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THE WEAK LEFSCHETZ PROPERTY FOR QUOTIENTS BY QUADRATIC MONOMIALS

JUAN MIGLIORE, UWE NAGEL and HAL SCHENCK

(*Dedicated to the memory of our friend Tony Geramita*)

Abstract

Michałek and Miró-Roig, in J. Combin. Theory Ser. A 143 (2016), 66–87, give a beautiful geometric characterization of Artinian quotients by ideals generated by quadratic or cubic monomials, such that the multiplication map by a general linear form fails to be injective in the first nontrivial degree. Their work was motivated by conjectures of Ilardi and Mezzetti, Miró-Roig and Ottaviani, connecting the failure to Laplace equations and classical results of Togliatti on osculating planes. We study quotients by quadratic monomial ideals, explaining failure of the Weak Lefschetz Property for some cases not covered by Michałek and Miró-Roig.

1. Introduction

The Hard Lefschetz Theorem [10] is a landmark result in algebraic topology and geometry: for a smooth n-dimensional projective variety X, cup product with the k'th power of the hyperplane class L gives an isomorphism between $H^{n-k}(X)$ and $H^{n+k}(X)$. A consequence of this is that multiplication by a generic linear form is injective up to degree n, and surjective in degree $\geq n$. This result places strong constraints on the Hilbert function, and was used to spectacular effect by Stanley [24] in proving the necessity conditions in McMullen's conjecture [12] on the possible numbers of faces of a simplicial convex polytope. In short, Lefschetz properties are of importance in algebra, combinatorics, geometry and topology; their study is a staple in the investigation of graded Artinian algebras.

Definition 1.1. Let $I = \langle f_0, \dots, f_k\rangle \subseteq S = \mathbb{K}[x_0, \dots, x_r]$ be an ideal with $A = S/I$ Artinian. Then A has the *Weak Lefschetz Property* (WLP) if there is an $\ell \in S_1$ such that for all i, the multiplication map $\mu_\ell\colon A_i \to A_{i+1}$ has maximal rank. If not, for some i, we say that A fails WLP in degree i.

The set of elements $\ell \in S_1$ with the property that the multiplication map μ_ℓ has maximum rank is a (possibly empty) Zariski open set in S_1. Therefore,

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the existence of the Lefschetz element ℓ in the definition of the Weak Lefschetz Property guarantees that this set is nonempty and is equivalent to the statement that for a general linear form in S_1 the corresponding multiplication map has full rank. Throughout this paper, A will denote a standard graded, commutative Artinian algebra, which is the quotient of a polynomial ring S as above by a homogeneous ideal I. The Weak Lefschetz Property depends strongly on $\mathrm{char}(\mathbb{K})$. For quadratic monomial ideals there is a natural connection to topology, and homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients plays a central role in understanding WLP.

Results of [8] show that WLP always holds for $r \leq 1$ in characteristic zero. For $r = 2$ WLP sometimes fails, but is known to hold for many classes:

- ideals of general forms [1];
- complete intersections [8];
- ideals with semistable syzygy bundle [4];
- almost complete intersections with unstable syzygy bundle [4];
- level monomial ideals of type 2 [2], [5];
- ideals generated by powers of linear forms [22], [16];
- monomial ideals of small type [5], [2].

Nevertheless, the $r = 2$ case is still not completely understood, and there remain intriguing open questions: for example, does every Gorenstein A have WLP? For $r \geq 3$ far less is known; it is open if every A which is a complete intersection has WLP. The survey paper [18] contains many questions and conjectures. Our basic reference for Lefschetz properties is the book of Harima, Maeno, Morita, Numata, Wachi and Watanabe [7]. In this paper, we study WLP for the class of quadratic monomial ideals using tools from homology and topology.

1.1. Laplace equations and Togliatti systems

One particularly interesting case occurs when the generators of I are all of the same degree. In this case, since $V(I)$ is empty, I defines a basepoint free map

$$\mathbb{P}^r \xrightarrow{\phi_I} \mathbb{P}^k,$$

and it is natural to ask how WLP connects to the geometry of ϕ_I. (See [3] for an unexpected connection between this approach and Hesse configurations, in the first non-trivial case for the Gorenstein problem mentioned above.) First, we need some preliminaries.

Definition 1.2. For an r-dimensional variety $X \subseteq \mathbb{P}^m$ and $p \in X$ such that $\mathcal{O}_{X,p}$ has local defining equations f_i, the d'th osculating space $T^d(X, p)$ is the linear subspace spanned by p and all $(\partial(f_i)/\partial x^\alpha)(p)$, with $|\alpha| \leq d$.

At a general point $p \in X$, the expected dimension of $T^d(X, p)$ is

$$\min\left\{m, \binom{r+d}{d} - 1\right\}.$$

If for some positive δ the dimension at a general point is $\binom{r+d}{d} - 1 - \delta < m$, then X is said to satisfy δ Laplace equations of order d. In [26], [27], Togliatti studied such systems, and showed that the map $\mathbb{P}^2 \to \mathbb{P}^5$ defined by

$$\{xy^2, yx^2, zy^2, yz^2, xz^2, zx^2\}$$

is the only smooth projection from the triple Veronese surface to $\mathbb{P}^5$ which satisfies a Laplace equation of order two. In [11], Ilardi studied rational surfaces of sectional genus one in $\mathbb{P}^5$ satisfying a Laplace equation, and conjectured that Togliatti's example generalizes to be the only map to $\mathbb{P}^{r(r+1)-1}$ via cubic monomials with smooth image.

