ON THE RADIAL BOUNDARY VALUES OF SUBHARMONIC FUNCTIONS

BJÖRN E. J. DAHLBERG

1. Introduction.

The maximum principle says that if \( u \) is subharmonic in the unit disc \( D \) and if

\[
\limsup_{z \to w} u(z) \leq 0 \quad \text{for all } w \in T,
\]

where \( T = \{ z : |z| = 1 \} \), then \( u \leq 0 \). In applications it is sometimes desirable to relax this condition to the weaker assumption

\[
(1.1) \quad \limsup_{r \to 1} \sup_{w \in T} u(rw) \leq 0 \quad \text{for all } w \in T
\]

and still get the conclusion \( u \leq 0 \). As the harmonic function \( v(re^{i\theta}) = \sum_{n=1}^{\infty} nr^n \sin n\theta \) shows, for which \( \lim_{r \to 1} v(rw) = 0 \) for all \( w \in T \), condition (1.1) alone is not sufficient to give that \( u \leq 0 \). The object of this paper is to discuss the kind of restrictions of the growth which together with (1.1) imply that \( u \leq 0 \).

We shall use the following notation. If \( u \) is subharmonic in \( D \) we put

\[
u^*(w) = \limsup_{r \to 1} u(rw), \quad u_*(w) = \liminf_{r \to 1} u(rw)
\]

and

\[
M(r, u) = \max \{ u(rw) : w \in T \}.
\]

We let

\[
P(r, \theta) = (2\pi)^{-1} + \pi^{-1} \sum_{n=1}^{\infty} r^n \cos n\theta
\]

denote the Poisson kernel. If \( f \in L^1(T) \) we put

\[
Pf(re^{i\theta}) = \int_{0}^{2\pi} f(e^{i\varphi}) P(r, \theta - \varphi) \, d\varphi.
\]

We start by discussing the following special case of our main result.

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Theorem 1. Let $u$ be subharmonic in $D$ and suppose that

\begin{equation}
(1.2) \quad u^*(w) < \infty \text{ for all } w \in T,
\end{equation}

\begin{equation}
(1.3) \quad \text{there is a } g \in L^1(T) \text{ such that } u_* \leq g \text{ a.e. on } T,
\end{equation}

\begin{equation}
(1.4) \quad M(r, u) = O[(1 - r)^{-2}] \text{ as } r \to 1.
\end{equation}

Then $u^* \in L^1(T)$ and $u_* = u^*$ a.e. on $T$. In addition $u \leq P(u^*)$.

Considering the function $u(re^{i\theta}) = \sum_{n=1}^{\infty} nr^n \sin n\theta$ again, for which $M(r, |n|) = O[(1 - r)^{-2}]$ we see that condition (1.4) can not be weakened to $O[(1 - r)^{-2}]$.

Professor H. S. Shapiro has asked the following question: Let $f$ be analytic in $D$ and assume $\lim_{r \to 1} f(rw) = g(w)$ exists for all $w \in T$ and $g \in L^1(T)$. What growth conditions should one impose on $f$ in order to deduce that $f \in H^1$? In this direction we have the following consequence of Theorem 1.

**Corollary 1.** Let $f$ be analytic in $D$. Suppose that $\lim \sup_{r \to 1} |f(rw)| < \infty$ for all $w \in T$ and there exists a $g \in L^1(T)$ such that $\lim_{r \to 1} f(rw) = g(w)$ a.e. If $\log M(r, |f|) = O[(1 - r)^{-2}]$ as $r \to 1$ then $f \in H^1$.

**Proof.** Putting $u = \log |f|$ we have that $u^*(w) < \infty$ for all $w \in T$ and $u_* (w) \leq \log^+ |g| \in L^1(T)$. Since $u$ is subharmonic in $D$ and satisfies (1.4) it follows from Theorem 1 that $\log |g| \in L^1(T)$ and $u \leq P(\log |g|)$. It follows from Jensen’s inequality that $|f| = \exp u \leq P|g|$. Hence $f \in H^1$ which proves the Corollary.

We remark that we cannot weaken the growth condition of the Corollary.

For the function

\begin{equation}
(1.5) \quad f(z) = \exp(iz(1-z)^{-2})
\end{equation}

satisfies $\lim \sup_{r \to 1} |f(rw)| = 1$ for all $w \in T$, $\lim_{r \to 1} f(rw) = g(w)$ exist a.e. on $T$ and $g \in L^\infty(T)$. In addition $\log M(r, |f|) = O[(1 - r)^{-2}]$ as $r \to 1$ but $f \notin H^1$.

Diederich [3] has considered a variant of Theorem 1 where radial boundary values are replaced by certain averages.

We shall use a growth condition, which is more general than (1.4). Let $\sigma > 0$, $\omega \in T$ and put $\sigma(z, w) = (1 - |z|)|z - w|^{-1}$. Suppose $u$ is subharmonic in $D$. We say that $u$ is of type $G(w, \sigma)$ if there are functions $a$ and $b$, both non-increasing and non-negative such that $a(t) = o(1)$ as $t \to 0$,

\begin{equation}
\int_0^1 \log (b(t) + 1) \, dt < \infty
\end{equation}

and
\begin{equation}
(1.5) \quad u(z) \leq a(|z-w|) b(\sigma(z,w))|z-w|^{-\epsilon}.
\end{equation}

We have the following relation between conditions of the form (1.4) and (1.5).

**Proposition 1.** Let $u$ be subharmonic in $D$ and suppose $M(r, u) = o[(1-r)^{-\epsilon}]$ for some $q > 0$. Then $u$ is of type $G(w, q)$ for all $w \in T$.

**Proof.** There is no loss in generality by assuming $M(r, u) \geq 0$ for $0 \leq r < 1$. Put
\[ a(t) = \sup \{ s^q M(1-s, u) : 0 < s \leq t \}. \]
Then $a(t) = o(1)$ as $t \to 0$. If $w \in T$ and $z \in D$ then
\[
u(z) \leq M(|z|, u) = M(1-|z-w|\sigma(z,w)) \leq (\sigma(z,w))^{-q}|z-w|^{-q}a(|z-w|).\]
Choosing $b(t) = t^{-\epsilon}$ we see that condition (1.5) is satisfied. Since $w$ was arbitrary the proposition follows.

