COMPLEX PREDUALS OF $L_1$ AND
SUBSPACES OF $l^n_\infty(C)$

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Introduction.

In the present paper we investigate the structure of complex preduals of $L_1$
and the problems concerning norm preserving extensions of compact
operators. Most of the results are known in the real case, but the complex case
does not follow from these in a straightforward manner; in fact, in many
respects the complex case is much more complicated and requires often
different proofs. Many of the proofs here give other methods to show the
Corresponding real result was proved by Lazar and Lindenstrauss [16]. The
proof given here provides an alternative way of proving the real result. The
main brick in the proof is a complex version of the Lazar selection theorem,
recently proved in [26]. The result is then used to give a new and very short
proof of the Hirsberg–Lazar characterization of preduals of $L_1$, whose unit ball
contains an extreme point.

Section 2 is devoted to the study of norm preserving extensions of compact
operators. We first prove that if $E$ is a finite dimensional space, whose unit ball
is the absolutely convex hull of finitely many points, then every point in $B_E$ can
be represented as a combination of extreme points so that the coordinate
functions are continuous. The real case is due to Kalman [12]. While his proof
is geometric, the proof here uses an argument on extension of operators, based
on the main theorem of section 1, but in the real case we do not need this

Received April 23, 1976.
The previous results of the paper are then used to characterize those preduals $X$ of $L_1$ with the property that every compact operator with image in $X$ can be extended preserving the norm. The corresponding real result was proved by Lazar and our proof follows his ideas. We end the section by proving that every predual of $l_1$ is isomorphic to an $L_{\infty,1}$-space, a result due to W. B. Johnson and the first named author.

**0. Notations and preliminaries.**

In this paper all Banach spaces are assumed to be complex spaces unless otherwise stated, and throughout the paper we shall use the notation and terminology commonly used in Banach space theory as it appears in [22], let us just here recall that if $X$ and $Y$ are Banach spaces, then the Banach distance $d(X, Y)$ is defined by

$$d(X, Y) = \inf \{ \| S \| \| S^{-1} \| \mid S \text{ is isomorphism of } X \text{ onto } Y \}$$

and if $X$ and $Y$ are not isomorphic we put $d(X, Y) = \infty$.

For every natural $n$ we let $\{ e_j \mid 1 \leq j \leq n \}$ denote the unit vector basis of $l_1^n$ and $\{ e^*_j \mid 1 \leq j \leq n \}$ its biorthogonal system, i.e. the unit vector basis of $l_\infty^n$. Further we let $T$ be the unit circle in $\mathbb{C}$, and if $x$ and $y$ are vectors in a Banach space, then we say that $x$ and $y$ are $T$-equivalent, if there is a $t \in T$ so that $x = ty$.

If $X$ is a Banach space we let $B_X$ denote the unit ball of $X$, and for a convex set $K \subseteq X$ we let $\partial_e K$ denote the extreme points. A compact absolutely convex set $K \subseteq X$ is called a complex polytope, if there exists a finite set $A \subseteq \partial_e K$ so that $\partial_e K = T \cdot A$.

Let $E$ and $F$ be locally convex spaces and denote by $c(F)$ the set of all non-empty convex subsets of $F$. If $K \subseteq E$ is convex and $\varphi : K \to c(F)$, then $\varphi$ is called convex, provided:

$$\lambda \varphi(x_1) + (1 - \lambda) \varphi(x_2) \subseteq \varphi(\lambda x_1 + (1 - \lambda)x_2)$$

for all $x_1, x_2 \in K$ and $\lambda \in [0, 1]$.

The map $\varphi$ is said to be lower semicontinuous if $\{ x \in K \mid \varphi(x) \cap U \neq \emptyset \}$ is open for every open subset $U$ of $F$. Finally when $K$ is absolutely convex, we say that $\varphi$ is $T$-symmetric, if $\varphi(tx) = t \varphi(x)$ for all $t \in T$ and $x \in K$. By a selection for $\varphi$ we mean a map $f : K \to F$ such that $f(x) \in \varphi(x)$ for all $x \in K$. Else our general reference in convexity is Alfsen’s book [1].

**1. Structure theorems for preduals of $L_1$.**

Before we prove the main theorems of this section mentioned in the introduction, we need the following easy proposition on complex polytopes:
1.1. Proposition. Let $E$ be a finite dimensional Banach space, then $B_E$ is a complex polytope, if and only if there is an $n \in \mathbb{N}$ and an operator $l_1^n \to E$ taking the unit ball of $l_1^n$ onto the unit ball of $E$.

Proof. Assume first $B_E$ is a complex polytope, and let $x_1, x_2, \ldots, x_n$ be extreme points of $B_E$, mutually non $T$-equivalent, and so that $\partial_B B_E = T \cdot \{x_1, x_2, \ldots, x_n\}$. If we define $S: l_1^n \to E$ by:

$$S \left( \sum_{j=1}^{n} t_j e_j \right) = \sum_{j=1}^{n} t_j x_j; \quad t_1, t_2, \ldots, t_n \in \mathbb{C};$$

then it is obvious that $S$ has the required properties.

If $S: l_1^n \to E$ satisfies the conditions in the proposition we put

$$A = \{e_j \mid S(e_j) \in \partial_B B_E\}.$$

If $x \in \partial_B B_E$, then $S^{-1}(x) \cap B_{l_1^n}$ is a compact face of $B_{l_1^n}$ and hence it contains an extreme point, thus there is an index $j$ and $t \in T$, so that $S(te_j) = x$, but then $e_j \in A$, and $x \in T \cdot S(A)$. Hence $\partial_B B_E = T \cdot S(A)$.

1.2. Corollary. Let $E$ be a finite dimensional Banach space. Then $E$ embeds isometrically into $l_{1,\infty}$ for some $n$ if and only if $B_{E^*}$ is a complex polytope.

We are now ready to state and prove the main theorem of this section.

1.3. Theorem. Let $X$ be a predual of $L_1$ and let $F_1$ and $F_2$ be finite dimensional subspaces of $X$ with $F_1$ isometric to a subspace of $l_{1,\infty}$ for some $k_0$. Then for every $\varepsilon > 0$ there is a subspace $E$ of $X$ with $F_1 \subseteq E$, $d(x, E) \leq \varepsilon$ for every $x \in B_{F_2}$ and so that $E$ is isometric to $l_{1,\infty}$ for suitable $m$.

