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STRONG LIMIT THEOREMS FOR SUPERCRITICAL IMMIGRATION-BRANCHING PROCESSES

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0. Abstract.

Consider an increasing sequence of random times and a corresponding sequence of random populations. Let these be the starting times and initial values of otherwise equivalent stochastic branching processes. The resulting superposition is an immigration-branching process. It is readily conjectured, that if the branching process is supercritical and the immigration dominated in some appropriate sense by the branching, the immigration-branching process, averaged and normalized exactly as the branching process, converges almost surely to a superposition of the limits of the composing branching processes. We verify this conjecture, satisfactorily sharpening and generalizing a number of results to be found in the literature. Our analysis shows that limit theorems for processes with immigration follow more directly and easily from corresponding results for processes without immigration, than is apparent from the literature. As underlying branching processes we admit either a general positively regular process in the sense of [1], or a Bellman–Harris process. Our auxiliary material on the Bellman–Harris process also contains improvements of results of [3] and [8].

1. Introduction.

Let X be an arbitrary set, $X^{(n)}$ the symmetrized n-fold direct product of X, θ some extra point, and $X^{(0)} = \{\theta\}$. Define

$$\hat{X} := \bigoplus_{n=0}^{\infty} X^{(n)}$$

$$\hat{x}+\hat{y} := \hat{x}; \qquad \hat{x} \in \hat{X},\ \hat{y} = \theta\,,$$

$$:= \langle x_1,\ldots,x_n, y_1,\ldots,y_m\rangle; \qquad \hat{x} = \langle x_1,\ldots,x_n\rangle,\ \hat{y} = \langle y_1,\ldots,y_m\rangle \in \hat{X}\,,$$

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$$\hat{x}[A] := 0; \qquad x = \theta,$$
$$:= \sum_{\nu=1}^{n} 1_A(x_\nu); \qquad \hat{x} = \langle x_1, \ldots, x_n \rangle \in \hat{X},$$
$$\hat{x}[\xi] := \int_X \xi(x)\,\hat{x}[dx],$$

for $A \subset X$ and every real valued function ξ on X.

Let $\mathfrak{A}$ be a σ-algebra on X and $\hat{\mathfrak{A}}$ the σ-algebra induced on $\hat{X}$ by $\mathfrak{A}$. Let either $T = \mathbf{N} = \{0, 1, 2, \ldots\}$, or $T = \mathbf{R}_+ = [0, \infty[$, and suppose to be given

(a) the *immigration process* $\{\tau_\nu, \hat{y}_\nu, P\}$, where $0 \leqq \tau_\nu \uparrow \infty$ is a sequence of (not necessarily finite) random times and $\{\hat{y}_\nu\}_{\nu \in \mathbf{N} \setminus \{0\}}$ is a random sequence in $(X, \mathfrak{A})$, both defined on the same space with probability measure P,
(b) the *Markov branching process* $\{\hat{x}_t, P^{\hat{x}}\}$, that is, an $(\hat{X}, \hat{\mathfrak{A}})$- valued Markov process with parameter set T and stationary transition probabilities satisfying the branching condition as in [1].

Denote by $\{\hat{x}_{\nu,t}; t \geqq \tau_\nu\}$ the branching process initiated at time τ_ν by $\hat{y}_\nu$ and set

$$N_t = \max\{\nu : \tau_\nu \leqq t\}.$$

The immigration-branching process $\{\hat{z}_t, \tilde{P}\}$ is then given by

$$\hat{z}_t = \sum_{\nu \leqq N_t} \hat{x}_{\nu,t}$$

and the corresponding probability measure $\tilde{P}$, with expectation functional $\tilde{E}$, defined on the appropriate product space. The formal construction is the same as in [5] except for the trivial adaptation to the more general, not necessarily Poissonian case considered here.

As branching processes we admit

(i) supercritical *positively regular* processes with a finite or infinite number of types, $T = \mathbf{N}$ or $T = \mathbf{R}_+$,
(ii) supercritical *Bellman–Harris* processes, identifying types with ages, $X = \mathbf{R}_+$.

A common feature of such processes is the existence of a real number $\varrho > 1$, a bounded function $\varphi \geqq 0$ on X, and a bounded measure $\varphi^*[1_A]$ on $(X, \mathfrak{A})$ such that

$$W_t := \varrho^{-t}\hat{x}_t[\varphi], \qquad t \geqq 0$$

is a non-negative martingale and for all η in a suitable class of averaging functions

$$\text{(1.1)} \qquad \lim_{t\to\infty} \varrho^{-t}\hat{x}_t[\eta] = \varphi^*[\eta]\,W_\infty \quad \text{a.s.},$$

where $W_\infty := \lim_{t\to\infty} W_t$ is non-degenerate if and only if a certain moment condition is satisfied.