We can now formulate our "radial" maximum principle.

**Theorem 2.** Let $u$ be subharmonic in $D$ and let $E \subset T$ be countable. Suppose
\begin{align}
(1.6) \quad & \text{for all } w \in T \text{ } u \text{ is of type } G(w, 2), \\
(1.7) \quad & \text{there is a } g \in L^1(T) \text{ such that } u_* \leq g \text{ a.e.} \\
(1.8) \quad & u_*(w) < \infty \text{ for } w \in T - E, \\
(1.9) \quad & u_+(rw) = o[(1-r)^{-1}] \text{ for } w \in E.
\end{align}
Then $u_* \in L^1(T)$ and $u_* = u_*$ a.e. In addition we have $u \leq Pu_*$. 

It follows from the example given in [13, p. 640] that $E$ can not be taken to be an uncountable Borelset if the other assumptions are unchanged. But if we make further restrictions of the growth we can allow larger exceptional sets $E$, see Theorem 4.

The question when a harmonic function in $D$ is determined by its radial limits has been extensively studied. For the most general results we refer to Wolf [16]. However, using Theorem 2 we get the following new result on this question.

**Theorem 3.** Let $u$ be a realvalued harmonic function in $D$ and let $E \subset T$ be countable. Suppose $u$ satisfies (1.6) and (1.7). If in addition
\begin{align}
\tag{1.10}
-\infty < u^*(w) < \infty \quad & \text{for } w \in E \\
\tag{1.11}
|u(rw)| = o[(1-r)^{-1}] \quad & \text{for } w \in T
\end{align}

then $u^* \in L^1(T)$, $u_* = u^*$ a.e. and $u = Pu_*$. 

We would like to point out that Theorem 3 neither implies nor is implied by the results of [16]. However, if we assume that $M(r, u) = O[(1-r)^{-2}]$ we get overlap with the results in [16]. The special cases when $M(r, u) = o[(1-r)^{-2}]$ and $M(r, u) = O[(1-r)^{2-\epsilon}]$ have been treated by Shapiro [13, 14] with other methods.

For the case when the exceptional set is no longer assumed to be countable we have the following result.

**Theorem 4.** Let $u$ be subharmonic in $D$ and let $0 < \alpha < 1$. Let $E \subset T$ be the countable union of closed sets of finite $\alpha$-dimensional Hausdorff measure. Suppose

\begin{align}
\tag{1.12}
M(r, u) = o[(1-r)^{-1+\epsilon}] \quad & \text{as } r \to 1 , \\
\tag{1.13}
u^*(w) < \infty \quad & \text{for } w \in T-E , \\
\tag{1.14}
\text{there is a } g \in L^1(T) \text{ such that } u_* \leq g \text{ a.e. .}
\end{align}

Then $u^* \in L^1$ and $u_* = u^*$ a.e. In addition $u \leq Pu_*$. 

Theorem 3 is sharp for if $E \subset T$ is a closed set of positive $\alpha$-dimensional Hausdorff measure then there exists by [2, p. 7] a probability measure $\mu$ concentrated on $E$ such that $\mu(\zeta : |\zeta - z| < r) \leq Cr^\alpha$ for all $z$. If $v = Pu$, then an integration by parts shows that $M(r, v) = O[(1-r)^{-1+\epsilon}]$ as $r \to 1$ and hence condition (1.10) can not be relaxed to $O[(1-r)^{-1+\epsilon}]$ as $r \to 1$.

As an application of Theorem 2 we have the following result on the “pointwise” normal derivatives.

**Theorem 5.** Let $u$ be harmonic in $D$ and let $E \subset T$ be countable. Suppose

\begin{align}
\tag{1.15}
u \text{ is of type } G(w, l) \text{ for all } w \in T , \\
\tag{1.16}
\text{there is a realvalued function } f \text{ on } T \text{ such that} \\
\limsup_{r \to 1} |f(w) - u(rw)|(1-r)^{-1} < \infty \quad & \text{for } w \in T-E , \\
\tag{1.17}
\text{there is a } g \in L^1(T) \text{ such that} \\
\lim_{r \to 1} (f(w) - u(rw))(1-r)^{-1} = g(w) \text{ a.e. ,}
\end{align}

\begin{align}
\tag{1.18}
u(rw) = o[\log(1-r)] \quad & \text{as } r \to 1 \quad \text{for } w \in E .
\end{align}

Then $f \in L^1(T)$, $u = Pf$ and $r \partial u / \partial r = Pg$. 


This theorem generalizes the pointwise saturation theorem of Berens [1] and Hedberg [7]. Their result is about the case when \( f \) is assumed to be in \( L^1(\mathbb{T}) \) and \( u = Pf \). We observe that in this case \( M(r,|u|) = o([1 - r]^{-1}) \) and hence we have from proposition 1 that \( u \) satisfies condition (1.15). Notice that we don’t assume \( f \) to be integrable.

2. The refined maximum principle.

We start with the following estimate for the growth of subharmonic functions.