Mezzetti, Miró-Roig and Ottaviani noticed that the "missing monomials" which correspond to the coordinate points from which the triple Veronese surface is projected are

$$\{x^3, y^3, z^3, xyz\}$$

whose quotient in $\mathbb{K}[x, y, z]$ was shown in [4] to fail WLP. The observation led to the paper [13], where they prove:

THEOREM 1.3 ([13]). *For a monomial ideal $I = \langle f_0, \dots, f_k \rangle$ generated in degree d with Artinian quotient A, let X denote the variety of the image of the map to $\mathbb{P}^k$ defined by I, and $X^\perp$ the variety of image of the complementary map defined by the monomials of $S_d \setminus I_d$. Then for $k + 1 \leq \binom{r+d-1}{r-1}$, the following are equivalent:*

- $\mu_\ell \colon A_{d-1} \to A_d$ *fails WLP;*
- $\{f_0, \dots, f_k\}$ *become linearly dependent in S/ℓ for a generic linear form ℓ;*
- *the variety $X^\perp$ satisfies at least one Laplace equation of order d.*

REMARK 1.4. The numerical hypothesis in Theorem 1.3 is equivalent to the condition that $\dim A_{d-1} \leq \dim A_d$; the condition that μ_ℓ fails WLP is only an assertion that injectivity fails. The classification theorem for injectivity in Theorem 1.5 below does not address surjectivity, and is one motivation for this paper.

The varieties X and $X^\perp$ are often called polar varieties, and can be defined more generally using inverse systems, but we will not need that here. Mezzetti, Miró-Roig and Ottaviani used Theorem 1.3 to produce counterexamples to

Ilardi's conjecture. Building on this work, Michałek and Miró-Roig showed in [14] that a suitable modification of Ilardi's conjecture is true. A monomial system is *Togliatti* if it satisfies the equivalent conditions of Theorem 1.3, *smooth* if $X^{\perp}$ is smooth, and *minimal* if no proper subset satisfies the equivalent conditions of Theorem 1.3. For quadrics, Michałek and Miró-Roig classify the failure of injectivity in degree one:

THEOREM 1.5 ([14]). *A smooth, monomial, minimal Togliatti system of quadrics $I \subseteq S$ is (after reindexing) $I = \langle x_0, \dots, x_i\rangle^2 + \langle x_{i+1}, \dots, x_r\rangle^2$, with $1 \le i \le r-2$.*

Maps defined by monomials of degree d are toric, and a key ingredient in [14] and [13] is Perkinson's classification in [20] of osculating spaces in terms of the lattice points corresponding to monomials.

1.2. Results of this paper

We explain failure of WLP for quadratic monomial quotients in some cases not covered by Theorems 1.3 and 1.5. For $\mathbb{K}[x_0, \dots, x_5]$, all failures of WLP are covered by Theorems 1.3 and 1.5, except Examples 1.6 and 1.7 below and Example 2.7. Failure of WLP in these cases is explained by our results.

EXAMPLE 1.6. Theorem 1.5 gives necessary and sufficient conditions for failure of WLP for I quadratic if $k+1 \le \binom{r+1}{2}$, since in this range μ_ℓ cannot be surjective in degree one. For $k+1 > \binom{r+1}{2}$, WLP can still hold if μ_ℓ is surjective. When $r = 5$, μ_ℓ is surjective and WLP holds for every Artinian monomial ideal with $k+1 \in \{16, \dots, 21\}$, except the ideal of all quadratic monomials save (up to relabelling) $\{x_0x_2, x_0x_3, x_1x_2, x_1x_3\}$, where surjectivity fails from degree 1 to degree 2.

EXAMPLE 1.7. Let $I = \langle x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_0x_1, x_2x_3, x_4x_5\rangle \subseteq \mathbb{K}[x_0, \dots, x_5]$; S/I has Hilbert series (1, 6, 12, 8), and fails WLP (surjectivity) in degree two.

By our Proposition 2.5, WLP fails for Example 1.6, and by our Proposition 3.4, WLP fails for Example 1.7. Failure in both examples also follows from Theorem 4.8, which generalizes Theorem 1.5. The following results will be useful:

PROPOSITION 1.8 ([15]). *If $\mu_\ell \colon A_i \twoheadrightarrow A_{i+1}$, then it is surjective for all $j \ge i$. If A is level, and if $\mu_\ell \colon A_i \hookrightarrow A_{i+1}$, then it is injective for all $j \le i$.*

PROPOSITION 1.9 ([15]). *For a monomial ideal I, the form $\ell = \sum_{i=0}^{r} x_i$ is generic.*

2. Squarefree monomial ideals

Let I be a monomial ideal of the form $I = J' + J_\Delta$, where J' is the ideal consisting of squares of variables, and J_Δ is a squarefree monomial ideal; J_Δ defines a Stanley-Reisner ideal [21, §5.2] corresponding to a simplicial complex Δ.

LEMMA 2.1. *If $A = S/I$ for I as above, then the Hilbert series of A is*

$$HS(A,t) = \sum f_{i-1}(\Delta)t^i.$$

PROOF. For the exterior Stanley-Reisner ring E/I_Δ, $HS(E/I_\Delta, t) = \sum f_{i-1}(\Delta)t^i$, and it is clear that $HS(A,t) = HS(E/I_\Delta, t)$, with $HF(A,i) = f_{i-1}$. Here $HF(A,i) = \dim A_i$ is the Hilbert function of A in degree i.

COROLLARY 2.2. *With the notation of Lemma 2.1, the algebra A is level if and only if the flag complex Δ is pure.*

PROOF. Denote by $m_1, \dots, m_t$ the monomials corresponding to the facets of Δ, that is, each m_i is the product of the variables corresponding to the vertices of the i-th facet. Let I' be the annihilator of $m_1, \dots, m_t$ in the sense of inverse systems, that is, I' consists of the monomials that annihilate each of the monomials $m_1, \dots, m_t$ under contraction. The minimal generators of J_Δ correspond to minimal non-faces of Δ. Hence, they are in I'. Clearly, J' is in I', which gives $I \subset I'$.

The $(i-1)$-dimensional faces of Δ correspond to divisors of one of the monomials $m_1, \dots, m_t$. Thus, $f_{i-1}(\Delta)$ is equal to the Hilbert function of R/I' in degree i. Hence Lemma 2.1 gives that A and R/I' have the same Hilbert function, which forces $I = I'$. Thus, we have shown that the inverse system of A is generated by the monomials corresponding to the faces of Δ. Now the claim follows.

