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ON LINNIK'S CONSTANT

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1. Introduction.

Let $p(q,a)$ denote the least prime in the arithmetic progression $nq+a$. A famous theorem of Linnik [11]-[12] states that for $(a,q)=1$

$$p(q,a) \ll q^L, \tag{1.1}$$

where L is a constant. All known proofs of (1.1) are based on two theorems concerning the distribution of the zeros of Dirichlet's L-functions: Linnik's density theorem and Linnik's theorem on the Siegel zero (the Deuring-Heilbronn phenomenon). For the numerical estimation of Linnik's constant L it is desireable to have good values for certain constants occuring in Linnik's theorems. Turán [20] and Knapowski [10] proved these theorems by Turán's power-sum method, and the present author [7]-[8] carried out the calculations with the result $L \leqq 550$.

Linnik's density theorem is very delicate near $\sigma=1$, where the usual Dirichlet polynomial method gives only trivial results. However, Selberg pointed out in [19] that a refined version of this method can be used in proving density estimates of the Linnik type. The key is the concept of a "pseudocharacter". Generally speaking, a square-free number r and a multiplicative arithmetic function f determine a periodic multiplicative function f_r, defined by

$$f_r(n) = f((r,n)), \tag{1.2}$$

which we call a pseudocharacter. Selberg used pseudocharacters for $f(n) = \mu(n)\varphi(n)$; in our proof of Linnik's second theorem we will choose $f(n) = \mu(n)2^{-\omega(n)}n$, where as usual $\omega(n)$ denotes the number of different prime factors of the number n.

Selberg's idea was to introduce pseudocharacters into the Dirichlet polynomial, detecting the zeros by the means of an identity which will be formulated as lemma 1 below. Averaging over r saves a logarithm in the zero-density estimate, and this is already enough for the proof of Linnik's density

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theorem. Furthermore, this argument gives surprisingly good numerical results. Let

$$R(\alpha, T) = \{\sigma + it \mid \alpha \leqq \sigma \leqq 1, |t| \leqq T\},$$

and let $N(\alpha, T, \chi)$ denote the number of zeros of Dirichlet's L-function $L(s, \chi)$ in the rectangle $R(\alpha, T)$; further let

$$N(\alpha, T, q) = \sum_{\chi \bmod q} N(\alpha, T, \chi),$$

$$N^*(\alpha, T, Q) = \sum_{q \leqq Q} \sum_{\chi \bmod q}{}^* N(\alpha, T, \chi),$$

where the asterisk indicates that the sum is over primitive characters. Selberg's method gave the estimates [15]

$$N(\alpha, T, q) \ll_\varepsilon (qT)^{(3+\varepsilon)(1-\alpha)}, \tag{1.3}$$

$$N^*(\alpha, T, Q) \ll_\varepsilon (Q^5 T^3)^{(1+\varepsilon)(1-\alpha)}, \tag{1.4}$$

where the constants implied by Vinogradov's symbol $\ll$ *depend on* ε. Density theorems of the type (1.3) and (1.4) are originally due to Fogels [3] and Gallagher [4]. Motohashi [17] refined these estimates for $\frac{4}{5} \leqq \alpha \leqq 1$ as follows:

$$N(\alpha, T, q) \ll_\varepsilon (q^2 T^3)^{(1+\varepsilon)(1-\alpha)}, \tag{1.5}$$

$$N^*(\alpha, T, Q) \ll_\varepsilon (Q^4 T^3)^{(1+\varepsilon)(1-\alpha)}. \tag{1.6}$$

We will prove a further refinement, extending the density hypothesis to the interval $[\frac{4}{5}, 1]$.

THEOREM 1. *For* $\frac{4}{5} \leqq \alpha \leqq 1$, $T \geqq 1$, *we have*

$$N(\alpha, T, q) \ll_\varepsilon (qT)^{(2+\varepsilon)(1-\alpha)}, \tag{1.7}$$

$$N^*(\alpha, T, Q) \ll_\varepsilon (Q^2 T)^{(2+\varepsilon)(1-\alpha)}. \tag{1.8}$$

A new feature of the proof is the application of Halaśz's inequality (in the form of lemma 7); in order that this argument be successful we need an additional device of integrating the inequality with respect to certain parameters.

Theorem 1 was proved in the unpublished paper [9]. After having finished it I learned from Dr. M. N. Huxley of a remarkable theorem of S. Graham [5], formulated as lemma 4 below, which is very useful in calculating the constants implied by the symbols $\ll_\varepsilon$. Also I succeeded in finding a new proof for Linnik's second theorem. After that all the necessary facts were available for the calculation of a substantially improved estimate for Linnik's constant, and I decided to write a new paper, pursuing a more complete treatment of the subject.

As regards Linnik's constant, a weak form of the estimate (1.7) is enough. Let τ be any real number, $D=q(|\tau|+1)$, and let $N(\lambda)$ denote the number of L-functions (mod q) having at least one zero in the rectangle $1-\lambda/\log D \leqq \sigma \leqq 1$, $|t-\tau| \leqq 1$.

THEOREM 1′. *For all $\lambda>0$ and D sufficiently large*

$$N(\lambda) \leqq 10e^{11\lambda}. \tag{1.9}$$

Linnik's second theorem gives a quantitative estimate for the effect of the possibly existing Siegel zero upon the other zeros. We will prove it in the following form.