Proof. It is enough to prove the theorem in the case dim $F_2 = 1$, the general case will then follow by induction. Hence let $\varepsilon > 0$, $\{y_i \mid 1 \leq i \leq n\}$ be a unit vector basis for $F_1$ and $z$ a unit vector in $F_2$. We define $R: B_{X^*} \to \mathbb{C} \times \mathbb{C}^n$ by

$$R x^* = (x^*(z), x^*(y_1), \ldots, x^*(y_n)); \quad x^* \in B_{X^*};$$

and put $W = RB_{X^*}$. Denote by $D_\varepsilon$ the disc in $\mathbb{C}$ with radius $\varepsilon$ and center 0 and let $P: \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ be the canonical projection. Since by our assumptions and corollary 1.2 $P(W)$ is a complex polytope and $\partial e W$ is totally bounded we can find mutually non $T$-equivalent extreme points $w_1, w_2 \ldots, w_m$ of $W$, so that if $W' = \text{conv} (T \cdot \{w_1, \ldots, w_m\})$, then $P(W) = P(W')$ and

$$\{z \in \mathbb{C} \mid (z, v) \in W\} \subseteq \{z \in \mathbb{C} \mid (z, v) \in W'\} + D_\varepsilon.$$

Define $S: l_{1,\infty} \to \mathbb{C} \times \mathbb{C}^n$ by

$$S = P \circ R.$$
(2) \[ S(e_j) = w_j; \quad 1 \leq j \leq m \]
\[ S(e_{m+1}) = (\varepsilon, 0, \ldots, 0). \]

Let \( B \) be the unit ball of \( C \oplus_{\infty} l^n_1(C) \), then by (1)
\[ S(B) \supseteq W. \]

Let \( \hat{c}(l^{m+1}_1) \) denote the set of all closed convex subsets of \( l^{m+1}_1 \) and define \( \psi, \psi_1, \psi_2: B_{X^*} \to \hat{c}(l^{m+1}_1) \) by:
\[ \psi_1(x^*) = S^{-1}(Rx^*) \]
\[ \psi_2(x^*) = \begin{cases} \{te_k\} & \text{if } Rx^* = tw_k, \ t \in T, \ k \leq m \\ B & \text{else} \end{cases} \]
\[ \psi(x^*) = \psi_1(x^*) \cap \psi_2(x^*). \]

It is easy to see that \( \psi \) is convex and \( T \)-symmetric. We wish to show that \( \psi \) is lower semicontinuous when \( B_{X^*} \) is equipped with the \( w^* \)-topology. If \( U \) is an open subset of \( l^{m+1}_1 \), then the sets
\[ A_j = \{ t \in T \mid te_j \notin U \}, \quad 1 \leq j \leq m \]
are compact, and since \( l^{m+1}_1 \) is finite dimensional, the set \( R^{-1}(SU) \) is a \( w^* \)-open subset of \( B_{X^*} \); therefore the set
\[ \{ x^* \mid \psi(x^*) \cap U \neq \emptyset \} = R^{-1}(SU) \setminus \bigcup_{j=1}^n \{ tR^{-1}(w_j) \mid t \in A_j \} \]
is \( w^* \)-open in \( B_{X^*} \). This proves that \( \psi \) is lower semicontinuous. By the complex analogy of Lazar's selection theorem [26, theorem 4.2] \( \psi \) admits an affine, \( T \)-symmetric and \( w^* \)-continuous selection \( \varphi \). For \( k = 1, 2, \ldots, m+1 \) we define \( x_k \in X \) by
\[ x^*(x_k) = e^*_k(\varphi(x^*)), \quad x^* \in B_{X^*}. \]

Let \( \alpha_1, \alpha_2, \ldots, \alpha_m \in C. \) By the definition of \( \psi_2 \) we now get:
\[
\left\| \sum_{i=1}^m \alpha_i x_i \right\| = \sup \left\{ \left\| x^* \left( \sum_{i=1}^m \alpha_i x_i \right) \right\| \mid x^* \in B_{X^*} \right\}
\]
\[
= \sup \left\{ \left\| \sum_{i=1}^m \alpha_i e^*_i(\varphi(x^*)) \right\| \mid x^* \in B_{X^*} \right\}
\]
\[
= \sup \left\{ \left\| \sum_{i=1}^m \alpha_i e^*_i(e) \right\| \mid e \in B_{r^i} \right\}
\]
\[
= \sup \{ |\alpha_i| \mid 1 \leq i \leq m \}.
\]
This gives that the linear span of \( \{x_k\}_{k=1}^m \) is isometric to \( l_\infty^m \). By (4) and the definition of \( \psi_1 \) we get for all \( x^* \in B_{X^*} \):

\[
(5) \quad Rx^* = S(\varphi(x^*)) = \sum_{k=1}^n x^*(x_k)w_k + x^*(x_{{m+1}})(\varepsilon, 0, \ldots, 0).
\]

If we put \( w_k = (w_k^j)_{j=1}^{n+1} \), we get by looking on (5) coordinatewise

\[
(6) \quad y_k = \sum_{j=1}^m w_k^{j+1}x_j, \quad 1 \leq k \leq n
\]

and

\[
(7) \quad \left\| z - \sum_{j=1}^m w_j^jx_j \right\| \leq \varepsilon;
\]

so the proof is complete.

As a corollary of the above theorem we can take out the next result, proved in the real case by Lazar and Lindenstrauss [16]. This is a slightly stronger version of the main result of Michael and Pelczynski in [23].

**1.4. Theorem.** Let \( X \) be a separable predual of \( L_1 \) and let \( F \subseteq X \) be a finite dimensional space which embeds isometrically into \( l_\infty^k \) for some \( k \). Then there exists an increasing sequence \( (E_n)_{n=1}^\infty \) of finite dimensional subspaces of \( X \) with \( X = \bigcup_{n=1}^\infty E_n \) and so that \( F \subseteq E_1, \dim E_{n+1} = 1 + \dim E_n \) and \( E_n \) isometric to \( l_\infty^1 \).