When A is an Artinian quotient by a quadratic monomial ideal I, then I always has a decomposition as above, and Δ is flag: it is defined by the non-edges, so is encoded by a graph. Removing a vertex or an edge from the graph gives rise to a short exact sequence, yielding an inductive tool to study WLP for quadratic monomial ideals.

2.1. Removing an edge or vertex

Let e_{ij} be the edge corresponding to monomial $x_i x_j$, and v_i the vertex corresponding to the variable x_i. Write S' for a quotient of S by some set of variables, which will be apparent from the context, and J' for the ideal of the squares of the variables in S'. For a face $\sigma \in \Delta$, $\mathrm{st}(\sigma) = \{\tau \in \Delta \mid \sigma \subseteq \tau\}$ and $\mathrm{lk}(\sigma) = \partial(\mathrm{st}(\sigma))$.

LEMMA 2.3. *For the short exact sequence*

$$0 \longrightarrow S(-1)/(I : x_i) \xrightarrow{\cdot x_i} S/I \longrightarrow S/(I + x_i) \longrightarrow 0,$$

we have

$$S/(I + x_i) \simeq S'/(J' + J_{\Delta'}),$$
$$S/(I : x_i) \simeq S'/(J' + J_{\Delta''}),$$

where $\Delta' = \Delta \setminus \mathrm{st}(v_i)$*, and* $\Delta'' = \mathrm{lk}(v_i)$.

PROOF. In $(I + x_i)$, all quadrics in I divisible by x_i are non-minimal, so the remaining quadrics are those not involving x_i, which are exactly the non-faces of $\Delta \setminus \mathrm{st}(v_i)$.

For Δ'', since $(J_\Delta + J') : x_i = I + \langle x_i \rangle + \langle x_k \mid x_i x_k \in I \rangle$ it follows that Δ'' is obtained by deleting all vertices not connected to v_i, as well as v_i itself, so what remains is $\mathrm{lk}(v_i)$.

LEMMA 2.4. *For the short exact sequence*

$$0 \longrightarrow S(-2)/(I : x_i x_j) \xrightarrow{\cdot x_i x_j} S/I \longrightarrow S/(I + x_i x_j) \longrightarrow 0,$$

we have

$$S/(I + x_i x_j) \simeq S'/(J' + J_{\Delta'}),$$
$$S/(I : x_i x_j) \simeq S'/(J' + J_{\Delta''}),$$

where $\Delta' = \Delta \setminus \mathrm{st}(e_{ij})$*, and* $\Delta'' = \mathrm{lk}(e_{ij})$.

PROOF. Since adding $x_i x_j$ to I corresponds to deleting e_{ij} from Δ, the description of Δ' is automatic. For Δ'', since

$$(J_\Delta + J') : x_i x_j = I + \langle x_i, x_j \rangle + \langle x_k \mid x_i x_k \text{ or } x_j x_k \in I \rangle,$$

it follows that Δ'' is obtained by deleting all vertices not connected to e_{ij}, as well as e_{ij} itself, so what remains is $\mathrm{lk}(e_{ij})$.

PROPOSITION 2.5. *Let*

- Δ' *denote* $\Delta \setminus \mathrm{st}(e_{ij})$ *or* $\Delta \setminus \mathrm{st}(v_i)$*,*
- Δ'' *denote* $\mathrm{lk}(e_{ij})$ *or* $\mathrm{lk}(v_i)$.

Let A''*,* A *and* A' *be the respective Artinian algebras* (*with multiplication map* μ *indexed by degree*)*, and* δ *the connecting homomorphism in equation* (2.1) *below. Then*

- μ_i *is injective if both* $\{\mu''_{i-2}$ *(edge) or* μ''_{i-1} *(vertex)*$\}$ *and* δ *are injective,*
- μ_i *is surjective if and only if both* μ'_i *and* δ *are surjective.*

Proof. Apply the snake lemma to the graded pieces of the short exact sequences of Lemma 2.3 or Lemma 2.4. For example, in the case of Lemma 2.4, this yields

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A''_{i-2} & \xrightarrow{\cdot x_i x_j} & A_i & \longrightarrow & A'_i & \longrightarrow & 0 \\ & & \downarrow \mu''_{i-2} & & \downarrow \mu_i & & \downarrow \mu'_i & & \\ 0 & \longrightarrow & A''_{i-1} & \xrightarrow{\cdot x_i x_j} & A_{i+1} & \longrightarrow & A'_{i+1} & \longrightarrow & 0 \end{array}$$

Letting K'', K and K' denote the kernels of μ_ℓ on the algebras A'', A and A', and similarly for the cokernels yields a long exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K''_{i-2} & \xrightarrow{\cdot x_i x_j} & K_i & \longrightarrow & K'_i & & \\ & \xrightarrow{\delta} & C''_{i-1} & \longrightarrow & C_{i+1} & \longrightarrow & C'_{i+1} & \longrightarrow & 0, \end{array} \tag{2.1}$$

and the result follows.

Corollary 2.6. *Equation 2.1 yields simple numerical reasons for failure of WLP for an edge (the vertex case is similar).*

- *If A'' fails injectivity in degree $i-2$, and $f_i(\Delta) \geq f_{i-1}(\Delta)$, then A fails WLP in degree i due to injectivity.*
- *If A' fails surjectivity in degree i, and $f_i(\Delta) \leq f_{i-1}(\Delta)$, then A fails WLP in degree i due to surjectivity.*

Example 2.7. The ideal $I = \langle x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_0x_1, x_2x_3\rangle \subseteq \mathbb{K}[x_0, \ldots, x_5]$ has quotient $A = S/I$ with Hilbert series (1, 6, 13, 12, 4). Applying Proposition 2.5 with the role of Δ' played by Example 1.7 shows that I fails WLP in degree two. A computer search shows this ideal and Examples 1.6, 1.7 are the only quadratic monomial ideals in $\mathbb{K}[x_0, \ldots, x_5]$ which fail WLP that are not covered by Theorem 1.3 or 1.5.