THEOREM 2. *Let χ_1 be a real non-principal character* (mod q), *$\beta_1=1-\delta_1$ a real zero of $L(s,\chi_1)$, χ a character* (mod q), *and $\varrho=\beta+i\tau=1-\delta+i\tau$ a zero of $L(s,\chi)$ with $\delta<\frac{1}{6}$, $\beta\leqq\beta_1$. Suppose that $D=q(|\tau|+1)$ is sufficiently large, that is, $D\geqq D_0(\varepsilon)$. Then*

$$\delta_1 \geqq (1-6\delta)D^{-(2+\varepsilon)\delta/(1-6\delta)}/8\log D\,. \tag{1.10}$$

Now theorems 1′ and 2 imply an estimate for L.

THEOREM 3. *Linnik's constant $L\leqq 80$.*

We do not appeal to Siegel's theorem, so that everything can be made explicit. In fact, it would not be too difficult to calculate a constant L_0 such that

$$p(q,a) \leqq q^{L_0}$$

for all $q\geqq 2$ and $(q,a)=1$.

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2. Lemmas for the proof of theorem 1.

We recall the notation (1.2) for a pseudocharacter. For the sake of brevity we will write $f_rf_{r'}(n)=f_r(n)f_{r'}(n)$, and $\sum'$ will denote a sum over square-free numbers coprime with q.

We will need several times the known estimate

$$L(s,\chi) \ll_\varepsilon E(\chi)/|s-1| + (q(|t|+1))^{\frac{1}{2}(1-\sigma)+\varepsilon}, \qquad 0 \leqq \sigma \leqq 1 .$$

where $E(\chi)=\varphi(q)/q$ if $\chi=\chi_0$, and $E(\chi)=0$ otherwise.

Our lemmas 1, 2, and 3 are partly generalized formulations of lemmas 2, 3 and 4 in Motohashi's paper [17]; for the sake of completeness we briefly sketch the proofs.

LEMMA 1. *Let*

$$(2.1) \qquad M(s,\chi,f_r) = \sum_{d=1}^{\infty} \xi_d \chi(d) f_r(d) d^{-s} \prod_{p \mid \frac{r}{(r,d)}} \{1+(f(p)-1)\chi(p)p^{-s}\},$$

where $\xi_d = O(1)$. *Then for* $\operatorname{Re} s > 1$

$$(2.2) \qquad L(s,\chi)M(s,\chi,f_r) = \sum_{n=1}^{\infty} \left(\sum_{d \mid n} \xi_d\right) \chi(n) f_r(n) n^{-s}.$$

PROOF. The right hand side of (2.2) is

$$(2.3) \qquad \sum_{n=1}^{\infty} \xi_d \chi(d) d^{-s} \sum_{n=1}^{\infty} \chi(n) f_r(nd) n^{-s}.$$

But since r is square-free and f is multiplicative,

$$f_r(nd) = f_r(d) f_t(n) ,$$

where $t=r/(r,d)$. Using this relation and the Euler product for the generating function of the multiplicative function $\chi(n)f_t(n)$, it is easily verified that (2.3) equals the left hand side of (2.2).

LEMMA 2. *Define the numbers* $h(d;r,r')$ *by the equation*

$$\prod_{\substack{p \mid rr' \\ p \nmid (r,r')}} (1+(f(p)-1)p^{-s}) \prod_{p \mid (r,r')} (1+(f^2(p)-1)p^{-s}) = \sum_{d=1}^{\infty} h(d;r,r')d^{-s}.$$

Then

$$f_r f_{r'}(n) = \sum_{d \mid n} h(d;\, r,r') .$$

PROOF. By the multiplicativity of the function $f_r f_{r'}(n)$ it is easily seen that

$$\sum_{n=1}^{\infty} f_r f_{r'}(n) n^{-s} = \zeta(s) \sum_{d=1}^{\infty} h(d;\, r,r') d^{-s} .$$

LEMMA 3. *If, in particular,* $f(n)=\mu(n)\varphi(n)$, *then*

$$\sum_d h(d;r,r')d^{-1} = \delta_{r,r'}\varphi(r)\,,$$

where $\delta_{r,r'}$ *is Kronecker's symbol. Further,*

$$\sum_d |h(d;r,r')| \leqq \prod_{p\mid r}(p+1)\prod_{p\mid r'}(p+1)\,.$$

PROOF. Follows from lemma 2.

The next lemma is related to Selberg's sieve method. Barban and Vehov [1] posed the following problem: to minimize the quadratic form

$$S = \sum_{n\leqq x}\Big(\sum_{d\mid n}\lambda_d\Big)^2 \tag{2.4}$$

under the conditions

$$\lambda_d = \begin{cases}\mu(d) & \text{for } 1\leqq d<z_1,\\ 0 & \text{for } d>z_2,\end{cases} \tag{2.5}$$

where $(1<)\ z_1<z_2$ are given numbers. The result of Barban and Vehov was that if

$$\lambda_d = \mu(d)\log(z_2/d)/\log(z_2/z_1), \qquad z_1\leqq d\leqq z_2\,, \tag{2.6}$$

then

$$S \ll x/\log(z_2/z_1)\,. \tag{2.7}$$

Motohashi [16] worked out the somewhat sketchy proof in more detail. Recently S. Graham [5] has sharpened the estimate (2.7) to an asymptotic formula, even with an estimate for the error. We state his result as

LEMMA 4. *In the notation of the equations* (2.4)–(2.6) *we have*

$$S = x/\log(z_2/z_1)+O(x/\log^2(z_2/z_1)) \quad \textit{for } x\geqq z_2\,,$$

$$S = x\log(x/z_1)/\log^2(z_2/z_1)+O(x/\log^2(z_2/z_1)) \quad \textit{for } z_1<x<z_2\,.$$