**Proof.** The result can be proved as in [16] by using our theorem 1.3 instead of their theorem 3.1. The fact that the \( E_n \)'s can be chosen to satisfy \( \dim E_{n+1} = 1 + \dim E_n \) follows from [23].

We pass now to give an alternative proof of a functional representation theorem for complex preduals of \( L_1 \) whose unit ball has an extreme point, due to Hirsberg and Lazar [10]. A simpler proof than the original one was given by Lima [17].

**1.5. Theorem.** Let \( X \) be a predual of \( L_1 \), whose unit ball has an extreme point \( e \). Let

\[
S = \{ x^* \in X^* \mid x^*(e) = 1 = \| x^* \| \}
\]

be equipped with the \( w^* \)-topology. If \( \psi : X \to C(S) \) is the natural map defined by \( \psi(x)(x^*) = x^*(x), x^* \in S \), then \( \psi \) is an isometry, which maps \( X \) onto the space of \( w^* \)-continuous complex affine functions on \( S \) and \( \psi(e) = 1 \).
PROOF. Clearly \( \psi(e) = 1 \) and \( \|\psi(x)\| \leq \|x\| \) for all \( x \in X \). Let \( y \in X \) with \( \|y\| = 1 \). We wish to show that \( \|\psi(y)\| \geq 1 \). If \( \varepsilon > 0 \) is arbitrary, then by theorem 1.3 we can find a finite dimensional space \( E \subseteq X \) so that \( e \in E \) and \( d(y, E) < \varepsilon \) and \( E \) isometric to \( l^\infty_n \) for some \( n \). Let \( (x_j)_{j=1}^n \) be a basis for \( E \) isometrically equivalent to the unit vector basis of \( l^\infty_n \) and let \( (x_j^*)_{j=1}^n \subseteq X^* \) be a sequence biorthogonal to \( (x_j)_{j=1}^n \) with \( \|x_j^*\| = 1 \), \( 1 \leq j \leq n \). By the above there is an \( x \in E \) with \( \|x\| = 1 \) and \( \|y - x\| \leq 2\varepsilon \). Let \( j \) be chosen so that \( |x_j^*(x)| = 1 \), since \( e \) is an extreme point of \( B_X \) \( |x_j^*(e)| = 1 \) and therefore \( x_j^*(e)x_j^* \in S \), moreover:

\[
|x_j^*(e)x_j^*(y)| \geq |x_j^*(x)| - \|x_j^*\| \|x - y\| \geq 1 - 2\varepsilon
\]

hence \( \|\psi(y)\| \geq 1 \). An argument of [26] gives that \( \psi \) is onto.

We want to thank A. Lazar for suggesting this proof.

1.6. COROLLARY. Let \( X \) be a predual of \( L_1 \) and \( e \in X \) with \( \|e\| = 1 \). Put

\[
S = \{x^* \in X^* \mid x^*(e) = 1 = \|x^*\|\}.
\]

Then the following statements are equivalent:

(i) \( e \) is an extreme point of \( B_X \).
(ii) \( S \) is an maximal face of \( B_{X^{**}} \).
(iii) \( e \) considered as an element of \( B_{X^{**}} \) is an extreme point.

PROOF. Assume (i) and that \( S \) is not a maximal face in \( B_{X^{**}} \). Then there exist \( y^* \in B_{X^{**}} \) such that \( y^* \notin \text{conv} \{tS \mid t \in T\} \). By Hahn–Banach there exist a \( w^* \)-continuous functional \( x \), that is \( x \in X \), such that

\[
y^*(x) = 1 > \sup \{\text{Re } x(x^*) \mid x^* \in \text{conv} \{tS \mid t \in T\}\}
\]

\[
\geq \sup \{|x^*(x)| \mid x^* \in S\}
\]

which contradicts the fact that the map \( \psi \) in the preceding theorem is an isometry.

(ii) \( \Rightarrow \) (iii). Assume \( S \) is a maximal face in \( B_{X^{**}} \). We may identify \( X^* \) with \( L_1(Q, \mathcal{A}, m) \) for some positive measure space \( (Q, \mathcal{A}, m) \). First we observe that for any \( B \in \mathcal{A} \) there is \( f \in S \) with \( \|f \cdot \chi_B\| > 0 \). If not the norm would be additive on the set \( \text{conv} (S \cup \{m(B)^{-1}\chi_B\}) \) so by [2, lemma 2.1] we get a contradiction to the maximality of \( S \). Assume there is \( B \in \mathcal{A} \) with \( m(B) > 0 \) and \( |e(q)| < 1 \) a.e. on \( B \). By the above observation there is \( f \in S \) with \( \|f \cdot \chi_B\| > 0 \). Since \( S \) is a face \( \|f \cdot \chi_B\|^{-1}f \cdot \chi_B \in S \). But

\[
|e(\|f \cdot \chi_B\|^{-1}f \cdot \chi_B)| < 1
\]

contradicting the definition of \( S \).

(iii) \( \Rightarrow \) (i). Trivial.
Remarks. Functional representations of the type in theorem 1.5 were investigated and studied by Kadison (see [1, p. 78]) who represented the self adjoint part of a $C^*$-algebra $\mathcal{A}$ isometrically as the real affine $w^*$-continuous functions on the state space. In this situation one can no longer represent $\mathcal{A}$ isometrically as complex affine functions on the state space unless $\mathcal{A}$ is commutative. This is probably well known, but since we are unable to give a reference to this fact, we shall give a proof which was shown to us by Christian Skau.