**Lemma 1.** Let \( B(\varepsilon) = \{ z : |z - 1| < \varepsilon, \ \text{Im} z > 0 \} \) and let \( u \) be subharmonic in \( \Omega = D \cup B(\varepsilon) \). Suppose \( u \) is of type \( G(1, q) \) for some \( q > 0 \) and \( u^+(z) \leq u^+ (z^*) \) for \( z \in \Omega - D \), where \( z^* = z|z|^{-2} \). Then

\[
(2.1) \quad \sup \{ u^+ (z) : z \in \Omega, \ \text{Im} z > 0, \ |z - 1| = r \} = o(r^{-q}) \quad \text{as } r \to 0 .
\]

**Proof.** Let \( \sigma(z) = (1 - |z|)|z - 1|^{-1} \) and put

\[
D_r = \{ z : \ \text{Im} z > 0, \ 2^{-1} r < |z - 1| < 2r, \ |\sigma(z)| < \frac{1}{2} \} .
\]

It is easily seen there is a constant \( C > 1 \) such that if \( 0 < |z - 1| < \frac{1}{2} \) then

\[
(2.2) \quad |z - 1| \leq C|z^* - 1| \quad \text{and} \quad |\sigma(z)| \leq C|\sigma(z^*)| .
\]

From (1.5) and (2.2) we now have if \( z \in \Omega - D \) and \( |z - 1| < \frac{1}{2} \)

\[
u^+(z) \leq u^+ (z^*) \leq a(|z^* - 1|) b(\sigma(z^*))|z^* - 1|^{-q} \leq C^0 a(C^{-1}|z - 1| b(C^{-1}|\sigma(z)|)|z - 1|^{-q} .
\]

Putting \( a_1(t) = C^0 a(C^{-1}t) \) and \( b_1(t) = b(C^{-1}t) \) we therefore have

\[
(2.3) \quad u^+(z) \leq a_1(|z - 1|) b_1(|\sigma(z)|) \quad \text{for } z \in \Omega \cap B(\frac{1}{2}) .
\]

Let \( d(r) = \sup \{ a_1(z) : \frac{1}{2} r \leq t \leq 2r \} \) and put \( v_r(z) = (d(r))^{-1} u^+(z) \) for \( z \in D_r \). If

\[
F_r(z) = \sigma(z) + i2r^{-1}|z - 1|
\]

then \( F_r \) is a diffeomorphism of \( D_r \) onto

\[
R = \{ x + iy : |x| < \frac{1}{2} \quad \text{and} \quad 1 < y < 4 \} .
\]

Letting \( u_r = v_r \circ F_r^{-1} \) we claim there is a number \( C \) independent of \( r \) such that

\[
(2.4) \quad u_r(z) \leq C s^{-2} \int_{B(z,s)} u_r(\xi, \eta) \, d\xi \, d\eta , \quad z \in R ,
\]

whenever \( B(z, s) = \{ \xi : |\xi - z| < s \} \) is contained in \( R \). To prove (2.4) we first observe
(2.5) \[ |\text{grad} F_r(z)| \leq Cr^{-1} \quad \text{for} \quad z \in D_r, \]

and if \( J_r(z) \) denotes the Jacobian of the mapping \( F_r \), then

\[ |J_r(z)| = r^{-1}|z-1|^{-2}|z|^{-1}y, \quad z \in D_r. \]

If \( z = x + iy \in D_r \) then \( |\sigma(z)| < \frac{1}{2} \) and multiplying both sides of this inequality with \( |z-1| \) we have

\[ |2(1-x)|z-1|^{-1} - |z-1|| < \frac{1}{2}(|z|+1). \]

Hence if \( r \) is small enough \( |x-1| \leq c|z-1| \) where \( c < 1 \). Consequently \( |J_r(z)| \geq cr^{-2} \) for \( z \in D_r \) and some number \( c \) independent of \( r \). To prove (2.4) pick \( z' \in D_r \) such that \( F_r(z') = z \), and if \( c \) is small enough we have from (2.5) that \( F_r(B(z',crs)) \subset B(z,s) \). This gives

\[ \int_{B(z,s)} u_r(\xi,\eta) \, d\xi \, d\eta \geq \int_{B(z',crs)} v_r(\xi,\eta)|J_r(\xi + i\eta)| \, d\xi \, d\eta \]
\[ \geq cr^{-2} \int_{B(z',crs)} v_r(\xi,\eta) \, d\xi \, d\eta. \]

Since \( v_r \) is subharmonic we therefore have \( u_r(z) \geq cs^2v_r(z') = cs^2u_r(z) \). It follows from (2.3) that \( u_r(\xi,\eta) \leq b_1(\xi) \) and from [4, Theorem 3] we have \( u_r(\xi,2) \leq C \) for \( |\xi| \leq \frac{1}{3} \) with \( C \) independent of \( r \). This means

\[ \sup \{ u^+(z) : |z-1| = r, \text{Im} \ z > 0 \} \]
\[ \leq Cd(r) + \sup \{ u^+(z) : |z-1| = r, \text{Im} \ z > 0, 1 > |\sigma(z)| \geq \frac{1}{3} \}. \]

Since the both last terms are \( o(r^{-4}) \) as \( r \to 0 \) the conclusion of the lemma follows.

**Proof of Theorem 2.** Let \( u \) fulfil the assumptions of Theorem 2. If we put \( f = (u^+_*)_t \) then from (1.7) \( f \in L^1(\T) \). Let \( v = u^+-Pf \) and define

\[ \Omega = \left\{ w \in \T : \limsup_{z \to w} v(z) \leq 0 \right\}. \]

We first note that \( \Omega \) is open in \( \T \). To this end we use the following fact: If a function \( u \) is subharmonic in \( D \) and bounded from above in

\[ S(I) = \{ rw : 0 < r < 1, w \in I \}, \]

where \( I \subset \T \) is an open arc, then the condition \( u_*(w) \leq 0 \) a.e. in \( I \) implies

(2.6) \[ \limsup_{z \to w} u(z) \leq 0 \quad \text{for all} \quad w \in \T \]
Let \( h \) be the harmonic function in \( S(I) \) with boundary values zero on \( \overline{I} \) and boundary values \( u^+(z) \) for \( z \in \partial S(I) - \overline{I} \). Define \( v(z) = (u(z) - h(z)) \) for \( z \in S(I) \) and zero otherwise. Then \( v \) is bounded and subharmonic in \( D \). For some \( F \in L^\infty(T) \) PF is the least harmonic majorant of \( v \) in \( D \). Littlewoods theorem [15, p. 172] gives \( F \leq 0 \) a.e. in \( I \). Therefore \( \lim_{z \to w} v(z) = 0 \) for all \( w \in I \), which gives (2.6).