LEMMA 5. *For* $\log R\geqq(\log q)^{\frac{1}{4}}$ *and* $R\to\infty$ *we have*

$$\sum_{r\leqq R}{}' r^{-1} = 6\pi^{-2}\prod_{p\mid q}(1+p^{-1})^{-1}\log R(1+o(1))\,.$$

PROOF. Use the generating function

$$\sum_{r=1}^{\infty}{}' r^{-s} = \frac{\zeta(s)}{\zeta(2s)} \prod_{p|q} (1+p^{-s})^{-1}.$$

LEMMA 6. *Let χ be a non-principal character* (mod q), *and let $\varrho=\beta+i\tau$ be a zero of the function $L(s,\chi)$ in the rectangle $R(\alpha,T)$. Choose in lemma* 1

$$f(n) = \psi(n) = \mu(n)\varphi(n), \qquad \xi_d = \lambda_d,$$

where the numbers λ_d are defined in (2.5) *and* (2.6), *and write*

$$a(n) = \sum_{d|n} \lambda_d$$

Suppose that $\frac{1}{2}\leqq\alpha<1$, $T\geqq 1$, $\varepsilon>0$ and that the numbers $R\geqq 1$, $X\geqq 1$ satisfy

$$(\log q)^{\frac{1}{2}} \leqq \log R \ll \log(qT),$$

$$X^{\alpha} \geqq ((qT)^{\frac{1}{2}}Rz_2)^{1+\varepsilon}. \tag{2.8}$$

Let $x=X\log^2(qT)$,

$$g(s,\chi) = \sum_{z_1<n\leqq x} a(n)\chi(n)e^{-n/X}n^{-s} \sum_{r\leqq R}{}' r^{-1}\psi_r(n). \tag{2.9}$$

Then

$$|g(\varrho,\chi)| \geqq (1-\varepsilon)(\varphi(q)/q)(6/\pi^2)\log R, \tag{2.10}$$

provided that qT exceeds a certain bound, depending on ε.

PROOF. First of all note that $a_1=1$ and $a_n=0$ for $2\leqq n\leqq z_1$. Hence, multiplying the equation (2.2) by r^{-1} and summing over r, we obtain by Mellin's transformation (see [18, p. 380, Satz 3.2])

$$\begin{aligned} &e^{-1/X} \sum_{r\leqq R}{}' r^{-1} + \sum_{n>z_1} a(n)\chi(n)e^{-n/X}n^{-\varrho} \sum_{r\leqq R}{}' r^{-1}\psi_r(n) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} L(i(\tau+u),\chi)\Gamma(-\beta+iu)X^{-\beta+iu} \sum_{r\leqq R}{}' r^{-1}M(i(\tau+u),\chi,\psi_r)du\,. \end{aligned} \tag{2.11}$$

Since by (2.1)

$$\sum_{r\leqq R}{}' r^{-1}M(it,\chi,\psi_r) \ll z_2 \sum_{r\leqq R} r\varphi^{-1}(r) \ll Rz_2,$$

the right hand side of (2.11) is $\ll_{\varepsilon} 1$ if X satisfies (2.8). Also it is easily seen that if the series in (2.11) are cut at x, the rest is $\ll_\varepsilon 1$. Hence (2.10) follows by lemma 5.

LEMMA 7. *Let*

$$f(s,\chi) = \sum_{n=1}^{N} a_n \chi(n) n^{-s}.$$

Then

$$\left(\sum_{j=1}^{J} |f(s_j,\chi_j)|\right)^2 \leqq \sum_{n=1}^{N} |a_n|^2 b_n^{-1} \sum_{j,k=1}^{J} \bar{\eta}_j \eta_k B(\bar{s}_j + s_k, \bar{\chi}_j \chi_k),$$

where the η_j are certain complex numbers of absolute value 1, *and*

$$B(s,\chi) = \sum_{n=1}^{\infty} b_n \chi(n) n^{-s},$$

where the b_n are any non-negative numbers such that $b_n > 0$ whenever $a_n \neq 0$ and the series $B(s,\chi)$ is absolutely convergent for all pairs $(s,\chi) = (\bar{s}_j + s_k, \bar{\chi}_j \chi_k)$.

This is a modified form of Halasz's inequality (see [14, Lemma 1.6]); usually the coefficients η_j are eliminated by taking absolute values.

LEMMA 8. *The number of zeros of the function $L(s,\chi)$ in the square*

$$\alpha \leqq \sigma \leqq 1, \quad |t-T| \leqq \tfrac{1}{2}(1-\alpha)$$

is

$$\ll (1-\alpha)\log(q(T+1)) + 1.$$

This is the well-known density lemma of Linnik [18, p. 331].

3. Proof of theorem 1.

For $\frac{4}{5} \leqq \alpha \leqq 1-\varepsilon$ the assertions follow from [6]; hence we may suppose that $\alpha \geqq 1-\varepsilon$. We first consider the estimate (1.7) and after that sketch the modifications required for the proof of (1.8).

In view of known results about the zeros of $\zeta(s)$, we may restrict ourselves to the zeros of $L(s,\chi)$ with $\chi \neq \chi_0$.

Let $D = qT$, $\Delta = 1/\log D$, and split up the rectangle $R(\alpha, T)$ into smaller ones:

(3.1) $$\alpha \leqq \sigma \leqq 1, \quad \max(-T, k\Delta) \leqq t \leqq \min(T, (k+1)\Delta),$$

$k = 0, \pm 1, \ldots$ For each function $L(s,\chi)$, having zeros in the rectangle (3.1), choose arbitrarily one of these zeros. Considering the even and odd numbers k separately, we get two "Δ-well-spaced" systems. Let J denote the cardinality of the system containing at least a half of the selected zeros. In view of lemma 8, it is sufficient to prove an estimate of the type (1.7) for the number J.