3. Topology and $\mathbb{Z}/2\mathbb{Z}$ coefficients

Lemma 3.1. *If* $\mathrm{char}(\mathbb{K}) = 2$, *$(A, \mu_\ell)$ is a chain complex.*

Proof. By Proposition 1.9, we may assume $\ell = \sum x_i$, so since $\ell^2 = \sum x_i^2 + 2\sum_{i<j} x_i x_j$ the result follows. More is true: the multiplication map sends a monomial $x^\alpha \mapsto \sum_i x^\alpha x_i$. Dualizing yields the transpose of the simplicial boundary operator with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

When we focus on the WLP in a specific degree i, it makes sense to consider not the simplicial complex corresponding to J, but by an associated simplicial complex:

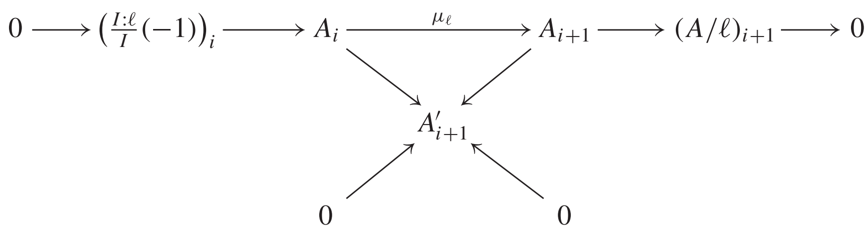
Definition 3.2. For the simplicial complex Δ associated to the squarefree quadratic monomial ideal J, and choice of degree i, let $\Delta(i)$ be the i-skeleton of Δ: delete all faces of dimension $\geq i+1$ from Δ. Algebraically, this corresponds to replacing I with $I(i) := I + \langle S_{i+2} \rangle$.

The next proposition shows the reason to define $\Delta(i)$: over $\mathbb{Z}/2\mathbb{Z}$, it gives a precise reason for the failure of WLP in degree i.

Proposition 3.3. *Let A be an Artinian quotient by a quadratic monomial ideal I, and ∂_i the simplicial boundary. Then over $\mathbb{Z}/2\mathbb{Z}$, A fails WLP in degree i if and only if*

- *surjectivity fails: $H_i(\Delta(i), \mathbb{Z}/2\mathbb{Z}) \neq 0$ and $f_i \leq f_{i-1}$, or*
- *injectivity fails:* $\operatorname{coker}(\partial_i) \neq 0$ *and $f_{i-1} \leq f_i$.*

Proof. Let $A' = S(-1)/(I : \ell)$, $k = \mathbb{Z}/2\mathbb{Z}$, and consider the factorization of the four term exact sequence



WLP fails in degree i if and only if $f_{i-1} \leq f_i$ and $\ker(\mu_\ell) \neq 0$ or $f_i \leq f_{i-1}$ and $\operatorname{coker}(\mu_\ell) \neq 0$. By Lemma 3.1, over $\mathbb{Z}/2\mathbb{Z}$, $\operatorname{coker}(\mu_\ell) \neq 0$ if and only if $\ker(\partial_i) \neq 0$.

Because we are considering WLP in degree i, replacing I with $I(i)$ has no impact, and the rank of μ_ℓ is the same as the rank of the dual map of $\mathbb{Z}/2\mathbb{Z}$ vector spaces

$$A_{i+1}^{\vee} \xrightarrow{\partial_i} A_i^{\vee}.$$

In particular, μ_ℓ is not surjective if and only if ∂_i has a kernel if and only if $H_i(\Delta(i), \mathbb{Z}/2\mathbb{Z}) \neq 0$. Similarly, $\ker(\mu_\ell) \neq 0$ if and only if $\operatorname{coker}(\partial_i) \neq 0$. This completes the proof.

3.1. Lifting to characteristic zero

Proposition 3.3 shows that over $\mathbb{Z}/2\mathbb{Z}$, the failure of WLP for quadratic monomial ideals is connected to topology. In Example 1.6, Δ is homotopic to S^1 and 2 points, and in Example 1.7, Δ is homotopic to S^2. The Bockstein spectral sequence shows that $\mathbb{Z}/2\mathbb{Z}$ homology abuts to integral homology; in certain circumstances this impacts the rank of μ_ℓ.

PROPOSITION 3.4. *For A an Artinian quotient by a quadratic monomial ideal I over a field of characteristic zero and degree i, consider $\Delta(i)$ of Definition 3.2.*

- *If the connecting map δ in* (3.1) *is not injective* (*hence $H_i(\Delta(i), \mathbb{Z}/2\mathbb{Z}) \neq 0$*)*, then μ_ℓ fails surjectivity; if also $f_i \leq f_{i-1}$, WLP fails.*
- *If the connecting map δ in* (3.1) *is injective* (*e.g. if $H_i(\Delta(i), \mathbb{Z}/2\mathbb{Z}) = 0$*)*, then μ_ℓ is surjective and WLP holds.*

PROOF. We work with the integral Stanley-Reisner ring $\mathbb{Z}[x_0, \ldots, x_r]$ and let B be the quotient by the ideal $I(i)$. The graded pieces of B are free $\mathbb{Z}$-modules, and so we may tensor the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

to obtain a commuting diagram with exact rows of $\mathbb{Z}$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_{i+1}^\vee & \xrightarrow{\cdot 2} & B_{i+1}^\vee & \longrightarrow & B_{i+1}^\vee \otimes \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_\ell^\vee} & & \downarrow{\scriptstyle \mu_\ell^\vee} & & \downarrow{\scriptstyle \partial_i} & & \\
0 & \longrightarrow & B_i^\vee & \xrightarrow{\cdot 2} & B_i^\vee & \longrightarrow & B_i^\vee \otimes \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0.
\end{array}$$