From (2.6) follows now that \( \Omega \) is open since \( v_* \leq 0 \) a.e. in \( T \). Define \( R = T - \Omega \). We want to show \( R = \emptyset \). We assume now \( R \neq \emptyset \). Let \( E = \{e_j\} \) and let

\[
F_j = \{w \in T : v(rw) \leq j \text{ for } 0 < r < 1\}.
\]

We claim \( F_j \) is closed for all \( j \). Let \( w_0 \) be a limit point of \( F_j \) and let

\[
S = \{rw_0 : 0 < r < 1\}.
\]

We notice that it follows from the Wiener criterion [8, p. 220] that \( F_j \) is not thin [8, p. 209] at any point of \( S \). Hence \( v(z) \leq j \) for \( z \in S \) and therefore \( F_j \) is closed. It follows from (1.8) and (1.9) that \( T = \bigcup_{j=1}^{\infty} F_j \cup E \). From the Baire category theorem follows the existence of an open arc \( I \) and an integer \( j \) such that \( I \cap R \neq \emptyset \) and \( I \cap R \subset \{e_j\} \) or \( I \cap R \subset F_j \). We will now show that in each case there is a contradiction.

Let \( I \cap R \subset \{e_j\} \). Pick an open arc \( J \) such that \( e_j \in J \subset \overline{J} \subset I \) and let the endpoints of \( J \) be \( b_1, b_2 \). Let

\[
P_j(z) = \Re[(e_j + z)(e_j - z)^{-1}]
\]

where \( \epsilon > 0 \). Then there is a number \( M > 0 \) such that \( v(re_j) \leq P_j(re_j) + M \) and \( v(rb_k) \leq M \) for \( k = 1, 2 \), and 0 < \( r < 1 \). If we define \( h(z) \) as \( (v(z) - M - P_j(z))^+ \) in \( S(J) \) and zero otherwise then \( h \) is subharmonic in \( D \). From Lemma 1 and (1.6) follows

\[
\sup \{h^+(z) : |z-e_j|=r\} = o(r^{-2}) \quad \text{as } r \to 0.
\]

Pick a point \( e \neq e_j \) in \( T \) and let \( J_1 \) and \( J_2 \) be the two arcs in \( T \) with endpoints \( e \) and \( e_j \). Mapping \( S(J_1) \) and \( S(J_2) \) respectively on the upper halfspace such that \( e_j \) corresponds to \( \infty \) then we find from the Phragmén–Lindelöf Theorem [5, p. 104] that \( h \leq 0 \). Put

\[
m(r) = \max \{v^+(z) : |z-e_j|=r, \ z \in D\}.
\]

Then \( \lim \sup_{r \to 0} rm(r) \leq C \epsilon \). Since \( \epsilon \) was arbitrary this gives \( m(r) = o(r^{-1}) \) as \( r \to 0 \) and a Phragmén–Lindelöf argument now gives \( \lim \sup_{z \to e_j} v(z) \leq 0 \) that is, \( e_j \in \Omega \) which is a contradiction.

Let \( I \cap R \subset F_j \) and \( I \cap R = I \). This means \( v \) is bounded from above in \( S(I) \) and from (2.6) we have that this is a contradiction.
Let \( I \cap R \subset F_j \) and \( I \cap R \neq I \). Without loss of generality we may assume the endpoints of \( I \) are in \( \Omega \) — otherwise we shrink \( I \). We can write \( I - I \cap R \) as a union of at most countably many pairwise disjoint open arcs \( I_n \). Our assumptions now imply the existence of a number \( M \geq j \) such that \( u(rw) \leq M \) whenever \( 0 < r < 1 \) and \( w \in F_j \) or \( w \) is an endpoint of some \( I_n \). From Lemma 1 follows

\[
\sup \{ u^+(z) : |z - a_n| = r \text{ or } |z - b_n| = r \} = o(r^{-2}) \quad \text{as } r \to 0.
\]

A Phragmén–Lindelöf argument gives now \( u \leq M \) in \( S(I_n) \). Hence \( u \leq M \) in \( S(I) \) which in view of (2.6) is a contradiction.

We have now proved \( \Omega = \mathbb{T}, \) that is, \( u^+ \leq Pf \). Hence there is a measure with nonpositive singular part such that \( P\mu \) is the least harmonic majorant of \( u \) in \( D \). The Littlewood theorem gives now \( u^* = u^* \) a.e. in \( \mathbb{T} \) and \( d\mu = u_+ dw + \mu_s \) and consequently \( u \leq Pu^* \). Theorem 2 is proved.

We shall now prove Theorem 3.

**Proof of Theorem 3.** From the proof of Theorem 2 we know \( u = Pu^* - P\lambda \) where \( \lambda \) is a nonnegative singular measure. It is sufficient to show \( \lambda = 0 \). Let \( I(w, r) \) be the open arc on \( \mathbb{T} \) with center \( w \) and length \( 2r \). Putting \( dm = (|u^*| + 1) dw \) one finds in the same way as [10, p. 159] that

\[
\lim_{r \to 0} \lambda(I(w, r))/m(I(w, r)) = \infty \quad \text{a.e. } [\lambda].
\]

From [10, p. 226] follows

\[
\liminf_{r \to 1} (-u(rw)) \geq \liminf_{r \to 0} (2r)^{-1}[\lambda(I(w, r)) - m(I(w, r))]
\]

and consequently \( u^*(w) = -\infty \) a.e. \([\lambda].\) Now (2.7) gives that \( \lambda \) is concentrated on the countable set \( E \) and (2.8) gives \( \lambda = 0 \). The Theorem is proved.

3. Exceptional sets.

Theorem 4 will be a consequence of the following lemma.