Let $g(s,\chi)$ be as in lemma 6, and let

$$R = D^{\varepsilon}, \quad z_1 = D^{\frac{1}{2}+7\varepsilon}, \quad z_2 = D^{\frac{1}{2}+8\varepsilon}, \quad X = D^{1+12\varepsilon}.$$

Then the conditions of lemma 6 are satisfied, and

$$|g(\varrho,\chi)| \gg_{\varepsilon} (\varphi(q)/q)\log D \tag{3.2}$$

for the zeros under consideration.

We apply lemma 7 to the Dirichlet polynomial $f(s,\chi)=g(s+\alpha,\chi)$, and choose

$$b_n = n^{-1}\Bigl(\sum_{r\leqq R}{}' r^{-1}\psi_r(n)\Bigr)^2 (e^{-n/N}-e^{-n/M}). \tag{3.3}$$

Actually we let M and N be variable,

$$M = e^{\xi}, \quad \xi \in [(1-\varepsilon)\log z_1, \log z_1], \tag{3.4}$$

$$N = e^{\eta}, \quad \eta \in [\log x, (1+\varepsilon)\log x], \tag{3.5}$$

because at the end of the proof we have to integrate with respect to ξ and η.

We consider $f(s,\chi)$ at the points $s_j=\varrho_j-\alpha$, where ϱ_j runs over the J selected zeros. Hence $0\leqq\varrho_j\leqq\varepsilon$, $|t_j|\leqq T$. Writing down the inequality of lemma 7, integrating with respect to ξ and η over the intervals given in (3.4)–(3.5), and dividing by $\log^2 D$, we get by (3.2)

$$\begin{aligned} &J^2(\varphi(q)/q)^2\log^2 D \\ &\ll_{\varepsilon} (\log D)^{-2}\sum_{z_1<n\leqq x}|a(n)|^2 n^{1-2\alpha}\sum_{j,k}\bar{\eta}_j\eta_k\iint B(\bar{s}_j+s_k,\bar{\chi}_j\chi_k)d\xi d\eta \\ &\ll_{\varepsilon} (\log D)^{-2}x^{2-2\alpha}\sum_{j,k}\bar{\eta}_j\eta_k\iint B(\bar{s}_j+s_k,\bar{\chi}_j\chi_k)d\xi\eta, \end{aligned} \tag{3.6}$$

the last step by lemma 4.

Now by lemma 2, for $0\leqq\sigma\leqq\frac{1}{2}$,

$$\begin{aligned} B(s,\chi) &= \sum_{n=1}^{\infty}\chi(n)(e^{-n/N}-e^{-n/M})n^{-1-s}\sum_{r,r'}{}'(rr')^{-1}\sum_{d|n}h(d;r,r') \\ &= \sum_{r,r'}{}'(rr')^{-1}\sum_{d}h(d;r,r')\chi(d)d^{-1-s}\sum_{m=1}^{\infty}\chi(m)m^{-1-s}(e^{-md/N}-e^{-md/M}). \end{aligned}$$

The sum over m is equal to

$$\frac{1}{2\pi i}\int_{\operatorname{Re} w=1} L(1+s+w,\chi)\big((N/d)^w-(M/d)^w\big)\Gamma(w)dw$$
$$= E(\chi)\big((N/d)^{-s}-(M/d)^{-s}\big)\Gamma(-s)+I_d(s,\chi)\,,$$

where $I_d(s,\chi)$ is the contour integral over the line $\operatorname{Re} w=-1+\varepsilon$. If $s=0$, then $((N/d)^{-s}-(M/d)^{-s})\Gamma(-s)$ is interpreted as $\log(N/M)$.

The integrals $I_d(s,\chi)$ give to (3.6) a negligible contribution. Indeed,

$$I_d(s,\chi) \ll_\varepsilon D^{\frac{1}{2}}(M/d)^{-1+\varepsilon},$$

whence the contribution is

$$\ll_\varepsilon J^2x^{2-2\alpha}D^{\frac{1}{2}}z_1^{-(1-\varepsilon)2}\sum_{r,r}{}' (rr')^{-1}\sum_d |h(d;r,r')|\ ;$$

this is by lemma 3 and by our choices of z_1, x, and R

$$\ll_\varepsilon J^2x^{2-2\alpha}D^{-5\varepsilon}R^2 \ll J^2\,.$$

Next consider the residues. In this case $\bar\chi_j\chi_k=\chi_0$, and the contribution of the residues to (3.6) is

$$(\log D)^{-2}(\varphi(q)/q)x^{2-2\alpha}\sum_{r,r'}{}' (rr')^{-1}\sum_d h(d;r,r')d^{-1}\sum_{j,k}{}'' \bar\eta_j\eta_k\times$$
$$\times\iint\big(e^{-\xi(\bar s_j+s_k)}-e^{-\eta(\bar s_j+s_k)}\big)\Gamma(-\bar s_j-s_k)d\xi d\eta\ ,$$

where the sum $\sum''$ is restricted by the condition $\bar\chi_j\chi_k=\chi_0$. Carrying out the integrations, then summing over d by using lemma 3 (after that only the terms with $r=r'$ survive), and finally summing over j,k and r, the above is seen to be

$$\ll_\varepsilon J(\varphi(q)/q)^2x^{2-2\alpha}\log^2 D\,.$$

Hence (3.6) takes the form

$$(\varphi(q)/q)^2J^2\log^2 D \ll_\varepsilon (\varphi(q)/q)^2Jx^{2-2\alpha}\log^2 D+J^2,$$

which implies (1.7).