It is our aim to prove corresponding results for immigration-branching processes, where the branching dominates the immigration in the sense that ϱ^{-t} remains the proper normalizing factor for $\hat{z}_t[\cdot]$. Setting

$$W_{\nu,t} := \varrho^{-(t-\tau_\nu)}\hat{x}_{\nu,t}[\varphi]; \qquad t \geqq \tau_\nu ,$$

$$W_{\nu,\infty} := \lim_{t\to\infty} W_{\nu,t}$$

$$\tilde{W}_t := \varrho^{-t}\hat{z}_t[\varphi] = \sum_{\nu \leqq N_t} \varrho^{-\tau_\nu} W_{\nu,t}$$

we want to show that $\tilde{W}_\infty := \lim_{t\to\infty} \tilde{W}_t$ exists and is finite a.s., that

$$\tilde{W}_\infty = \sum_{\nu=1}^{\infty} \varrho^{-\tau_\nu} W_{\nu,\infty} \quad \text{a.s.}$$

and that for a sufficiently large class of η

$$\lim_{t\to\infty} \varrho^{-t}\hat{z}_t[\eta] = \varphi^*[\eta]\,\tilde{W}_\infty \quad \text{a.s.}\,.$$

Some of our results follow immediately from the structure of $\{\hat{z}_t\}$ as a superposition of branching processes, others rely on special properties of the underlying branching model. The problem seems to be non-trivial only if $\inf\varphi = 0$, which is the case, for example, for a branching diffusion with absorbing barriers, or some Bellman–Harris processes.

Obviously we cannot have $\tilde{W}_\infty < \infty$ a.s. without conditions on the immigration process. The approach to be found in the literature is the assumption of some specific structure. (See however, K. B. Athreya, P. R. Parthasarathy and G. Sankaranarayanan, *Supercritical age-dependent branching processes with immigration*, J. Appl. Probability 11 (1974), 695–702). In case of a Bellman–Harris branching part, which has been extensively studied, usually only immigration with age zero is admitted, while $\{\tau_\nu\}$ in some papers is a Poisson process, in some a renewal process or even a summed ergodic process slightly more general than a renewal process, [7]. General positively regular processes have been treated with inhomogeneous Poisson immigration, [5]. Finally, components of a decomposable branching process ([9]) may be viewed as immigration-branching processes.

We shall not make any assumptions of this form and work instead with the general conditions

(1.2) $$\sum_{\nu=1}^{\infty} \varrho^{-\tau_\nu}\hat{y}_\nu[\varphi] < \infty \quad \text{a.s.}\,,$$

(1.3) $$\sum_{\nu=1}^{\infty} \varrho^{-\tau_\nu}\hat{y}_\nu[1] < \infty \quad \text{a.s.}\,.$$

More explicit properties are needed only to verify these estimates. Since $\tilde{W}_t$ is a submartingale conditioned upon $\mathfrak{I}=\sigma(\tau_1,\tau_2,\ldots,\hat{y}_1,\hat{y}_2,\ldots)$ with

$$\sup_t \tilde{E}(\tilde{W}_t\,|\,\mathfrak{I}) = \sum_{\nu=1}^{\infty} \varrho^{-\tau_\nu}\hat{y}_\nu[\varphi] \quad \text{a.s.},$$

Doob's condition is satisfied for almost all realizations of the immigration process if and only if (1.2) holds. Hence (1.2) implies that $\tilde{W}_\infty$ exists and is finite a.s.. In fact, we shall see that (1.2) is minimal for this result in a large class of models. However, if $\inf\varphi=0$, this being the only case in which (1.2) and (1.3) differ, it is not difficult to see that (1.2) does not ensure the well-behaviour of quantities like $\hat{z}_t[\mathbf{1}]$, the total population size at time t. It is here, where (1.3) comes in.

We now outline the rest of the paper. In section 2 we give the preliminaries on the underlying branching processes. In section 3 we study the structure of $\{\hat{z}_t\}$ as a superposition. The analysis solves the case $\inf\varphi>0$ completely. The additional results needed for the case $\inf\varphi=0$ are then obtained in section for positively regular processes and in section 5 for Bellman–Harris processes.

Throughout this paper $c, c_1, c_2,\ldots$ denote positive real constants.

2. Facts on the underlying branching processes.

Let $\mathscr{B}$ be the Banach algebra of all bounded, $\mathfrak{A}$-measurable functions with supremum-norm

$$\|\xi\| = \sup_{x\in X}|\xi(x)|,$$

and $\mathscr{B}_+$ the non-negative cone in $\mathscr{B}$.