**Lemma 2.** Suppose \( u \) is subharmonic in \( D \), \( 0 < \alpha < 1 \) and \( E \subset \mathbb{T} \) is a closed set of finite \( \alpha \)-dimensional Hausdorff measure. If \( \limsup_{z \to w} u(z) \leq 0 \) for \( w \in \mathbb{T} - E \) and

\[ M(r, u) = o[(1-r)^{\alpha-1}] \quad \text{as } r \to 1 \]

then \( u \leq 0 \).
PROOF. Let $L$ be the class nonnegative subharmonic functions in $D$ vanishing in a neighbourhood of the origin. For $v \in L$ we define

$$H_1 v(rw) = \int_0^r t^{-1} v(rw) \, dt, \quad 0 \leq r < 1, \ w \in \mathbb{T},$$

and $Hv = H_1 (H_1 v)$. If $v \in L$ and $v \in C^2(D)$ then

$$\Delta Hv(rw) = r^{-2} \int_0^r t^{-1} \left( \int_0^t s \Delta v(sw) \, ds \right) \, dt.$$  

If $\varphi \in C^\infty(\mathbb{T})$ we therefore have

$$\int_\mathbb{T} Hv(rw) \varphi''(w) \, dw = r^2 \int_\mathbb{T} \Delta Hv(rw) \varphi(w) \, dw - \int_\mathbb{T} r \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} Hv(rw) \right) \varphi(w) \, dw$$

$$= \int_{|\zeta| < r} \Delta v(\zeta) \varphi(\frac{\zeta}{|\zeta|^2}) \log (r|\zeta|^{-1}) \, d\xi \, d\eta - \int_\mathbb{T} v(rw) \varphi(w) \, dw,$$

where $\zeta = \xi + i\eta$. If $v \in L$ and is not assumed to be in $C^2(D)$ we can by [8, p. 114] find a sequence $\{v_n\}_n^\infty$ of twice continuously differentiable subharmonic functions such that $v_n \downarrow v$ and $\Delta v_n dx dy$ tends weakly to the Riesz measure $\mu$ associated to $v$. Hence we have

$$(3.1) \int_\mathbb{T} Hv(rw) \varphi''(w) \, dw = \int_{|\zeta| < r} \log (r|\zeta|^{-1}) \varphi(\frac{\zeta}{|\zeta|^2}) \, d\mu(\zeta) - \int_\mathbb{T} v(rw) \varphi(w) \, dw,$$

where $\mu$ is the Riesz measure associated to $v$.

Let

$$L_\alpha = \{ v \in L : M(r, v) = O[(1 - r)^{-1 + \alpha}] \text{ as } r \to 1 \},$$

where $0 < \alpha < 1$. Since $H_1 v \in L^\infty(D)$ if $v \in L_\alpha$ it follows

$$\lim_{r \to 1} Hv(rw) = K v(w)$$

exists for all $w \in \mathbb{T}$ and

$$\sup_{w \in \mathbb{T}} |K v(w) - Hv(rw)| = O(1 - r) \quad \text{as } r \to 1.$$  

Hence $K v$ is upper semicontinuous if $v \in L_\alpha$.

Let $u$ fulfill the assumptions of Lemma 2. Since $E \neq \mathbb{T}$ there is a point $w_0 \in \mathbb{T} - E$. Therefore there is an open arc $I$ and a number $M$ such that $w_0 \in I$ and $u(rw) \leq M$ if $0 < r < 1$ and $w \in I$ and $u(z) \leq M$ if $|z| \leq \frac{1}{2}$. Define $v = (u - M)^+$. Then $v \in L_\alpha$ and

$$(3.3) \quad K v(w) = 0 \quad \text{for } w \in I.$$
If \( \varphi \in C^\infty(T) \), \( \varphi \geq 0 \) and the support of \( \varphi \) lies in \( T - E \) then we get from (3.1) and (3.2):

\[
\int_T K v(w) \varphi''(w) dw = \lim_{r \to 1} \int_T K v(rw) \varphi''(w) dw \\
\geq \limsup_{r \to 1} \int_T v(rw) \varphi(w) dw = 0
\]

and therefore \( K v \) is convex in \( T - E \). We will show that \( K v \) is convex in \( T \).

There is a constant \( C \) such that for all \( \varepsilon > 0 \) there are finitely many open arcs \( I_j \subset T \), \( j = 1, \ldots, N \) with length \( \varepsilon_j < \varepsilon \), \( \bigcup_{j=1}^N I_j \supset E \) and \( \sum_{j=1}^N \varepsilon_j^2 \leq C \). By [6, p. 43] there are \( \varphi_j \in C^\infty(T) \), \( \varphi_j \geq 0 \) and the support of \( \varphi_j \) is in \( I_j^* \), where \( I_j^* \) is the open arc with the same center as \( I_j \) and the length of \( I_j^* \) is \( 2\varepsilon_j \), and

\[
0 \leq \varphi_j \leq 1, \quad \sum_{j=1}^N \varphi_j = 1 \quad \text{on} \ E, \\
\sup_{w} |\varphi_j^{(k)}(w)| \leq C_k \varepsilon_j^{-k}.
\]

Let \( \varphi \in C^\infty(T) \) and \( \varphi \geq 0 \). Since \( K v \) is convex in \( T - E \) we have

\[
\int_T K v \varphi'' dw \geq \sum_{j=1}^N \int_T K v g_j'' dw
\]

where \( g_j = \varphi_j \varphi \). We notice

\[
\sup \left\{ |g_j^{(k)}(w)| : w \in T \right\} \leq C \varepsilon_j^{-k}, \quad 0 \leq k \leq 2,
\]

where \( C \) only depends on \( \varphi \). Since \( v \in L_\alpha \) an integration by parts shows

\[
\sup_{w \in T} |K v(w) - 2H v(rw) + H v(2r - 1)w)| = A(1-r)(1-r)^{1+\alpha}
\]

where \( A(t) \) is an increasing function with \( \lim_{r \to 0} A(t) = 0 \). Hence

\[
\int_T K v g_j'' dw = \int_T (2H v(r_jw) - H((2r_j - 1)w)) g_j'' dw + R_j
\]

where \( |R_j| \leq CA(\varepsilon_j)\varepsilon_j^3 \). Let \( S_j \) denote the integral on the right hand side. From (3.1) we find

\[
S_j \geq -2 \int_T v(r_jw) g_j(w) dw \geq -CM(1-\varepsilon_j,v)\varepsilon_j.
\]