For the proof of the estimate (1.8) we may modify the above argument in the following way, suggested by M. N. Huxley. It is easily seen that Halász's lemma may be stated as the inequality

$$\left|\sum_{j=1}^J \eta_j\sum_{n=1}^N a_nC(n,\chi_j)\chi_j(n)n^{-s_j}\right|^2 \tag{3.7}$$
$$\leqq \sum_{n=1}^N |a_n|^2b_n^{-1}\sum_{j,k=1}^J \bar\eta_j\eta_k\sum_{m=1}^\infty \overline{C(m,\chi_j)}C(m,\chi_k)b_m\bar\chi_j(m)\chi_k(m)m^{-\bar s_j-s_k},$$

where the η_j are arbitrary complex numbers. Choose η_j so that $|\eta_j| = q_j/\varphi(q_j)$ (if χ_j is a character modulo q_j), and let

$$C(m, \chi_j) = \sum_{\substack{r \leq R \\ (r, q_j) = 1}} \mu^2(r)\psi_r(m)r^{-1}.$$

Note that $C(m, \chi)$ depends only on the modulus of χ. Because of primitivity, $\chi_j\chi_k$ is principal only when $\chi_j = \chi_k$. The estimate (1.8) is now easy to prove, starting from (3.7) and arguing as before.

In fact, as was pointed out by Huxley, this method gives for α near 1 the estimate

$$N^*(\alpha, T, Q) \ll_\varepsilon (Q^3T^2)^{(1+\varepsilon)(1-\alpha)}.$$

4. Proof of theorem 1′.

The estimate (1.9) requires a proof only if λ is less than a certain constant since otherwise the assertion follows from (1.7). As before we may restrict ourselves to non-principal characters.

Let χ_j, $j = 1, \ldots, J$ be the non-principal characters $(\bmod\, q)$ for which the corresponding L-function has a zero in the rectangle under consideration, and select for each index a zero ϱ_j. Now we may repeat the steps of the proof of (1.7), with considerable simplifications however. In particular, the integration device is needless, so that we may choose the numbers b_n as in (3.3), with $M = z_1$, $N = x$. The proof can be made explicit by using lemmas 4 and 5. Suppose that we have chosen

$$z_i = D^{a_i},\ i = 1,2;\ R = D^b,\ X = D^c,$$

where a_1, a_2, b and c are certain constants satisfying

$$\tfrac{1}{2} + a_2 + b - c < 0, \quad \tfrac{1}{2} - a_1 + 2b < 0.$$

Then the result is that for D sufficiently large

$$J \leqq (\pi^2/6)\frac{(c-a_1)^2}{b(a_2-a_1)}e^{2c\lambda}.$$

To prove (1.9) choose now $a_1 = 5/2$, $a_2 = 4$, $c = 11/2$, $b = 1 - \varepsilon$.

5. Lemmas for the proof of theorem 2.

Throughout this and the next section, χ_1 will denote a real non-principal character $(\bmod\, q)$. Let

$$a_n = \sum_{d|n} \chi_1(d)\,.$$

Then

$$a_n = \prod_{p^a \| n} (1+\chi_1(p)+\ldots+\chi_1(p^a))\,;$$

hence $a_n \geqq 0$. If n is square-free, then $a_n=0$ if there exists a prime divisor p of n such that $\chi_1(p)=-1$, and otherwise

$$a_n = 2^{\omega(n)}\,.$$

LEMMA 9. *Let χ be a Dirichlet character, f a multiplicative function, r and r' square-free numbers such that $\chi_1(p)=1$ for all prime divisors of rr', and define for* $\operatorname{Re} s>1$

$$G_{r,r'}(s,\chi) = \sum_{n=1}^{\infty} \mu^2(n)a_n\chi(n)f_rf_{r'}(n)n^{-s}\,.$$

Then

$$G_{r,r'}(s,\chi) = L(s,\chi)L(s,\chi\chi_1)P_{r,r'}(s,\chi)Q(s,\chi)\,, \tag{5.1}$$

where

$$P_{r,r'}(s,\chi) = \prod_{\substack{p|rr' \\ p\nmid(r,r')}} (1+2\chi(p)f(p)p^{-s}) \prod_{p|(r,r')} (1+2\chi(p)f^2(p)p^{-s}) \times$$
$$\times \prod_{p|rr'} (1+2\chi(p)p^{-s})^{-1}\,,$$

$$Q(s,\chi) = \prod_{\chi_1(p)=1} (1-\chi(p)p^{-s})^2(1+2\chi(p)p^{-s}) \prod_{\chi_1(p)=-1} (1-\chi^2(p)p^{-2s})\,.$$

PROOF. Let, for a moment, $\prod'$ denote a product over primes satisfying $\chi_1(p)=1$. The generating function of the multiplicative function $\mu^2(n)a_n\chi(n)f_rf_{r'}(n)$ is

$$\prod_p{}' (1+2\chi(p)f_rf_{r'}(p)p^{-s}) =$$

$$\prod_{p\nmid rr'}{}' (1+2\chi(p)p^{-s}) \prod_{\substack{p|rr' \\ p\nmid(r,r')}} (1+2\chi(p)f(p)p^{-s}) \prod_{p|(r,r')} (1+2\chi(p)f^2(p)p^{-s})\,.$$

If this is divided by the Euler product of $L(s,\chi)L(s,\chi\chi_1)$, the quotient is easily seen to be $P_{r,r'}(s,\chi)Q(s,\chi)$.