The snake lemma yields a long exact sequence

$$\begin{array}{l}
0 \longrightarrow \ker(\mu_\ell^\vee) \xrightarrow{\cdot 2} \ker(\mu_\ell^\vee) \longrightarrow H_i(\Delta(i), \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\delta} \\
\quad \longrightarrow \operatorname{coker}(\mu_\ell^\vee) \xrightarrow{\cdot 2} \operatorname{coker}(\mu_\ell^\vee) \longrightarrow \operatorname{coker}(\partial_i) \longrightarrow 0.
\end{array} \tag{3.1}$$

The connecting homomorphism δ is not injective if and only if it has a kernel. Because $\ker(\mu_\ell^\vee)$ is the kernel of a map between free $\mathbb{Z}$-modules, it is torsion free and therefore free. As the leftmost map in the long exact sequence is induced by $\cdot 2$, it follows that $\ker(\mu_\ell^\vee)$ is a nonzero free $\mathbb{Z}$ module if and only if δ is not injective. Localizing at 0 gives an exact sequence of $\mathbb{Q}$ vector spaces, and so in characteristic zero, A fails WLP due to surjectivity in degree i if and only if δ is not an injection on $H_i(\Delta(i), \mathbb{Z}/2\mathbb{Z})$ and $f_i \le f_{i-1}$. The proof of the second assertion is similar.

EXAMPLE 3.5. When Δ is a n-cycle, WLP fails if n is even, and holds if n is odd. To see this, consider the map $\mu_\ell^\vee$, and $i = 1$. Then $\mu_\ell^\vee$ has the form

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \cdots & \cdots & 1 & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

For both cases, $H_1(\Delta(1), \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$. However, when n is even δ is zero, whereas when n is odd δ is the identity. The determinant of $\mu_\ell^\vee$ is 0 for n even, and 2 for n odd, confirming our calculation.

4. Generalizing the condition of Michałek and Miró-Roig

There is an interesting connection between Example 1.7, where WLP fails in degree two due to topology, and the criterion for failure of injectivity in degree one appearing in Theorem 1.5. The key observation is that letting $V_1 = \{x_0, x_1\}$, $V_2 = \{x_2, x_3\}$, $V_3 = \{x_4, x_5\}$ be bases for the trio of vector spaces V_i, then the algebra A in Example 1.7 can be written as

$$\bigotimes_{i=1}^{3} \mathrm{Sym}(V_i)/V_i^2 \qquad \text{where } V_i \text{ is the irrelevant ideal of } \mathrm{Sym}(V_i).$$

Theorem 1.5 deals with quadratic algebras which have a tensor decomposition with only two factors of type $\mathrm{Sym}(V_i)/V_i^2$. We show that quadratic algebras with more than two factors of type $\mathrm{Sym}(V_i)/V_i^2$ also always fail to have WLP. In fact, we prove an even more general result on tensor product algebras; the next lemma plays a key role:

LEMMA 4.1 ([2, Lemma 7.8]). *Let $A = A' \otimes A''$ be a tensor product of two graded Artinian k-algebras A' and A''. Let $L' \in A'$ and $L'' \in A''$ be linear elements, and set $L := L' + L'' = L' \otimes 1 + 1 \otimes L'' \in A$. Then*

(a) *If the multiplication maps $\times L' \colon A'_{i-1} \to A'_i$ and $\times L'' \colon A''_{j-1} \to A''_j$ are both not surjective, then the multiplication map*

$$\times L \colon A_{i+j-1} \to A_{i+j}$$

is not surjective.

(b) *If the multiplication maps* $\times L' \colon A'_i \to A'_{i+1}$ *and* $\times L'' \colon A''_j \to A''_{j+1}$ *are both not injective, then the multiplication map*

$$\times L \colon A_{i+j} \to A_{i+j+1}$$

is not injective.

THEOREM 4.2. *Let*

$$A = \bigotimes_{i=1}^{n} \mathrm{Sym}(V_i)/V_i^{k_i}, \qquad \text{with } \dim(V_i) \geq 2 \text{ and } n, k_i \geq 2.$$

Then the algebra A does not have the WLP.

PROOF. Let $k = \sum_{i=1}^{n}(k_i - 1)$. First, assume $\dim A_{k-1} \geq \dim A_k$. Since $\dim V_i \geq 2$, the multiplication map

$$(\mathrm{Sym}(V_i)/V_i^{k_i})_{k_i-2} \xrightarrow{\cdot \ell_i} (\mathrm{Sym}(V_i)/V_i^{k_i})_{k_i-1},$$

is not surjective, where ℓ_i is the sum of the variables in $\mathrm{Sym}(V_i)$. Hence Lemma 4.1 gives that $\times \ell \colon \dim A_{k-1} \to \dim A_k$ is not surjective, where $\ell = \ell_1 + \cdots + \ell_n$. Thus, A does not have the WLP.

Second, assume $\dim A_{k-1} < \dim A_k$. Consider the residue class $0 \neq \overline{m} \in A_{k-1}$ of $m_1 - m_2 := \ell_1^{k_1-2}\ell_2^{k_2-1} \cdots \ell_n^{k_n-1} - \ell_1^{k_1-1}\ell_2^{k_2-2}\ell_3^{k_3-1} \cdots \ell_n^{k_n-1}$. Then

$$\overline{m} \cdot \ell = \overline{m_1} \cdot \ell_1 - \overline{m_2} \cdot \ell_2 = 0.$$

Thus, $\mu_\ell \colon \dim A_{k-1} \to \dim A_k$ is not injective, so A does not have the WLP.

When we allow one-dimensional V_i, interesting things can happen:

THEOREM 4.3. *For A Artinian with $A_d \neq 0$ and $A_{d+1} = 0$, $C = A \otimes \mathbb{K}[z]/z^j$ has WLP when $j \geq d + 1$.*

PROOF. Choose bases $A_0 z^i \oplus A_1 z^{i-1} \oplus \cdots \oplus A_i$ for C_i and $A_0 z^{i+1} \oplus A_1 z^i \oplus \cdots \oplus A_{i+1}$ for C_{i+1}. Then when $i + 1 \leq d$, the multiplication map is given by a block matrix

$$\mu_i^C = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ d_0 & I & 0 & \cdots & \vdots \\ 0 & d_1 & I & \cdots & \vdots \\ \vdots & 0 & d_2 & \cdots & 0 \\ \vdots & \ddots & 0 & \cdots & I \\ 0 & 0 & \cdots & \cdots & d_i \end{bmatrix},$$

where we write d_i for the multiplication map μ_ℓ from $A_i \to A_{i+1}$; μ_i^C is clearly injective. When $k = i + 1 - d > 0$, we truncate the last k row blocks and rightmost $k - 1$ column blocks of the matrix, and so the matrix still has full rank. This completes the proof.