Since \( E \) has finite \( \alpha \)-dimensional Hausdorff-measure it now follows \( \int K v \varphi'' dw \geq o(1) \) as \( \varepsilon \to 0 \) and therefore \( K v \) is convex and hence constant. From (3.3) follows \( K v = 0 \) and hence \( v = 0 \), that is, \( u \leq M \). Using (2.6) we have \( u \leq 0 \) and the lemma is proved.
Proof of Theorem 4. Let \( u \) fulfill the assumptions of Theorem 4 and let

\[
\Omega = \left\{ w \in T : \limsup_{z \to w} (u(z) - Pg^+(z)) \leq 0 \right\}.
\]

Put \( R = T - \Omega \), and assume \( R \neq \emptyset \). Assume \( E = \bigcup_j E_j \), where \( E_j \) is a closed set of finite \( \alpha \)-dimensional Hausdorff measure. Arguing as in the proof of Theorem 2 there is an open arc \( I \subset T \) and an integer \( j \) such that \( I \cap R \neq \emptyset \) and \( I \cap R \subset E_j \). We may assume the endpoints of \( I \) are in \( \Omega \), otherwise we shrink \( I \). Therefore there is a number \( M \) such that \( u(z) - Pg^+(z) \leq M \) when \( z \in \partial S(I) - I \). Let \( v(z) = (u(z) - Pg^+(z) - M)^+ \) when \( z \in S(I) \) and zero otherwise. Then \( v \) is subharmonic in \( D \) and \( \lim_{z \to w} v(z) = 0 \) for \( z \in T - E_j \). Lemma 2 gives \( v \leq 0 \), hence \( u - Pg^+ \) is bounded from above in \( S(I) \). From (2.6) follows \( I \subset \Omega \), which is a contradiction. Therefore \( \Omega = T \) and the conclusion follows now from Littlewood's Theorem.


We will deduce Theorem 5 from the following lemma.

Lemma 3. Suppose \( u \) is harmonic in \( D \) and \( I_0 \subset T \) is an open arc and \( E \subset I_0 \) is countable. If

\[
|u| \text{ is of type } G(w, 1) \text{ for all } w \in I_0,
\]

\[
\limsup_{r \to 1} (1-r)^{-1} |f(w) - u(rw)| < \infty \text{ for all } w \in I_0 - E,
\]

\[
\text{there is a } g \in L^1(T) \text{ such that } \lim_{r \to 1} (1-r)^{-1} (f(w) - u(rw)) = g(w) \text{ in } I_0,
\]

\[
|u(rw)| = o(\log (1-r)) \text{ as } r \to 1 \text{ for } w \in E,
\]

then \( f \) is locally integrable in \( I_0 \) and for all \( \varphi \in C^\infty(T) \) with support in \( I_0 \) we have

\[
\lim_{r \to 1} \int_T \varphi(w) \frac{\partial u}{\partial r}(rw) \, dw = \int_T \varphi(w) g(w) \, dw.
\]

For the proof we will study a certain type of kernels. Let \( I \subset T \) be an open, nonempty arc. Put

\[
D(I) = \{ rw : \frac{1}{2} < r < 1, w \in I \}, \quad D^\star(I) = \{ rw : \frac{1}{2} < r < 2, w \in I \}
\]

and let \( g(z, \zeta; I) \) be the Green function of \( D^\star(I) \), normalized by
$g(z, \zeta; I) + (2\pi)^{-1} \log |z - \zeta|$ is harmonic in $D^*(I)$ as a function of $z$. For $z \neq 0$ let $z^* = z|z|^{-2}$ and define for $z, \zeta \in D(I)$:

$$K(z, \zeta; I) = g(z, \zeta; I) + g(z^*, \zeta; I).$$

Let $\Gamma(I) = \partial D(I) - \bar{I}$, $f \in L^1(\Gamma(I), ds)$ where $ds$ is the element of arc length, and let $\mu$ be a measure supported on $\overline{I}$. Then we put

$$N(f, \mu; I) = \int_{\Gamma(I)} \frac{\partial}{\partial n_\zeta} (z, \zeta; I) f(\zeta) \, ds + \int_I K(z, \zeta; I) \, d\mu(\zeta),$$

where $\partial/\partial n_\zeta$ denotes differentiation with respect to the unit inward normal of $\Gamma(I)$.

We now have for all $\varphi \in C^\infty(\mathbb{T})$ with support in $I$:

$$(4.5) \quad \lim_{r \to 1} \int_I \frac{\partial}{\partial r} N(f, \varphi; I)(rw) \varphi(w) \, dw = \int \varphi \, d\mu.$$  

To see this put $u = N(f, 0; I)$ and $v = N(0, \mu; I)$. Noticing $u$ has a harmonic extension to $D^*(I)$ such that $u(z) = u(z^*)$ we have

$$(4.6) \quad \frac{\partial u}{\partial r}(w) = 0 \quad \text{for } w \in I.$$  

Let $w \in I$ and put

$$g_w(z) = (2\pi)^{-1} \int_{\partial D^*(I)} \log |z - \zeta| \frac{\partial}{\partial n_\zeta} g(w, \zeta; I) \, ds.$$  

Then $g_w$ is harmonic in $D^*(I)$. If we put $h_w(z) = g_w(z) + g_w(z^*)$ then

$$K(z, w; I) = b_w(z) - (2\pi)^{-1} \log |z-w||z^*-w|.$$  

If $V$ is an open set and $\overline{V} \subset D^*(I)$ then

$$\sup \{|h_w(z)| : w \in I, z \in v\} = c_v < \infty.$$  

Since $h_w$ is harmonic and $h_w(z) = h_w(z^*)$ we find

$$(4.7) \quad r \frac{\partial}{\partial r} K(re^{i\theta}, e^{it}) = P(r, \theta - t) + S(r, \theta, t)$$

where

$$\sup \{|S(r, \theta, t)| : e^{it} \in K, e^{it} \in I\} = o(1 - r) \quad \text{as } r \to 1.$$  

for all compact sets $K \subset I$. The relation (4.5) follows from (4.6) and (4.7).