LEMMA 10. *In the preceding lemma, choose*

$$f(n) = \mu(n)2^{-\omega(n)}n\,,$$

and suppose also that $L(\chi_1, \beta_1)=0$, *where* $\beta_1 = 1-\delta_1$ *is a real number satisfying* $\frac{3}{4}<\beta_1<1$. *Then for the sum*

$$T = \sum_{n=1}^{\infty}{}' a_n e^{-n/Y} n^{-\beta_1}\Big(\sum_{r\leqq R}{}' a_r f_r(n) r^{-1}\Big)^2$$

we have the asymptotic formula

$$T = (\varphi(q)/q)Q(1,\chi_0)L(1,\chi_1)\Gamma(\delta_1)Y^{\delta_1}S + O_\varepsilon(Rq^{1/4}Y^{1/2-\beta_1+\varepsilon}), \tag{5.2}$$

where χ_0 *is the principal character* (mod q), *and*

$$S = \sum_{r\leqq R}{}' a_r r^{-1}.$$

PROOF. By Mellin's transformation and lemma 9

$$2\pi i T = \sum_{r,r'\leqq R}{}' (rr')^{-1} a_r a_{r'} \int_{\operatorname{Re} s=1} G_{r,r'}(s+\beta_1,\chi_0)\Gamma(s)Y^s ds .$$

By (5.1) the functions $G_{r,r'}(s,\chi)$ are meromorphic in the half-plane $\operatorname{Re} s>\frac{1}{2}$. Move the integration to the line $\operatorname{Re}(s+\beta_1)=\frac{1}{2}+\varepsilon$. The zero $s=\beta_1$ of $L(s,\chi_1)$ compensates the pole $s=0$ of $\Gamma(s)$, and the pole $s=1$ of $L(s,\chi_0)$ gives the main term in (5.2); note that

$$P_{r,r'}(1,\chi_0) = 0 \quad \text{for } r\neq r',$$

$$P_{r,r}(1,\chi_0) = ra_r^{-1}.$$

To estimate the contour integral observe that by lemma 9

$$G_{r,r'}(\tfrac{1}{2}+\varepsilon+it,\chi_0) \ll_\varepsilon (q(|t|+1)^2)^{1/4}(rr'(r,r'))^{1/2-\varepsilon/4}$$

and that

$$\sum_{r,r'\leqq R}{}' a_r a_{r'}(rr')^{-1}(rr'(r,r'))^{1/2-\varepsilon/4} \ll_\varepsilon R .$$

LEMMA 11. *We have*

$$S \geqq (\varphi(q)/q)Q(1,\chi_0)L(1,\chi_1)\delta_1^{-1} + O_\varepsilon(R^{-1/2+\varepsilon}q^{1/4+\varepsilon}) .$$

PROOF. The generating function of $\mu^2(n)a_n\chi_0(n)$ is

$$F(s) = L(s,\chi_1)L(s,\chi_0)Q(s,\chi_0) ;$$

hence

$$S \geqq R^{-\delta_1} \sum_{r\leqq R}{}' a_r r^{-\beta_1}$$

$$= (R^{-\delta_1}/2\pi i) \int_{a-iRq}^{a+iRq} F(s+\beta_1)R^s s^{-1} ds + O(R^{-1/2}) ,$$

where $a=\delta_1+1/\log(qR)$. The proof is completed by moving the integration to the line $\mathrm{Re}\,(s+\beta_1)=\frac{1}{2}+\varepsilon$, where the integral is estimated by using Hölder's inequality and mean fourth power estimates.

LEMMA 12. *Let β_1 be as in the preceding lemmas, and suppose also that $L(\varrho,\chi)=0$, where χ is a character* (mod q), *and $\varrho=\beta+i\tau$, $\frac{3}{4}<\beta<\beta_1$. Put $D=q(|\tau|+1)$. Then in the case $\chi\neq\chi_0$, χ_1, we have*

$$T \geqq S^2(1+Y^{(1+\varepsilon)(\beta-\beta_1)})+O_\varepsilon(RD^{1/2}Y^{1/2-\beta_1+\varepsilon}) . \tag{5.3}$$

If $\chi=\chi_0$ or χ_1, then either

$$T \geqq S^2(1+\tfrac{1}{2}Y^{(1+\varepsilon)(\beta-\beta_1)})+O_\varepsilon(RD^{1/2}Y^{1/2-\beta_1+\varepsilon}) , \tag{5.4}$$

or

$$\delta_1 \geqq \tfrac{1}{2}Y^{\beta-1}|\Gamma(1-\varrho)|^{-1}\{1+O_\varepsilon(R^{-1/2+\varepsilon}q^{1/4+\varepsilon})\} . \tag{5.5}$$

PROOF. Assume first that $\chi\neq\chi_0,\chi_1$. Consider the series

$$T_\chi = \sum_{n=1}^{\infty} \mu^2(n)a_n\chi(n)e^{-n/Y}n^{-\varrho}\Big(\sum_{r\leqq R}{}' a_r f_r(n) r^{-1}\Big)^2 ,$$

where $f_r(n)$ is as in lemma 10. As in the proof of lemma 10, T_χ can be expressed by the means of the functions $G_{r,r'}(s,\chi)$, but now the integrand is regular between the lines $\mathrm{Re}\,s=1$ and $\mathrm{Re}\,(s+\beta)=\frac{1}{2}+\varepsilon$. Hence

$$T_\chi \ll_\varepsilon RD^{1/2}Y^{1/2-\beta+\varepsilon} . \tag{5.6}$$

On the other hand, write for a moment

$$T_\chi = \sum_{n=1}^{\infty} \alpha_n e^{-n/Y}n^{-\beta} .$$

Then $\alpha_1=S^2$, so that by (5.6)