Let $a \in \mathcal{A}$. By assumption there is a pure state $p$ such that $\|a\| = |p(a)|$. Let $\pi_p$ be the corresponding representation with cyclic vector $\xi$. Then we have

$$\|a\| = |p(a)| = |\langle \pi_p(a)\xi, \xi \rangle| \leq \|\pi_p(a)\xi\| \|\xi\| \leq \|a\|.$$  

By Schwartz's equality $\pi_p(a)\xi = p(a)\xi$. If $b \in \mathcal{A}$, then

$$p(ba) = \langle \pi_p(ba)\xi, \xi \rangle = \langle \pi_p(b)\pi_p(a)\xi, \xi \rangle = p(a)\langle \pi_p(b)\xi, \xi \rangle = p(a)p(b).$$

Similarly we get $p(ab) = p(a)p(b)$. It follows that the spectral radius coincides with the norm on $\mathcal{A}$, so [4, theorem 4.7] gives that $\mathcal{A}$ is commutative. (The above result is incorrect for Banach algebras, consider the bounded operators on a predual of $L_1$ [4, p. 87]).

On the other hand functional representations of Banach algebras will always be isomorphisms due to the Bohnenblust–Karlin theorem [4], and for $C^*$-algebras the onto argument is still valid, in fact this gives a characterization of the $C^*$-algebras among the Banach algebras. This is just a restatement of the Azimov–Ellis geometric interpretation of the Vidav–Palmer theorem [3].

Let $\mathcal{A}$ be a Banach algebra with unit and assume $\mathcal{A}$ is complex predual of $L_1$. Then the map $\psi$ of theorem 1.5 is onto, so $\mathcal{A}$ is a $C^*$-algebra. Since $\psi$ is an isometry, $\mathcal{A}$ is commutative, so $\mathcal{A}$ is isometric to $C(S)$ for some compact Hausdorff space $S$. This result was proved by Hirsberg and Lazar for function algebras [10] and in general by Ellis [8].

2. Norm preserving compact extensions.

In the real theory of norm preserving extensions of compact operators the subspaces of the spaces $l^n_\infty$ play a central role. The same is the case in the complex theory, as our results in section 1 indicate; however, there is one major difference. In the real theory the subspaces of the $l^n_\infty$'s are exactly the spaces, whose unit ball is a polytope (this follows for example from corollary 1.2 and the fact that the unit ball of a real Banach space is a polytope if and only if the unit ball of the dual space is a polytope [13]); this is not so in the complex case as the example $l^n_\infty$ shows.
We recall that a function $f$ on a circled subset $K$ of a Banach space is called $T$-homogeneous if $f(tx) = tf(x)$ for all $x \in K$, $t \in T$.

Before we can prove our main results we need the following:

2.1. **PROPOSITION.** Let $K$ be a compact metric space, $x_0, x_1, \ldots, x_n \in K$; $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$; so that $\sum_{j=1}^{n} |\lambda_j| \leq 1$. Then the subspace $X$ of $C(K)$ consisting of those $f \in C(K)$ for which $f(x_0) = \sum_{j=1}^{n} \lambda_j f(x_j)$ is a predual of $L_1$.

**PROOF.** We shall assume $|\lambda_j| < 1$ for all $j$, since else $X$ is a G-space and hence is a predual of $L_1$ [25]. It is immediate that

$$\partial_e B_{X^*} = \{ t \delta_x \mid x \in K, x \neq x_0, t \in T \}.$$ 

Let $\varphi : T \times K \to \partial_e B_{X^*}$ be the onto map defined by $\varphi(t, x) = t \delta_x$, $t \in T$, $x \in K$; and let $\mu$ and $v$ be two boundary measures on $B_{X^*}$ with the same barycenter. According to a theorem of Effros [7], it is enough to show $\mu(f) = v(f)$ for all $T$-homogeneous $f \in C(B_{X^*})$. By the Hahn–Banach theorem there exist Radon probabilities $\mu_1$ and $v_1$ on $T \times K$ so that $\varphi(\mu_1) = \mu$, $\varphi(v_1) = v$. By maximality $\mu(T\{\delta_{x_0}\}) = v(T\{\delta_{x_0}\}) = 0$, hence $\mu_1$ and $v_1$ are concentrated on $T \times (K \setminus \{x_0\})$. Let $f \in C(B_{X^*})$ be $T$-homogeneous, and let $\varepsilon > 0$. By regularity we can find a compact subset $K_1 \subseteq K \setminus \{x_0\}$ containing $x_1, \ldots, x_n$, so that

$$|\mu_1 - v_1|(T \times K_1) \geq \|\mu_1 - v_1\| - \varepsilon.$$ 

By Tietze's extension theorem we can find $\tilde{f} \in C(K)$ so that $\tilde{f}(x) = f(\delta_x)$, $x \in K_1$, $\|\tilde{f}\| = \|f\|$, and $\tilde{f}(x_0) = \sum_{j=1}^{n} \lambda_j f(\delta_{x_j})$; clearly $\tilde{f} \in X$ and hence:

$$|u(f) - v(f)| = \left| \int_{T \times K} f(t \delta_x) - t \tilde{f}(x) d(u_1 - v_1)(t, x) \right|$$

$$\leq \int_{T \times (K \setminus K_1)} |f(\delta_x) - \tilde{f}(x)| d|u_1 - v_1|(t, x) \leq 2\|f\| \varepsilon.$$ 

Since $\varepsilon$ was arbitrary $\mu(f) = v(f)$.

We are now able to prove the following theorem on complex polytopes.

2.2. **THEOREM.** Let $E$ be a finite dimensional Banach space whose unit ball is a complex polytope, and let $x_1, x_2, \ldots, x_n$ be the extreme points modulo $T$. If $x_0 \in B_E$ with $x_0 = \sum_{j=1}^{n} \lambda_j^0 x_j$, $\sum_{j=1}^{n} |\lambda_j^0| \leq 1$, then there exist functions $\lambda_1, \lambda_2, \ldots, \lambda_n \in C(B_E)$ so that $\sum_{j=1}^{n} |\lambda_j(x)| \leq 1$, $x = \sum_{j=1}^{n} \lambda_j(x) x_j$ for all $x \in B_E$, and $\lambda_j(x_0) = \lambda_j^0$, $1 \leq j \leq n$.