Let $h \in L^1(\mathbb{T})$ and let $h$ be lower semicontinuous. Since $h$ is bounded from below by a constant we have from (4.7) that
\[
\lim_{r \to 1} N(0, h ; I)(rw) = h^*(w)
\]
exists for all \( w \in I \) and the monotone convergence theorem gives
\[
h^*(w) = \int_{I} K(w, \zeta ; I) h(\zeta) d\zeta, \quad w \in I.
\]
Moreover, since \( K(w, \zeta ; I) \geq 0 \), this expression for \( h^* \) shows that \( h^* \) has a lower semicontinuous extension to \( \bar{I} \). Since \( h \) is lower semicontinuous we have
\[
\liminf_{r \to 1} \int P(r, \theta - t) h(e^{it}) dt \geq h(e^{i\theta}).
\]
Therefore we have from (4.7)
\[
(4.8) \quad \liminf_{r \to 1} (h^*(w) - N(0, h ; I)(rw))(1 - r)^{-1} \geq h(w).
\]

**Proof of Lemma 3.** Let
\[
\Omega = \{ w \in I_0 : \text{there is an open arc } I, w_0 \in I \subset I_0 \text{ and } u = N(u, g ; I) \text{ in } D(I) \}
\]
and put \( F = I_0 - \Omega \). Then \( F \) is relatively closed in \( I_0 \) and it is sufficient to prove \( F = \emptyset \). We therefore assume \( F \neq \emptyset \). Let
\[
F_j = \{ w \in I_0 : |u(rw) - u(sw)| \leq j(2 - r - s) \text{ for } 0 < r, s < 1 \}
\]
and let \( E = \{ e_j \} \). Since \( F_i \) is closed in \( I_0 \) for all \( i \), the Baire category theorem implies the existence of an open arc \( I \) and integer \( j \) such that \( I \cap F \neq \emptyset \) and \( I \cap F \subset \{ e_j \} \) or \( I \cap F \subset F_j \). We will show that each case leads to a contradiction.

Let \( I \cap F \subset \{ e_j \} \). We may without loss of generality assume the endpoints of \( I \) are in \( \Omega \), otherwise we make \( I \) smaller. Put \( v = u - N(u, g ; I) \). It follows from (4.5) and the reasoning in [11] that we can extend \( v \) to a function harmonic in the set
\[
S = \{ rw : w \in I, 0 < r < \infty \} - \{ e_j \}
\]
such that \( v(z) = v(z^*) \) in \( S \). It is easy to see that \( |v| \) is of type \( G(w, 1) \) for \( w \in \bar{I} \). Putting
\[
m(r, v) = \sup \{ v^*(z) : |z - e_j| = r \}
\]
it follows from Lemma 1 that \( m(r, v) = o(r^{-1}) \) as \( r \to 1 \). Since \( v(ra_j) = o[\log (r - 1)] \) as \( r \to 1 \) a Phragmén–Lindelöf argument gives
\[
(4.9) \quad m(r, v) = o(\log r) \quad \text{as } r \to 0.
\]
The assumptions are symmetrical with respect to \( v \) and \( -v \). This gives \( m(r, |v|) = o(\log r) \) as \( r \to 0 \) and consequently the singularity at \( e_j \) is removable. We
have $v(z)=0$ for $z \in \partial D^*(I)-\overline{I}$. It now follows from Lemma 1, (4.1) and the Phragmén–Lindelöf theorem that $v=0$ in $D^*(I)$. Hence $e_j \in \Omega$ which is a contradiction.

**Remark 1.** Notice that in the proof of (4.9) we only used that $u$ was of type $G(w,1)$ for $w \in I_0$.

Let $I \cap F \subset F_j$. There is by (4.2), (4.3) and [12, p. 73] a lower semicontinuous function $h$ in $L(T)$ such that

$$
\limsup_{r \to 1} (1-r)^{-1}(f(w)-u(rw)) \leq h(w)
$$

for all $w \in I_0$. In addition we can make $\int_T |h-g|dw$ as small as we want. Let $J$ be an open nonempty open arc such that $\overline{J} \subset I$. Let $v=u-N(u,h;J)$. It follows from the choice of $h$ and the definition of $F_j$ that $\lim_{r \to 1} v(rw)=H(w)$ exists for all $w \in J$. Notice also that $v$ has a subharmonic extension across $I-I \cap F$. Therefore the restriction of $H$ to $J-J \cap F$ is upper semicontinuous. It follows from the definition of $F_j$ that the restriction of $H$ to $J \cap F$ is upper semicontinuous. We now claim $H$ is upper semicontinuous in $J$. To show this it is sufficient to show that if $\eta \in J \cap F$, $\{w_k\} \subset J-J \cap F$ and $w_k \to \eta$ then $\limsup_{k \to \infty} H(w_k) \leq H(\eta)$. Let $I_k$ be the maximal open arc in $J-J \cap F$ containing $w_k$. Pick $\varepsilon > 0$. Then there is a $\delta > 0$ and a neighbourhood $V$ of $\eta$ in $T$ such that $v(rw)<H(\eta)+\varepsilon$ if $1-\delta<r<1$ and $w \in J \cap F \cap V$. Let

$$
S_k = \{rw : (1-\delta)<r<(1-\delta)^{-1}, \ w \in I_k\},
$$

and $\gamma_{k,j} = \{r a_{k,j} : 1-\delta<r<(1-\delta)^{-1}\}$, where $a_{k,j}, j=1,2$, are the endpoints of $I_k$ and $\gamma_{k,3} = \partial S_k - (\gamma_{k,1} \cup \gamma_{k,2})$. Let $\sigma_{k,j}$ be the harmonic measure of $\gamma_{k,j}$ with respect to $S_k$. Since $v$ is lower semicontinuous it follows that

$$
\sup \{v(z) : z \in \gamma_{k,j}\} = A_{k,j} < \infty \quad \text{for all} \ k,j.
$$

