma) = n_+(\beta, \gamma) \) for all \((\beta, \gamma) \in \Lambda^+ \times (R_- + R_+)\) such that \( \beta + \gamma \in \Lambda^+ \), \( \lambda_p > 0 \).

Noting that \( \lambda'_p = \lambda_p \) or \( \lambda'_p = \lambda_p + 1 \) when \( \lambda = \alpha_{ij} + \lambda \) and \( \alpha_{ij} \in R_+ \), we get the following result:

**Theorem 3.3.** For the pair \((g, l) = (C_{p+q}, C_p \oplus C_q)\), \( A_0^+ \) is equal to the set \( \{ \lambda \in A_p^+ \mid \lambda_p > 0 \} \) and it is stable.

c) \((g, l) = (D_{p+q}, D_p \oplus D_q)\).

If we think of \( D_n \) as the Lie algebra of complex antisymmetric \( 2n \times 2n \)-matrices, then we have the following subalgebras in \( g = D_{p+q} \):

\[
D_p = \{ a \in g \mid a_{ij} = 0 \text{ when } i > 2p \text{ or } j > 2p \},
\]

\[
D_q = \{ a \in g \mid a_{ij} = 0 \text{ when } i \leq 2p \text{ or } j \leq 2p \}.
\]

We set \( f = D_p \oplus D_q \). Let \( \mathfrak{h} \subset f \) be a Cartan subalgebra of \( g \) and let

\[
\{ \alpha_{ij} \mid |i| < |j|; \ i, j = \pm 1, \pm 2, \ldots, \pm (p+q) \}
\]

be the set of roots for \((g, \mathfrak{h})\) such that

\[
\{ \alpha_{ij} \mid |i| < |j| \leq p \} \cup \{ \alpha_{ij} \mid |j| > |i| \leq p + 1 \}
\]

is the set of roots for \((f, \mathfrak{h})\). There exists a basis \( \{ h_1, \ldots, h_{p+q} \} \) in \( \mathfrak{h} \) such that

\[
\alpha_{ij}(h_k) = \delta_{ik} + \delta_{jk} - \delta_{-ik} - \delta_{-jk}.
\]

As the set of simple roots for \((f, \mathfrak{h})\) we take \( \{ \alpha_1, \ldots, \alpha_{p+q} \} \) where

\[
\alpha_i = \alpha_{i, -(i+1)} \quad \text{when } 1 \leq i \leq p - 1 \text{ or } p + 1 \leq i \leq p + q - 1,
\]

\[
\alpha_p = \alpha_{p-1, p}, \quad \alpha_{p+q} = \alpha_{p+q-1, p+q}.
\]

We denote \( \lambda_i = \lambda(h_i) \). Then

\[
\Lambda = \{ \lambda \in \mathfrak{h}^* \mid \lambda_i \in \mathbb{Z} \forall 1 \leq i \leq p \text{ or } \lambda_i + \frac{1}{2} \in \mathbb{Z} \forall 1 \leq i \leq p; \lambda_i \in \mathbb{Z} \forall p + 1 \leq i \leq p + q \text{ or } \lambda_i + \frac{1}{2} \in \mathbb{Z} \forall p + 1 \leq i \leq p + q \},
\]

\[
\Lambda^+ = \{ \lambda \in \Lambda \mid \lambda_1 \geq \ldots \geq \lambda_{p-1} \geq \lambda_p \geq -\lambda_{p-1}; \lambda_{p+1} \geq \ldots \geq \lambda_{p+q-1} \geq \lambda_{p+q} \geq -\lambda_{p+q-1} \}.
\]

Any \( \lambda \in \Lambda \) can be written in the form \( \lambda = \sum \lambda^i \alpha_i \), where

\[
\lambda^i = \sum_{j=1}^{i} \lambda_{jp}, \quad 1 \leq i \leq p - 2,
\]
\[ \lambda^{p-1} = \frac{1}{2} \left( \sum_{j=1}^{p-1} \lambda_j - \lambda_p \right), \quad \lambda^p = \frac{1}{2} \sum_{j=1}^{p} \lambda_j, \]

\[ \lambda^i = \sum_{j=p+1}^{i} \lambda_j, \quad p + 1 \leq i \leq p + q - 2, \]

\[ \lambda^{p+q-1} = \frac{1}{2} \left( \sum_{j=p+1}^{p+q-1} \lambda_j - \lambda_{p+q} \right), \quad \lambda^{p+q} = \frac{1}{2} \sum_{j=p+1}^{p+q} \lambda_j. \]

We define again a lexicographical ordering "<" in \( \Lambda \) with respect to the basis \( \{\alpha_1, \ldots, \alpha_{p+q}\} \). Now

\[ R_+ = \{ \alpha_{ij} \mid 1 \leq i \leq p - 1 \text{ or } i = -p; \ |j| \geq p + 1 \}, \]

\[ R_- = \{ \alpha_{ij} \mid 1 - p \leq i \leq -1 \text{ or } i = p; \ |j| \geq p + 1 \}. \]

The proof of the following theorem is a simple counting of different types of pairs \((\alpha_-, \alpha_+) \in R_- \times R_+\).

**Theorem 3.4.** For the pair \((g, \mathfrak{f}) = (D_{p+q}, D_p \oplus D_q)\), \( \Lambda^+_0 \) is equal to the set \( \{ \lambda \in \Lambda^+ \mid \lambda_{p-1} > \lambda_p \} \) and it is stable.

**Remark.** Let \( N = 2q \) for the cases a), b) and let \( N = 4q \) for c). Set

\[ \delta = \frac{1}{N} \sum_{\alpha \in R_+} \alpha. \]

Then \( \Lambda^+_0 = \Lambda^+ + \delta \). This kind of rule seems to be more generally valid; for example, when \( g = G_2 \) (exceptional simple algebra of rank 2) and \( \mathfrak{f} = A_2 \), then it is found that

\[ \Lambda^+_0 = \Lambda^+ + \delta \quad \text{for} \quad \delta = \frac{1}{2} \sum_{\alpha \in R_+} \alpha. \]

**References**