**PROOF.** Let $X$ be the subspace of $C(B_E)$ consisting of those $f$ for which $f(x_0)$
\(= \sum_{j=1}^{n} \lambda_j^0 f(x_j)\) and let \(S: l_1^n \rightarrow E\) be the operator from Proposition 1.1. Further let \(I: E^* \rightarrow C(B_E)\) be the canonical embedding; clearly \(I(E^*) \subseteq X\). Since \(E^*\) is isometric to a subspace of \(l_\infty^n\) (via \(S^*\)) it follows from theorem 1.3 and proposition 2.1 that there is an \(m\) and a subspace \(F\) of \(X\) isometric to \(l_\infty^m\) with \(I(E^*) \subseteq F\). Since \(F\) is a \(\mathcal{P}_1\)-space there is a norm one operator \(\tilde{I}: l_\infty^n \rightarrow F\) so that \(I = \tilde{I} S^*\). Put \(\lambda_j = \tilde{I} e_j^*\), \(1 \leq j \leq n\). If \(x \in B_E\), then

\[
\left(\sum_{j=1}^{n} |\lambda_j(x)| \right) = \left\| \sum_{j=1}^{n} e_j^*(\tilde{I}^* \delta_x)e_j \right\|_{l_1^n} = \|\tilde{I}^* \delta_x\|_{l_1^n} \leq 1 .
\]

For all \(y^* \in E^*\) we obtain:

\[
y^* \left( \sum_{j=1}^{n} \lambda_j(x)x_j \right) = (S^* y^*) \left( \tilde{I}^* \delta_x \right)
\]

\[
= (\tilde{I} S^* y^*)(x) = y^*(x)
\]

and hence

\[
x = \sum_{j=1}^{n} \lambda_j(x)x_j .
\]

Since \(tx_j\) is an extreme point for every \(t \in \mathbb{T}\), we have \(\lambda_j(tx_j) = t\) and

\[
\lambda_j(x_0) = \sum_{k=1}^{n} \lambda_k^0 \lambda_j(x_k) = \lambda_j^0 , \quad 1 \leq j \leq n .
\]

2.3. **Corollary.** Under the same conditions as in 2.2 there exist \(T\)-homogeneous functions \(f_1, \ldots, f_n \in C(B_E)\) so that \(x = \sum_{j=1}^{n} f_j(x)x_j\) and \(\sum_{j=1}^{n} |f_j(x)| \leq 1\) for all \(x \in B_E\).

**Proof.** Let \(\lambda_j, 1 \leq j \leq n\) be as in theorem 2.2. Define for each \(j, 1 \leq j \leq n\),

\[
f_j(x) = \int_{\mathbb{T}} t^{-1} \lambda_j(tx) dt , \quad x \in B_E ,
\]

where \(dt\) is the normalized Haar measure on \(\mathbb{T}\). It is easily checked that the \(f_j\)'s satisfy the requirements.

**Remark.** A slightly weaker form of corollary 2.3 was proved in the real case by Kalman [12] using geometric arguments. The real version of theorem 2.2 was proved by Lazar [15] by modification of Kalmans proof. Note that our proof of 2.2 and 2.3 with obvious changes gives an alternative proof in the real case, without using theorem 1.3, in fact it is easy to see that a \(C(K)\)-space has the finite binary intersection property, then argue as in [20, proof of theorem 5.5].
2.4. PROPOSITION. Let \((\Omega, \mathcal{B}, \mu)\) be a measure space and let \(F\) be a closed face of \(B_{L_1(\mu)}\) and put \(E=\text{span} (F)\). Then:

(i) \(E\) is an \(L\)-ideal

(ii) \(E \cap B_{L_1(\mu)} = \text{conv} (T \cdot F)\)

(iii) If \(L_1(\mu)\) is a dual space and \(F\) is \(w^*\)-closed, then \(E\) is \(w^*\)-closed.

PROOF. Since \(F\) is contained in a maximal proper face whose cone defines an order in \(L_1(\mu)\) which makes it an abstract \(L\)-space, it is no loss of generality to assume that \(F\) is contained in the positive cone of \(L_1(\mu)\).

If we let \(C\) denote the cone generated by \(F\), then it is readily verified that \(E = (C-C) + i (C-C)\), and since \(C\) is hereditary [2, lemma 2.7] it follows that if \(f \in E\) then \(f \geq 0\) if and only if \(f \in C\). Since a face cone in an \(L\)-space is a lattice cone we get by the above

\[
E = \{f \in L_1(\mu) \mid |f| \in C\}.
\]

Since \(C\) is closed and the lattice operations are continuous, \(E\) is closed by (1). This relation also gives that \(E\) is an \(L_1\)-space under the induced order. To prove (i) we first observe that by (1) \(E\) is a solid subspace of \(L_1(\mu)\) in the sense that \(f \in E\), \(|g| \leq |f|\) implies \(g \in E\). Since \(E\) is an \(L_1\)-space under the induced order, monotone, norm bounded nets in \(E\) converge [24], [27], and hence \(E\) is a band in \(L_1(\mu)\). It follows that \(E\) is an \(L\)-ideal.

(ii) will follow from

\[
B_{L_1(\mu)} \cap E = \{f \mid |f| \in \text{conv} (F, \{0\})\} = \text{conv} (T \cdot F).
\]

The first equality in (2) is obvious by (1). If \(f \in L_1(\mu)\) with \(|f| \in \text{conv} (F \cup \{0\})\) and \(\varepsilon > 0\), then we can find a simple function \(g\) with \(|g| \leq |f|\) and \(|g - f| < \varepsilon\). Hence \(g \in E\). If \(g = \sum_{j=1}^{m} a_j \chi_{A_j}\) with \(A_j \cap A_i = \emptyset, \ i \neq j\), then \(|\chi_{A_j}|^{-1} \chi_{A_j} \in F\), \(1 \leq j \leq n\). Furthermore

\[
1 \geq \|g\| = \sum_{j=1}^{m} |a_j| \|\chi_{A_j}\|
\]

and thus \(g \in \text{conv} (T \cdot F)\). The inclusion \(\text{conv} (T \cdot F) \subseteq B_{L_1(\mu)} \cap E\) is trivial.