Put $M=\sup \{v(z) : |z|=1-\delta\}$. Then (4.1), Lemma 1 and the Phragmén–Lindelöf theorem gives:

$$
v(z) \leq A_{k,1} \sigma_{k,1}(z) + A_{k,2} \sigma_{k,2}(z) + M \sigma_{k,3}(z), \quad z \in S_k.
$$

There are now two cases to consider. If $I_k = I_{k_0}$ for all $k \geq k_0$ then $\eta = a_{k_0,j}$ for some $j$. In this case (4.11) gives

$$
\limsup_{k \to \infty} H(w_k) \leq A_{k_0,j} \leq H(\eta) + \varepsilon.
$$

Otherwise $\lim_{k \to \infty} \text{diam } (I_k)=0$ and $I_k \subset V$ for $k \geq k_0$. From (4.11) follows now

$$
H(w_k) \leq H(\eta) + \varepsilon + M \sigma_{k,3}(w_k), \quad k \geq k_0.
$$
Since it is straightforward to show
\[ \lim_{k \to \infty} \left( \sup_{w \in I_k} \{ \sigma_{k,3}(w) : w \in I_k \} \right) = 0 \]

it follows that \( H \) is upper semicontinuous. In the same way it follows that
\[ \limsup_{w \to w_0} H(w) \leq 0 \]
whenever \( w_0 \) is an endpoint of \( J \). We claim \( v \leq 0 \). To show this put \( p(z) = v^+(z) \) for \( z \in S(J) \) and zero otherwise. Then \( p \) is subharmonic in \( D \) and \( p \) is of type \( G(w,1) \) for all \( w \in T \). Theorem 2 gives \( p \leq PH_1 \), where \( H_1(w) = 0 \) when \( w \notin J \) and \( S = H^+(w) \) when \( w \in J \). Since \( H_1 \) is upper semicontinuous there is a point \( w_0 \in I \) such that \( \max_{z \in D} p(z) = H_1(w_0) \). Suppose \( H_1(w_0) > 0 \). Then it follows from [9, p. 67] that
\[ u(rw_0) - N(u,h;J)(rw) \leq -c(1-r) + H(w_0) \quad \text{for some } c > 0. \]

But this contradicts (4.8) and (4.10). Hence \( v \leq 0 \). Letting \( h \to g \) in \( L^1 \)-norm we find
\[ (4.12) \quad u \leq N(u,g;J). \]

Since the argument can be carried out with \(-u\) as well it follows \( u = N(u,g;J) \). This contradiction shows \( \Omega = I_0 \) and the lemma is proved.

**Remark 2.** In the proof of (4.12) we only used that \( u \) was of type \( G(w,1) \) for \( w \in I_0 \).

**Lemma 4.** Suppose \( u \) fulfils the assumptions of Theorem 5. If for some open arc \( I \subset T, I \neq \varnothing \), we have \( f^+ \in L^1(I) \), then \( f \in L^1_{\text{loc}}(I) \) and \( |u| \) us of type \( G(w,1) \) for all \( w \in I \).

**Proof.** Let \( h \) be the harmonic function in \( S(I) \) with boundary values equal to \( u^+(z) \) when \( z \in \partial S(I) - \bar{I} \) and zero elsewhere. Put \( v = (u^+ - h)^+ \). Then \( v \) is subharmonic in \( D \) and of type \( G(w,1) \) for all \( w \in T \). From Theorem 2 we now have \( v \leq PF \) for some \( F \in L^1(T) \). This means \( u \mid S(I) \) is equal to the difference of two positive harmonic functions. Let \( \Phi \) be a conformal map of \( D \) onto \( S(I) \). From Fatou's theorem follows that \( u \circ \Phi \) has a nontangential limit \( G(w) \) a.e. in \( T \) and \( G \in L^1(T) \). From Poisson's representation formula follows
\[ M(r,|u \circ \Phi|) = O[(1-r)^{-1}] \quad \text{as } r \to 1. \]

Going back to \( u \) this means \( f \in L^1_{\text{loc}}(I) \) and \( |u| \) is of type \( G(w,2) \) for all \( w \in I \). Let be an open arc such that \( J \neq \varnothing \) and \( J \subset I \). Then \( f \in L^1(J) \). Let \( h_1 \) be the harmonic function in \( S(J) \) with boundary values equal to \( |u| \) on \( S(J) - \bar{J} \) and zero elsewhere. Arguing as in the beginning of the proof, it follows \( (|u| - h_1)^+ \leq PF \) for some \( F \in L^1(T) \) and hence \( |u| \) is of type \( G(w,1) \) for all \( w \in J \). Since \( J \) was arbitrary the Lemma follows.
Proof of Theorem 5. Let $u$ fulfil the assumptions of Theorem 5. Let
\[ F_j = \{ w \in \mathcal{T} : |u(rw) - u(sw)| \leq j(2 - r - s) \text{ for } 0 < r, s < 1 \}, \]
and
\[ E = \{ e_j \}_{j=1}^{\infty} \]
and let $u = N(u, g; I)$. Then as above $R$ is closed and the Baire category theorem implies the existence of an open arc $I$ and an integer $j$ such that $I \cap R \neq \emptyset$ and $I \cap R \subset \{ e_j \}$ or $I \cap R \subset F_j$. If $I \cap R \subset \{ e_j \}$ it follows from (4.9) and Remark 1 that $f^+ \in L^1_{\text{loc}}(I)$. If $I \cap R \subset F_j$ it follows from (4.12) and Remark 2 that $f^+ \in L^1_{\text{loc}}(I)$. Hence we have from Lemma 4 that in both cases $u$ fulfils the assumptions of Lemma 3 on $I$ and consequently $I \subset \Omega$. This contradiction shows $\Omega = \mathcal{T}$ which yields Theorem 5.

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CHALMERS UNIVERSITY OF TECHNOLOGY

AND

UNIVERSITY OF GÖTEBORG
DEPARTMENT OF MATHEMATICS
FACK
S-402 20 GÖTEBORG
SWEDEN