When $j \leq d$, things are more complicated: if $k' = i + 1 - d > 0$, then we truncate the matrix above by the top k' row blocks, and leftmost $k' - 1$ column blocks, resulting in a matrix of the form

$$\begin{bmatrix} d_{k'} & I & 0 & \cdots & \vdots \\ 0 & d_{k'+1} & \ddots & \cdots & \vdots \\ \vdots & 0 & \ddots & \cdots & 0 \\ \vdots & \ddots & 0 & \cdots & I \\ 0 & 0 & \cdots & \cdots & d_i \end{bmatrix},$$

and WLP depends on properties of A. We now return to the quadratic case.

4.1. Tensor algebras of quadratic quotients

COROLLARY 4.4. *For an Artinian A which is a quotient by quadratic monomials with $A_{i+1} \neq 0$, $C = A \otimes \mathbb{K}[z]/z^2$ has WLP in degree i if and only if $\cdot \ell^2$ has full rank on A_{i-1}, where ℓ is the sum of the variables of A.*

PROOF. Choose bases $A_{i-1}z \oplus A_i$ for C_i and $A_i z \oplus A_{i+1}$ for C_{i+1}, and let I_j be the identity on A_j. If $A_{i+1} \neq 0$, then $C_i \stackrel{\mu_\ell}{\to} C_{i+1}$ is given by the block matrix

$$\begin{bmatrix} d_{i-1} & I_i \\ 0 & d_i \end{bmatrix},$$

and changing basis for the row and column space reduces the matrix to

$$\begin{bmatrix} d_i d_{i-1} & 0 \\ 0 & I_i \end{bmatrix}.$$

As $d_i \cdot d_{i-1} = \mu_{\ell^2}$, the result follows.

COROLLARY 4.5. *If $A_2 \neq 0$ and $\operatorname{char}(\mathbb{K}) = 2$, then $A \otimes \mathbb{K}[z]/z^2$ does not have WLP.*

PROOF. By Lemma 3.1, μ_ℓ is a differential so $d_i d_{i-1} = 0$; apply Corollary 4.4.

PROPOSITION 4.6. $(\operatorname{Sym}(V)/V^2) \bigotimes_{i=1}^n \mathbb{K}[z_i]/z_i^2$ *has WLP if* $\operatorname{char}(\mathbb{K}) \neq 2$.

PROOF. Let $A = \bigotimes_{i=1}^n \mathbb{K}[z_i]/z_i^2$, $C = \operatorname{Sym}(V)/V^2$ and $B = C \otimes A$. By [25], A has WLP; choose a basis for $B_i \simeq (C_1 \otimes A_{i-1}) \oplus A_i$ respecting the direct sum, and similarly for B_{i+1}. Let d_i^B denote the multiplication map $B_i \to B_{i+1}$, and let d_i^A and I_i^A denote the multiplication and identity maps on A_i. Then, for $1 < i < n$,

$$d_i^B = \begin{bmatrix} d_{i-1}^A & 0 & \dots & 0 & I_i^A \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & d_{i-1}^A & I_i^A \\ 0 & \dots & \dots & 0 & d_i^A \end{bmatrix} \tag{4.1}$$

is a $\dim(B_{i+1}) \times \dim(B_i)$ matrix; if i is 0 or n then d_i^B changes shape, but has full rank at these values. When n is odd, the graded components of A of maximal dimension occur in degrees $(n-1)/2$ and $(n+1)/2$, while if n is even there is a unique maximal graded component of A in degree $n/2$.

If $\dim(A_{i-1}) \le \dim(A_i) \le \dim(A_{i+1})$ then, since A has WLP, both d_{i-1}^A and d_i^A are injective, hence so is d_i^B, and similarly for surjectivity. So it suffices to study the behavior of B at the peak; a check shows that $\dim(B_i)$ is maximal at $m = \lceil (n+1)/2 \rceil$, with

$$\dim(B_{m-1}) < \dim(B_m) \quad \text{and} \quad \dim(B_{m+1}) < \dim(B_m),$$

hence we need only show that $B_{m-1} \hookrightarrow B_m$ and $B_m \twoheadrightarrow B_{m+1}$.

When n is odd, this is automatic: there are always two consecutive degrees in A where d_{i-1}^A and d_i^A are both injective or both surjective, and from the structure of d_i^B above, this means d_i^B is also injective or surjective.

When n is even, the result is more delicate. Let $d = d_{(n/2)-1}^A$ and $I = I_{A_{(n/2)-1}}$. Then because A is Gorenstein, the matrices for the multiplication maps are symmetric. The easiest way to see this is to note that these maps are the maps on the Koszul complex of $\Lambda(\mathbb{K}^n)$, but with all signs positive: if $\mathrm{char}(\mathbb{K}) = 2$ then $A \simeq \Lambda(\mathbb{K}^n)$. So if n is even, we may write the matrix of Equation (4.1) as

$$d_i^B = \begin{bmatrix} d & 0 & \dots & 0 & I \\ 0 & \ddots & \ddots & \vdots & I \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & d & I \\ 0 & \dots & \dots & 0 & d^T \end{bmatrix},$$

and it suffices to show that

$$\phi = \begin{bmatrix} d & I \\ 0 & d^T \end{bmatrix},$$

has full rank. Since d is injective, ordering bases so the top left $\dim(A_{m-1}) \times \dim(A_{m-1})$ block of d has full rank, we also have that the bottom right $\dim(A_{m-1}) \times \dim(A_{m-1})$ block has full rank, so that ϕ has rank $\dim(A_m) + \dim(A_{m-1})$ and therefore d_i^B has full rank. This completes the proof.