$$\Big|\sum_{n=2}^{\infty} \alpha_n e^{-n/Y}n^{-\varrho}\Big| \geqq S^2+O_\varepsilon(RD^{1/2}Y^{1/2-\beta+\varepsilon}) ;$$

hence

$$\begin{aligned} S^2+O_\varepsilon(RD^{1/2}Y^{1/2-\beta+\varepsilon}) &\leqq \sum_{n=2}^{\infty} |\alpha_n|e^{-n/Y}n^{-\beta} \\ &\leqq Y^{(1+\varepsilon)(\beta_1-\beta)} \sum_{n=2}^{\infty} |\alpha_n|e^{-n/Y}n^{-\beta_1}+O(Y^{-1}) , \end{aligned}$$

provided that Y is sufficiently large. The proof is now completed by observing that

$$T = \sum_{n=1}^{\infty} |\alpha_n| e^{-n/Y} n^{-\beta_1}.$$

If $\chi = \chi_0$ or χ_1, the above argument does not work since the function $L(s,\chi)L(s,\chi\chi_1)$ is not regular at $s=1$. Instead of (5.6) we have

$$T_\chi = (\varphi(q)/q)Q(1,\chi_0)L(1,\chi_1)\Gamma(1-\varrho)Y^{1-\varrho}S + O_\varepsilon(RD^{1/2}Y^{1/2-\beta+\varepsilon}).$$

Hence in T_χ

$$\left|\sum_2^\infty\right| \geqq |S^2 - (\varphi(q)/q)Q(1,\chi_0)L(1,\chi_1)|\Gamma(1-\varrho)|Y^{1-\beta}|$$
$$+ O_\varepsilon(RD^{1/2}Y^{1/2-\beta+\varepsilon}) = A + O_\varepsilon(\ldots),$$

say. There are two possibilities:

$$A \geqq \tfrac{1}{2}S^2, \tag{5.7}$$

$$A < \tfrac{1}{2}S^2. \tag{5.8}$$

In the case (5.7) we get (5.4) by the same argument as above. If (5.8) is true, then

$$S = \theta(\varphi(q)/q)Q(1,\chi_0)L(1,\chi_1)|\Gamma(1-\varrho)|Y^{1-\beta}, \tag{5.9}$$

where $2/3 \leqq \theta \leqq 2$. The assertion now follows by comparing (5.9) and lemma 11.

6. Proof of theorem 2.

The idea of the proof is to compare the results of lemmas 10 and 12. We may suppose that $\delta_1 \ll 1/\log D$. The parameters Y and R will be chosen below; for the moment we suppose only that $\log RY \ll \log D$. Let us consider the case $\chi \neq \chi_0, \chi_1$ first.

The estimates (5.2) and 5.3) imply that

$$(\varphi(q)/q)Q(1,\chi_0)L(1,\chi_1)\Gamma(\delta_1)Y^{\delta_1}S \geqq S^2(1+Y^{-(1+\varepsilon)\delta}) + O_\varepsilon(RD^{1/2}Y^{1/2+\varepsilon}).$$

Cancelling this inequality and substituting for S its estimate from lemma 11, we obtain

$$Y^{\delta_1} \geqq (\delta_1\Gamma(\delta_1))^{-1}(1+Y^{-(1+\varepsilon)\delta}) + O_\varepsilon(RD^{1/2}Y^{-1/2+\varepsilon}) + O_\varepsilon(R^{-1/2+\varepsilon}D^{1/4+\varepsilon}).$$

Since $\delta_1\Gamma(\delta_1) = 1 + O(\delta_1)$, it follows that

$$Y^{\delta_1} - 1 \geqq Y^{-(1+\varepsilon)\delta} + O(\delta_1) + O_\varepsilon(RD^{1/2}Y^{1/2+\varepsilon}) + O_\varepsilon(R^{-1/2+\varepsilon}D^{1/4+\varepsilon}). \tag{6.1}$$

Now choose

$$Y = D^{2/(1-6\delta)+\varepsilon_1}, \qquad R = D^{(1+2\delta)/(2-12\delta)+\varepsilon_2},$$

where ε_1 and ε_2 are small numbers depending on ε, in order that the terms $O_\varepsilon(\ldots)$ in (6.1) be of a lower order of magnitude than $Y^{-(1+\varepsilon)\delta}$. Then (6.1) implies that

$$\delta_1 \geqq (1-\varepsilon)Y^{-(1+\varepsilon)\delta}/K(Y,\delta_1)\,, \tag{6.2}$$

where

$$K(Y,\delta_1) = \delta_1^{-1}(Y^{\delta_1}-1) = e^\alpha \log Y, \qquad \alpha \in (0,\delta_1 \log Y)\,. \tag{6.3}$$

We may suppose that $\delta_1 \log Y < \frac{1}{2}$ since otherwise theorem 2 clearly holds. Hence $\alpha < \frac{1}{2}$, so that the inequality (1.10) follows from (6.2) and (6.3).

If $\chi = \chi_0$ or χ_1, we have the two possibilities (5.4) and (5.5) of lemma 12. In the case (5.4) the argument is as above, and the resulting lower bound for δ_1 is half of that in (6.2).

In the case (5.5) we make use of the estimate

$$|1-\varrho| \geqq 0.28/\log q\,. \tag{6.4}$$

This is trivial in the case $\chi = \chi_0$, and otherwise it follows either from lemma 12 of [13] (ϱ real) or from lemma 3b of [7] (ϱ non-real). Now (5.5) and (6.4) together imply a sufficiently good lower bound for δ_1, and the proof of theorem 2 is complete.