Finally assume that \(L_1(\mu)\) is a dual space and \(F\) is \(w^*\)-closed. According to the Banach–Dieudonné theorem it is enough to prove that \(E \cap B_{L_1(\mu)}\) is \(w^*\)-
closed. It follows immediately that \( C \cap B_{L_1} \) is \( w^* \)-compact. If \((f_i) \subseteq E \cap B_{L_1} \) is a \( w^* \)-convergent net with limit \( f \), then by a compactness argument we may assume that the nets \((\Re f_i), \((\Im f_i) \) and \((\Im f_i) \) all converge to elements in \( C \cap B_{L_1} \). Thus \( f \in E \) and trivially \( \|f\| \leq 1 \).

The next lemma is actually one of the implications in our main theorem, but we have taken it out separately of technical reasons.

2.5. Lemma. Let \( X \) be a predual of \( L_1 \). If \( B_X \) has an infinite dimensional \( w^* \)-closed face, then \( X \) contains a subspace isometric to \( c \).

Proof. Let \( F \) be an infinite dimensional \( w^* \)-closed face of \( B_X \) and put \( N = \text{span}(F) \). By proposition 2.4, \( N \) is a \( w^* \)-closed \( L \)-ideal of \( X^* \) with \( F \) as a maximal proper face of \( B_N \). If \( Z = X/N^0 \) then \( Z^* = N \), and since \( F \) is split in \( \text{conv}(F \cup -iF) \) every \( f \in A(F) \) (here \( A(F) \) denotes the complex, affine, \( w^* \)-continuous functions on \( F \)) can be extended to an element in \( Z \) [25]; hence the map \( \psi : Z \rightarrow A(F) \) defined by \( \psi z(x^* = x^*(z) \) is an isometry onto. By Zippin’s result [28] there is an isometric embedding \( U : \Re c \rightarrow \Re A(F) \). If \( W : c \rightarrow A(F) \) is defined by \( W(x + iy) = Ux + iUy \) \( x, y \in \Re c \) then \( W \) is an isometric embedding. In fact let \( s \in F \) with

\[
\|W(x + iy)\| = |W(x + iy)(s)|
\]

and choose \( t \in T \), \( t = u + iv \), \( u, v \in R \) so that

\[
\|W(x + iy)\| = t((Us)(s) + i(Uy)(s)) = u(Us)(s) - v(Uy)(s)
\]

\[
= U(\Re t(x + iy))(s) \leq \|U(\Re t(x + iy))\| \leq \|x + iy\|.
\]

In a similar manner we get \( \|W(x + iy)\| \geq \|x + iy\| \). (The last argument was shown to us by Å. Lima.) Hence we have shown that \( Z \) contains a subspace \( Y \) isometric to \( c \).

If \( (x_n) \subseteq Y \) is a dense sequence, then we can define a metric on \( Y^* \) with the aid of this sequence, so that it generates the \( w^* \)-topology on \( Y^* \). Let us denote \( Y^* \)'s completion in this metric by \( \hat{Y}^* \), clearly \( Y^* \) can be considered topologically as a subset of \( \hat{Y}^* \).

Let \( \varphi : B_N \rightarrow B_Y \) be defined by:

\[
(\varphi x^*)(y) = x^*(y), \quad y \in Y, \ x^* \in N.
\]

From [26, theorem 4.2] there is an affine, \( T \)-symmetric \( w^* \)-continuous map \( \Phi : B_X \rightarrow B_Y \), so that \( \Phi | B_N = \varphi \). Define \( S : Y \rightarrow X \) by

\[
x^*(Sy) = (\Phi x^*)(y), \quad y \in Y, \ x^* \in B_X.
\]

Clearly \( S \) is an isometry and hence \( c \) embeds isometrically into \( X \).
We recall that a Banach space $X$ is called an $\mathcal{L}_{\infty,1}$-space, if for every finite dimensional subspace $E \subseteq X$ there is a finite dimensional subspace $F \subseteq X$ with $E \subseteq F$ and $d(F, \ell_\infty^{\dim F}) \leq \lambda$. It is well-known that a Banach space $X$ is a predual of $L_1$ if and only if it is an $\mathcal{L}_{\infty,1+\varepsilon}$-space for all $\varepsilon > 0$ [22]. The Banach space $c_0$ is an $\mathcal{L}_{\infty,1}$-space, while $c$ is not, as it is seen from:

2.6. Lemma. There is a two dimensional subspace $E$ of $c$, which does not embed isometrically into $l_\infty^n$ for any $n$.

Proof. Let

$$x^1 = (\cos k^{-1})_{k=1}^\infty, \quad x^2 = (\sin k^{-1})_{k=1}^\infty$$

and put $E = \text{span} (x_1, x_2)$. For the element $x_k = \cos k^{-1}x^1 + \sin k^{-1}x^2$, $k \in \mathbb{N}$, we get

$$x_k(n) = \cos (n^{-1} - k^{-1}), \quad n, k \in \mathbb{N};$$

and hence $x_k(k) = 1$, $|x_k(n)| < 1$ when $n + k$. This shows that $\delta_n \in E^*$ defined by $\delta_n(x) = x(n)$ for all $x \in E$, $n \in \mathbb{N}$ is an extreme point. Corollary 1.2 now completes the proof.

In [15] Lazar characterized those real Banach spaces $X$ which have the property that every compact operator with image in $X$ can be extended preserving the norm. A similar result is true in the complex case; the proof of it goes along the lines of [15, proof of theorem 3].

2.7. Theorem. Let $X$ be a predual of $L_1$. The following statements are equivalent:

(i) $X$ is a $\mathcal{L}_{\infty,1}$-space.

(ii) No subspace of $X$ is isometric to $c$.

(iii) $B_{X^*}$ has no infinite dimensional $w^*$-closed faces.

(iv) For all Banach spaces $Y$ and $Z$ with $Y \subseteq Z$ and every compact operator $S: Y \to X$, there is a compact extension $\tilde{S}: Z \to X$ with $\|\tilde{S}\| = \|S\|$.

(v) For all Banach spaces $Y$ and $Z$ with $Y \subseteq Z$ and every operator $S: Y \to X$ with $\dim S(Y) \leq 2$, there is a compact extension $\tilde{S}: Z \to X$ with $\|S\| = \|\tilde{S}\|$.

Proof. (i) $\Rightarrow$ (ii): follows from lemma 2.6.