Proposition 4.7. *If* $\operatorname{char}(\mathbb{K}) \neq 2$ *and* $\dim(V_1) = a \geq 2$, $\dim(V_2) = b \geq 2$, *then*

$$\operatorname{Sym}(V_1)/V_1^2 \otimes \operatorname{Sym}(V_2)/V_2^2 \otimes \bigotimes_{i=1}^{n} \mathbb{K}[z_i]/z_i^2$$

has WLP if and only if n is odd.

Proof. We first show that if $n = 2k$ is even then the algebra does not have the WLP. Theorem 1.5 (or Theorem 4.2, taking $k_1 = k_2 = 2$ and $n = 2$) shows that

$$\operatorname{Sym}(V_1)/V_1^2 \otimes \operatorname{Sym}(V_2)/V_2^2$$

fails both injectivity and surjectivity from degree 1 to degree 2. On the other hand, $\bigotimes_{i=1}^{n} \mathbb{K}[z_i]/z_i^2$ is a complete intersection of quadrics, so it reaches its unique peak in degree k. Thus it fails injectivity from degree k to $k+1$ and fails surjectivity from degree $k-1$ to k. Then Lemma 4.1 shows that our tensor product fails both injectivity and surjectivity from degree $k+1$ to $k+2$, so WLP fails.

Now, let

$$A = \bigotimes_{i=1}^{n} \mathbb{K}[z_i]/z_i^2 \quad \text{with } n \text{ odd}, \quad C = \operatorname{Sym}(V_1)/V_1^2 \otimes \operatorname{Sym}(V_2)/V_2^2,$$

and write $B = C \otimes A$. Choose bases for $B_i \simeq (C_2 \otimes A_{i-2}) \oplus (C_1 \otimes A_{i-1}) \oplus A_i$ and for $B_{i+1} \simeq (C_2 \otimes A_{i-1}) \oplus (C_1 \otimes A_i) \oplus A_{i+1}$ respecting the direct sums. Let d_i^A and I_i^A denote, respectively, the multiplication and identity maps on A_i, with similar notation for B and C. Then with respect to the bases above,

$$d_i^B = \begin{bmatrix} I_2^C \otimes d_{i-2}^A & d_1^C \otimes I_{i-1}^A & 0 \\ 0 & I_1^C \otimes d_{i-1}^A & d_0^C \otimes I_i^A \\ 0 & 0 & d_i^A \end{bmatrix}.$$

As in the proof of Proposition 4.6, the key will be to use the fact that A has WLP and is a complete intersection. From the structure of d_i^B, if we choose i such that

$$\dim(A_{i-2}) \leq \dim(A_{i-1}) \leq \dim(A_i) \leq \dim(A_{i+1}),$$

then d_i^B is injective; and similarly for surjectivity. Because n is odd, by symmetry the only problematic case is when $i = (n+1)/2$, so that

$$\dim(A_{i-2}) < \dim(A_{i-1}), \quad \dim(A_{i-1}) = \dim(A_i), \quad \dim(A_{i+1}) < \dim(A_i).$$

Using that $a, b \geq 2$, a dimension count shows we need to prove that d_i^B is injective. Since A has WLP, the map d_{i-1}^A is an isomorphism, and as in the proof of Proposition 4.6, d_i^A is the transpose of d_{i-2}^A. However, in contrast to Proposition 4.6, the map d_1^C plays a role.

If $\{x_1, \ldots, x_a\}$ is a basis for V_1 and $\{y_1, \ldots y_b\}$ a basis for V_2, then $d_1^C(\sum x_i - \sum y_j) = 0$, and an Euler characteristic computation shows that this is the only element of the kernel of d_1^C. This means that with respect to the ordered bases

$$\left\{x_1, \ldots, x_a, y_1, \ldots, y_{b-1}, \sum x_i - \sum y_j\right\} \quad \text{for } C_1$$
$$\text{and} \quad \{x_1y_1, \ldots, x_1y_b, x_2y_1, \ldots x_2y_b, \ldots x_ay_b\} \quad \text{for } C_2,$$

the matrix $d_1^C \otimes I_{i-1}^A$ has a block decomposition with rightmost $\dim(C_2) \cdot \dim(A_{i-1}) \times \dim(A_{i-1})$ submatrix zero. For example, when $a = 2$ and $b = 3$, with $I = I_{i-1}^A$ the matrix takes the form

$$\begin{bmatrix} I & 0 & I & 0 & 0 \\ I & 0 & 0 & I & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & I & 0 & 0 \\ 0 & I & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 \end{bmatrix}.$$

The block of d_i^B corresponding to the submatrix $d_0^C \otimes I_i^A$ consists of $\dim(C_1)$ stacked copies of $I_i^A = I_{i-1}^A$, while the block of d_i^B corresponding to $I_1^C \otimes d_{i-1}^A$ consists of $\dim(C_1)$ diagonal copies of the invertible matrix U representing d_{i-1}^A; this follows because n is odd so $\dim(A_{i-1}) = \dim(A_i)$ and A has WLP.