7. Linnik's constant.

As in [8], we use a formula of Turán [20]. Let $k \geqq 2$,

$$K(w) = e^{kw}(e^w - e^{-w})/2w, \qquad K_1(w) = K(2w \log q)\,,$$

$$R(n) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, w=2} K_1^2(w) n^{-w}\, dw\,.$$

Then (see [20])

$$R(n) = 0 \quad \text{if } 1 \leqq n \leqq q^{4k-4}, \text{ or } n \geqq q^{4k+4},$$

$$R(n) \ll 1/\log q \quad \text{if } q^{4k-4} < n < q^{4k+4},$$

and for $(a,q)=1$

(7.1)

$$\sum_{\substack{q^{4k-4}<n<q^{4k+4} \\ n \equiv a \pmod q}} \Lambda(n) R(n) n^{-1} = \frac{1}{\varphi(q)}\left\{1 - \sum_\chi \bar{\chi}(a) \sum_{\varrho_\chi} K_1^2(\varrho_\chi - 1)\right\} + O(q^{-2})\,,$$

where ϱ_χ for each character $\chi \pmod q$ runs over the non-trivial zeros of $L(s,\chi)$.

For $w=(-\lambda+i\tau)/\log q$ we have

$$|K_1^2(w)| \leqq e^{-(4k-4)\lambda} \min\{1, (4(\lambda^2+\tau^2))^{-1}\}, \tag{7.2}$$

since for $\operatorname{Re} w \leqq 0$

$$|(e^{2w}-1)/2w| \leqq \min(1, |w|^{-1}).$$

Let us first consider the case when there is no Siegel zero. For a suitably chosen number k the quantity $\{\ldots\}$ in (7.1) will be bigger than a positive constant, and then obviously

$$p(q,a) \leqq q^{4k+4}.$$

For sufficiently large q, the region

$$\sigma \geqq 1-\frac{1}{15\log q}, \quad |t| \leqq q^\varepsilon \tag{7.3}$$

is free of zeros of all L-functions $\pmod q$. This follows from a more general theorem of Miech [13] except that Miech had the constant 20 instead of 15. The widened zero-free region (7.3) is obtained by using in Miech's argument the estimate of Burgess [2] for L-functions.

Subdivide the strip $0 \leqq \sigma \leqq 1$ into rectangles $R_0, R_{\pm 1}, R_{\pm 2}, \ldots$ by the horizontal lines $t = \pm 1/\log q$, $\pm 2/\log q$, $\pm 3/\log q, \ldots$ Consider the zeros in the rectangles

$$1-\lambda/\log q \leqq \sigma \leqq 1, \quad |t| \leqq 1/\log q \tag{7.4}$$

and

$$1-\lambda/\log q \leqq \sigma \leqq 1, \quad \nu/\log q \leqq t \leqq (\nu+1)/\log q, \quad |\nu| \leqq q^\varepsilon \log q. \tag{7.5}$$

By lemma 3b of [7], the number of zeros of any single L-function $\pmod q$ in (7.4) is $\leqq 3$ for $\lambda \leqq 2$, and $\leqq e^\lambda$ for $\lambda \geqq 2$; also, the number of zeros in (7.5) is $\leqq 2$ for $\lambda \leqq 1$ and $\leqq e^\lambda$ for $\lambda \geqq 1$. Hence by (7.2) and theorem 1′ the contribution of the zeros in R_0 to the sum in (7.1) is at most

$$30e^{-(4k-15)/15} + 330\int_{1/15}^{2} e^{-(4k-15)\lambda} d\lambda + 10e^{-(4k-16)2} +$$

$$+120\int_2^\infty e^{-(4k-16)\lambda} d\lambda < \left(30+\frac{330}{4k-15}\right)e^{-(4k-15)/15}$$

$$+\left(10+\frac{120}{4k-16}\right)e^{-2(4k-16)}.$$

Similarly, the zeros in the rectangles R_ν with $1 \leqq |\nu| \leqq q^\varepsilon \log q$ contribute all in all at most

$$(\pi^2/12)\left\{\left(20+\frac{220}{4k-15}\right)e^{-(4k-15)/15}+\left(10+\frac{120}{4k-16}\right)e^{-(4k-16)}\right\}.$$

Hence the total contribution of the zeros with $|\mathrm{Im}\,\varrho| \leqq q^\varepsilon$ is <1 in absolute value if $k=19$. It is easily seen that the zeros with $|\mathrm{Im}\,\varrho| > q^\varepsilon$ can be neglected, so that by our previous remark theorem 3 holds at least in the case when there is no Siegel zero for any L-function (mod q).

If a Siegel zero $\beta_1 (> 1-1/15 \log q)$ does exist, then in any case the region

$$\sigma \geqq 1-0.3/\log q, \quad |t| \leqq q^\varepsilon \tag{7.6}$$

is by theorem 2 free of the other zeros. Since the region (7.6) is considerably wider than (7.3), the estimation of the sum in (7.1) is less delicate than was above. Using theorems 2 and 1′ it can be seen by crude estimations that the term

$$1-|K_1^2(\beta_1-1)|$$

dominates the sum over the non-exceptional zeros. We do not enter into the details, but note only that it is convenient to consider separately the cases $\delta_1 \geqq q^{-\varepsilon}$, $(q^{\frac{1}{2}} \log q)^{-1} \ll \delta_1 < q^{-\varepsilon}$. Of course, the latter possibility could be avoided by appealing to Siegel's theorem, but this is not necessary. So all estimations are in principle effective.

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