(ii) $\Rightarrow$ (iii): is lemma 2.5.

(iii) $\Rightarrow$ (iv): Assume that (iii) holds and let $S: Y \to X$ be compact with $\|S\| = 1$. It follows that $S^*$ is continuous from $B_{X^*}$ to $B_{Y^*}$, when the first ball is equipped with the $w^*$-topology and the latter with the norm topology. We wish to construct an affine, $T$-symmetric map $\varphi: B_{X^*} \to B_{Y^*}$, continuous
when the sets are equipped with the \( w^* \)-topology, respectively the norm
topology, so that

\[
\varphi x^* \mid Y = S^* x^* \quad \text{for all } x^* \in B_{X^*}.
\]

Put \( K = S^* B_{X^*} \). Arguing as Lazar [15, p. 360] we get that there are finitely
many non \( T \)-equivalent extreme points \( u_1^*, u_2^*, \ldots, u_n^* \) of \( K \) such that

\[
\partial_e K \cap \{ y^* \in Y^* \mid \|y^*\| = 1 \} = T \cdot \{ u_1^*, \ldots, u_n^* \}.
\]

We also get that there is a \( \beta \) \( 0 < \beta < 1 \), so that if \( y^* \in \partial_e K \) with \( \|y^*\| > \beta \), then
\( \|y^*\| = 1 \). Put

\[
K_{\beta} = \{ y^* \in K \mid \|y^*\| \leq \beta \}.
\]

The Krein–Milman theorem gives that

\[
K = \text{conv} (K_{\beta} \cup T \cdot \{ u_1^*, \ldots, u_n^* \})
\]

and

\[
K \cap \{ y^* \in Y^* \mid \|y^*\| = 1 \} \subseteq \text{conv} (T \cdot \{ u_1^*, \ldots, u_n^* \}) = K_1.
\]

Let for \( j = 1, 2, \ldots, n \); \( z_j^* \in Z^* \) be Hahn–Banach extensions of \( u_j^* \). We define
a map \( \psi \) of \( K \) into the closed convex subsets of \( B_{Z^*} \) by

\[
\psi(y^*) = \{ z^* \in B_{Z^*} \mid z^* \mid Y = y^* \} \quad \text{for } y^* \in K \text{ and } \|y^*\| < 1
\]

\[
\psi(y^*) = \left\{ \sum_{j=1}^{n} \lambda_j z_j^* \mid y^* = \sum_{j=1}^{n} \lambda_j u_j^*, \sum_{j=1}^{n} |\lambda_j| = 1 \right\}
\]

for \( y^* \in K, \|y^*\| = 1 \).

Clearly \( \psi(y^*) \neq \emptyset \) for all \( y^* \in K \) and it is readily verified that \( \psi \) is convex and
\( T \)-symmetric. We shall prove \( \psi \) is lower semicontinuous, when \( K \) and \( B_{Z^*} \) are
equipped with the norm topologies. Hence let \( U \) be an open subset of \( Z^* \) and let

\[
y_0^* \in \{ y^* \in K \mid \psi(y^*) \cap U \neq \emptyset \} = K_2.
\]

If \( \|y_0^*\| < 1 \), then we argue like Lazar [15] to get that \( y_0^* \) is an interior point of
\( K_2 \). Next suppose that \( \|y_0^*\| = 1 \) and let \( z_0 \in \psi(y_0^*) \cap U \) with \( z_0 = \sum_{j=1}^{n} \lambda_j^0 z_j^* \),
where \( y_0^* = \sum_{j=1}^{n} \lambda_j^0 u_j^* \) and \( \sum_{j=1}^{n} |\lambda_j| = 1 \). By theorem 2.2 there are functions
\( \lambda_1, \lambda_2, \ldots, \lambda_n \in C(K_1) \) so that \( y^* = \sum_{j=1}^{n} \lambda_j(y^*) u_j^* \), \( \sum_{j=1}^{n} |\lambda_j(y^*)| \leq 1 \) for all
\( y^* \in K_1 \) and \( \lambda_j(y_0^*) = \lambda_j^0, 1 \leq j \leq n \). Let \( \varepsilon > 0 \) be given so that the ball with center
\( z_0^* \) and radius \( \varepsilon \) is contained in \( U \), and let \( W_1 \) be a neighborhood of \( y_0^* \) so that

\[
\left\| \sum_{j=1}^{n} \lambda_j(y^*) z_j^* - z_0^* \right\| \leq 3^{-1} \varepsilon \quad \text{for } y^* \in W_1 \cap K_1.
\]
It is easy to see that there is a neighborhood $W$ of $y^*_1$ so that if $y^* \in W \cap K$ and

$$y^* = \alpha y^*_1 + (1 - \alpha) y^*_2,$$

where $y^*_1 \in K_\beta$, $y^*_2 \in K_1$, and $\alpha \in [0, 1]$, then $\alpha < 3^{-1} \varepsilon$ and $y^*_2 \in W_1$. Let $y^* \in W$, then

$$y^* = \alpha y^*_1 + (1 - \alpha) y^*_2 \quad \text{with} \quad y^*_1 \in K_\beta, \ y^*_2 \in K_1$$

and let $z^* \in \psi(y^*_1)$. Put

$$v^* = \alpha z^* + (1 - \alpha) \sum_{j=1}^{n} \lambda_j(y^*_2) z_j^*.$$

By the convexity of $\psi$, $v^* \in \psi(y^*)$ and furthermore

$$\|z_0^* - v^*\| \leq \alpha \|z_0^* - z^*\| + (1 - \alpha) \left\| \sum_{j=1}^{n} \lambda_j(y^*_2) z_j^* - z_0 \right\| \leq \frac{2}{3} \varepsilon + \frac{1}{3} \varepsilon = \varepsilon$$

so $v^* \in \psi(y^*) \cap U$, which gives that $y^*_0$ is an interior point of $K_2$, and thus we have proved that $\psi$ is lower semicontinuous.