Continuing with the $a = 2$, $b = 3$ example, and writing $d = d^A_{i-2}$, $I = I^A_i = I^A_{i-1}$ and U for the invertible matrix d^A_{i-1}, the matrix representing d^B_i is

$$\begin{bmatrix}
d & 0 & 0 & 0 & 0 & 0 & I & 0 & I & 0 & 0 & 0\\
0 & d & 0 & 0 & 0 & 0 & I & 0 & 0 & I & 0 & 0\\
0 & 0 & d & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & d & 0 & 0 & 0 & I & I & 0 & 0 & 0\\
0 & 0 & 0 & 0 & d & 0 & 0 & I & 0 & I & 0 & 0\\
0 & 0 & 0 & 0 & 0 & d & 0 & I & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & U & 0 & 0 & 0 & 0 & I\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & U & 0 & 0 & 0 & I\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U & 0 & 0 & I\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U & 0 & I\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U & I\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^T
\end{bmatrix}.$$

Now, since U is invertible of the same size as I, we may use it to row reduce the matrix above. Because the bottom row block of $d^C_1 \otimes I^A_{i-1}$ has only one nonzero entry, this allows row reduction of the matrix for d^B_i in the example to

$$\begin{bmatrix}
d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2U^{-1}\\
0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2U^{-1}\\
0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2U^{-1}\\
0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2U^{-1}\\
0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & -2U^{-1}\\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & -1U^{-1}\\
0 & 0 & 0 & 0 & 0 & d & I & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & d & 0 & I & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & I & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & I & 0 & 0\\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & I & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^T
\end{bmatrix}.$$

So we need only show that

$$\begin{bmatrix} d & -U^{-1}\\ 0 & d^T \end{bmatrix}$$

has full rank, which follows as in Proposition 4.6. The result follows.

THEOREM 4.8. *If* $\mathrm{char}(\mathbb{K}) \neq 2$*, and*

$$C = \bigotimes_{i=1}^{n} \mathrm{Sym}(V_i)/V_i^2 \quad \textit{with} \quad \dim(V_i) = d_i,$$

then C has WLP if and only if one of the following holds:

(1) $d_2, \ldots, d_n$ *are* 1*;*

(2) $d_3, \ldots, d_n$ *are* 1*, and n is odd.*

PROOF. From our work above, we have that:

(1) WLP holds by Proposition 4.6;

(2) WLP holds by Proposition 4.7.

Proposition 4.7 also covers failure of WLP in case (2) when n is even, so what remains is to show WLP fails when three or more $d_i \geq 2$; if no $d_i = 1$, this follows from Theorem 4.2.

Assume for convenience that $d_1 \geq d_2 \geq \ldots \geq d_n$. We first consider the case $d_{n-1} > d_n = 1$; of course then we have $n - 1 \geq 3$ by assumption. Let

$$A' = \bigotimes_{i=1}^{n-2} \mathrm{Sym}(V_i)/V_i^2, \quad A'' = \mathrm{Sym}(V_{n-1}) \otimes \mathbb{K}[z]/z^2, \quad C = A' \otimes A''.$$

By the proof of Theorem 4.2, A' fails both injectivity and surjectivity from degree $n-3$ to degree $n-2$. The Hilbert function of A'' is $(1, d_{n-1}+1, d_{n-1})$ so clearly it fails surjectivity from degree 0 to degree 1, and it fails injectivity from degree 1 to degree 2. Then by Lemma 4.1, C fails surjectivity from degree $n-2$ to degree $n-1$, and it fails injectivity in the same degree, hence it fails to have WLP.

Now let

$$B' = \bigotimes_{i=1}^{m} \mathrm{Sym}(V_i)/V_i^2, \quad B'' = \bigotimes_{i=1}^{p} \mathbb{K}[z_i]/z_i^2, \quad C = B' \otimes B'',$$

where $\dim V_i \geq 2$ for $1 \leq i \leq m$, $m \geq 3$, and $m + p = n$. We will show that C fails WLP.

Case 1. $p = 2k$ is even, $k \geq 1$.

We will show that C fails WLP from degree $m - 1 + k$ to $m + k$. By Theorem 4.2, we know that the multiplication on B' from degree $m-1$ to m is neither surjective nor injective. Since B'' is a complete intersection of quadrics with even socle degree $2k$, it fails injectivity from degree k to $k+1$, and it fails surjectivity from degree $k-1$ to k. Then Lemma 4.1 applies as before to show that C fails WLP as claimed.

Case 2. $p = 2k+1$ is odd.

We will again show that C fails WLP from degree $m+k-1$ to $m+k$. Our proof is by induction on k, having just shown the case $k=0$. So assume $k \geq 1$. We rewrite C as follows:

$$C' = \bigotimes_{i=1}^{m} \mathrm{Sym}(V_i)/V_i^2 \otimes \bigotimes_{i=1}^{2k-1} \mathbb{K}[z_i]/z_i^2,$$

$$C'' = \mathbb{K}[x, y]/(x^2, y^2), \quad C = C' \otimes C''.$$

By induction, C' fails both injectivity and surjectivity from degree $m+k-2$ to $m+k-1$. Clearly C'' fails injectivity from degree 1 to degree 2, and fails surjectivity from degree 0 to degree 1. Then Lemma 4.1 again gives the result.

COROLLARY 4.9. *If $K_{a,b}$ is the complete bipartite graph and $K_{a,b,r}$ the cone over $K_{a,b,r-1}$ with $K_{a,b,0} = K_{a,b}$, then the Stanley-Reisner ring of $K_{a,b,r}$ plus squares of variables has WLP if and only if r is odd.*

REMARK 4.10. We close with some directions for future work.

- Quadratic monomial ideals are always Koszul but may fail to have WLP. In [17], Migliore-Nagel conjectured that all quadratic Artinian Gorenstein algebras have WLP; this was recently disproved by Gondim-Zappala [6]. It should be interesting to investigate the confluence between Koszulness, quadratic Gorenstein algebras, and WLP.
- Use the syzygy bundle techniques of [4], combined with Hochster's and Laksov's work on syzygies [9] to study WLP. This will mean studying the WLP for products of linear forms, and we are at work [19] on this.
- Connect to the work of Singh-Walther [23] on the Bockstein spectral sequence and local cohomology of Stanley-Reisner rings.

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MATHEMATICS DEPARTMENT
UNIVERSITY OF NOTRE DAME
SOUTH BEND
IN 46566
USA
E-mail: migliore.1@nd.edu

MATHEMATICS DEPARTMENT
UNIVERSITY OF KENTUCKY
LEXINGTON
KY 40506
USA
E-mail: uwe.nagel@uky.edu

MATHEMATICS DEPARTMENT
IOWA STATE UNIVERSITY
AMES
IA 50014
USA
E-mail: hschenck@iastate.edu