The map $\psi \circ S^*$ is $w^*$-lower semicontinuous and $T$-symmetric and therefore by [26] it has a $w^*$-continuous affine selection $\varphi$. Define $\tilde{S} : Z \to X$ by

$$x^*(\tilde{S}z) = \varphi(x^*)(z), \quad z \in Z, \ x^* \in B_{X^*}.$$

By the properties of $\varphi$, $\tilde{S}$ is compact and it is an extension of $S$ with $\|\tilde{S}\| = 1$.

(iv) $\Rightarrow$ (i). The proof of this implication is essentially the same as the proof of theorem 7.9 in [20], but let us give it for the sake of completeness. Assume (iv) and let $E \subseteq X$ be finite dimensional. By assumption there is a compact operator $S$ in $X$, so that $Sx = x$ for $x \in E$ and $\|S\| = 1$. Put for every $n \in \mathbb{N}$

$$S_n = n^{-1} \left( \sum_{k=0}^{n} S^k \right).$$

By the ergodic theorem on compact operators [6, p. 711] $S_n$ converges to a finite dimensional projection $P$ with $\|P\| = 1$ and with image

$$F = \{ x \in X \mid Sx = x \}.$$

Since $X^{**}$ is a $\mathcal{P}_1$-space and $P^{**}(X^{**}) = F$, it follows that $F$ is a $\mathcal{P}_1$-space and hence $F$ is isometric to $l^\infty_\text{dim} F$. Clearly $E \subseteq F$.

(iv) $\Rightarrow$ (v): is trivial.

(v) $\Rightarrow$ (ii): Assume (v). Then the proof of the implication (iv) $\Rightarrow$ (i) shows that every two dimensional subspace of $X$ embeds isometrically into $l^n_\infty$ for suitable $n$, and hence according to lemma 2.6 $X$ does not contain $c$ isometrically.
Remark. We do not know whether the condition (v) implies that \( X \) is a predual of \( L_1 \). Lima [17, theorem 4.10] proved that the answer is positive for real spaces, and in [18, theorem 4.1] he proved that the answer is positive in the complex case if we in (v) require \( \dim S(Y) \leq 3 \) instead of \( \dim S(Y) \leq 2 \).

As a corollary to theorem 2.7 we get as in the real case:

2.8. Theorem. If \( X \) is an \( \mathcal{L}_{\infty,1} \)-space, then \( X^* = l_1(\Gamma) \) for some \( \Gamma \).

Proof. Assume first that \( X \) is separable. If \( \partial_c B_{X^*} \) is uncountable modulo \( T \), then \( \partial_c B_{X^*} \) has an infinite compact subset \( E \) (one may even choose \( E \) to be the Cantor set) with \( E \cap tE = \emptyset \) for all \( t \in T \setminus \{1\} \). By [25, lemma 22] \( F = \operatorname{conv}(E) \) is a \( w^* \)-closed face of \( B_{X^*} \) which contradicts theorem 2.7. Hence \( \partial_c B_{X^*} \) is countable modulo \( T \) and thus \( X^* = l_1 \).

The general case follows from this together with [14, theorem 6, p. 227] (the implication we need is also proved for the complex case, although this is not stated explicitly; it is also likely, using the result of [26], that this theorem carries over to the complex case).

The final result of this section due to W. B. Johnson and the first named author shows that every predual of \( l_1 \) is isomorphic to an \( \mathcal{L}_{\infty,1} \)-space.

2.9. Theorem. Let \( X \) be a real or complex Banach space with \( X^* = l_1 \). Then there exists an \( \mathcal{L}_{\infty,1} \)-space \( Y \) which is isomorphic to \( X \).

Proof. Let \( (x_n^*) \subseteq X^* \) be a basis isometrically equivalent to the unit vector basis of \( l_1 \). Put for every natural number \( n \) \( E_n = \operatorname{span}\{x_1^*, \ldots, x_n^*\} \) and define a new norm on \( X^* \) by:

\[
|||x^*||| = \|x^*\| + \sum_{n=1}^{\infty} 2^{-n} d(x^*, E_n), \quad x^* \in X^*.
\]

(This renorming technique was used in [5].) Since the \( E_n \)'s are finite dimensional the unit ball determined by \( ||| \cdot ||| \) is \( w^* \)-closed and hence \( ||| \cdot ||| \) is the dual norm of a norm \( ||| \cdot ||| \) on \( X \) which is readily seen to be equivalent to \( \| \cdot \| \).

Put \( Y = (X, ||| \cdot |||) \). We shall show that \( Y \) is a \( \mathcal{L}_{\infty,1} \)-space. Put for every natural number \( n \) \( y_n^* = |||x^*|||^{-1} x_n^* \) and let \( k \) be a natural number and \( t_1, t_2, \ldots, t_k \) scalars. Then

\[
\left\| \sum_{n=1}^{k} t_n y_n^* \right\| = \left\| \sum_{n=1}^{k} t_n |||x^*|||^{-1} x_n^* \right\|
\]
\[
\begin{align*}
= \sum_{n=1}^{k} 2^{-1} |t_n| (1 - 2^{-n})^{-1} \\
+ \sum_{n=1}^{k-1} 2^{-n} \left( \sum_{j=n+1}^{k} 2^{-1} |t_j| (1 - 2^{-j})^{-1} \right) \\
= 2^{-1} \sum_{j=1}^{k} \sum_{n=0}^{j-1} 2^{-n} |t_j| (1 - 2^{-j})^{-1} \\
= \sum_{j=1}^{k} |t_j|
\end{align*}
\]
which show \( Y^* = l_1 \).

If we show that every \( w^* \)-limit point of the sequence \( (y_n^*) \) has norm strictly less than 1, then it will follow from theorem 2.7 for the complex case and Lazar [15, theorem 3] for the real case that \( Y \) is an \( \mathcal{L}_{\infty, 1} \)-space. Hence let \( x^* \in X^* \) and let \( (y_n^*) \) be a sequence with \( y_n^* \xrightarrow{w^*} x^* \). Since \( \|x_n^*\| \to 2 \) we get that \( \|x^*\| \leq 2^{-1} \) and therefore for \( n \) sufficiently large \( d(x^*, E_n) < 2^{-1} \). This gives
\[
\|x^*\| < 2^{-1} + 2^{-1} \sum_{n=1}^{\infty} 2^{-n} \leq 1.
\]

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