We call $\{\hat{x}_t, \boldsymbol{P}^{\hat{x}}\}$ *positively regular* if it satisfies the following condition:

(M) *The first moment semigroup $\{\boldsymbol{E}^{\langle\cdot\rangle}\hat{x}_t[\cdot]\}_{t\in T}$ exists and can be represented in the form*

(2.1) $$\boldsymbol{E}^{\langle x\rangle}\hat{x}_t[\eta] = \varrho^t\varphi^*[\eta]\varphi(x)+Q_t^{\langle x\rangle}[\eta], \qquad x\in X,\ t\in T,\ \eta\in\mathscr{B},$$

with $\varrho\in\,]0,\infty[$, $\varphi\in\mathscr{B}_+$, and φ^* a non-negative bounded linear functional on $\mathscr{B}$ such that

(2.2) $$\varphi^*[\varphi] = 1,$$

$$\varphi^*[Q_t^{\langle\cdot\rangle}[\cdot]] \equiv 0, \qquad Q_t^{\langle\cdot\rangle}[\varphi] \equiv \mathbf{0},$$

$$|Q_t^{\langle x\rangle}[\eta]| \leqq \alpha_t\varphi^*[\eta]\varphi(x), \qquad x\in X,\ \eta\in\mathscr{B}_+,\ t>0,$$

for some $\alpha_\cdot$: $T\to[0,\infty[$ satisfying $\varrho^{-t}\alpha_t\to 0$, $t\to\infty$.

Clearly (M) implies that φ^* is a measure. For convenience, we take $\varphi^*[\mathbf{1}]=1$.

For a finite set of types our definition of positive regularity is the usual one, in the infinite case its motivation derives from branching diffusions, [1].

In case $T=\mathsf{R}_+$, we shall need some additional structure in order to get beyond the consideration of discrete skeletons:

(C) *The set X is a separable metric space, $\mathfrak{A}$ is the topological Borel σ-algebra, and $\{\hat{x}_t, \boldsymbol{P}^{\hat{x}}\}$ is rightcontinuous.*

As in [1], we assume *supercriticality*, that is, $\varrho > 1$. The significance of φ and φ^* for the limit theory has been indicated in section 1. The following facts on supercritical positively regular processes are quoted from [1]:

A necessary and sufficient condition for non-degeneracy of W_∞ is

$$(x \log x) \qquad \varphi^*[\boldsymbol{E}^{\langle\cdot\rangle} W_t \log W_t] < \infty \quad \text{for some } t>0\,.$$

If this condition is satisfied, $\boldsymbol{E}^{\langle x\rangle} W_\infty = \varphi(x)$, $\forall\, x \in X$, otherwise $W_\infty = 0$ a.s. $[\boldsymbol{P}^{\hat{x}}]$, $\forall\, \hat{x} \in \hat{X}$.

In the following, $\mathscr{U}$ will be a class of averaging functions $\eta \in \mathscr{B}$ such that (1.1) holds when $\eta \in \mathscr{U}$. If $T=\mathsf{N}$, we can take $\mathscr{U}=\mathscr{B}$. If $T=\mathsf{R}_+$ and φ is lower semi-continuous a.e. $[\varphi^*]$, we can take $\mathscr{U}$ as the class of all functions of the form $\varphi\xi$ with $\xi \in \mathscr{B}$ continuous a.e. $[\varphi^*]$.

To obtain a more satisfactory statement on $\mathscr{U}$ if $T=\mathsf{R}_+$ and $\inf \varphi = 0$, we need additional structure:

(C*) *There exist random variables $H^t \geqq 0$, $t>0$, such that*

$$\hat{x}_s[1] \leqq H^t, \; 0 \leqq s \leqq t, \qquad \|\boldsymbol{E}^{\langle\cdot\rangle} H^t\| \downarrow 1, \; t \to 0\,.$$

This is automatic, e.g., for branching diffusions, see § of [1]. If (C*) is satisfied, we can take $\mathscr{U}$ as the class of all $\xi \in \mathscr{B}$ which are continuous a.e. $[\varphi^*]$. This has not been stated explicitly in [1], but it is the essence of the proof of Theorem 1″ of [1]. A proof can also be obtained by specializing the argument given below in section 4 to processes without immigration.

Let us now turn to Bellman–Harris processes. The basic facts are to be found, e.g., in [2] or [4].

Let F be the offspring distribution and G the lifetime distribution, which as usual is assumed to be non-lattice with $G(0)=0$. Supercriticality now amounts to

$$1 < m = \int_0^\infty x\, dF(x) < \infty\,.$$

Standard renewal techniques, see, e.g., Lemma 1 of [3] or the proof of Lemma 7 of [8], lead to a representation of the first moment semigroup as that of (M) with parameters $\varrho > 1, \varphi, \varphi^*$ given by

$$m \int_0^\infty \varrho^{-y} dG(y) = 1 ,$$

$$\varphi^*[1_{dx}] = dA(x) = \frac{\varrho^{-x}(1-G(x))dx}{\int_0^\infty \varrho^{-y}(1-G(y))\,dy} ,$$

$$\varphi(x) = \frac{V(x)}{\int_0^\infty V(y)\,dA(y)}, \qquad V(x) = \frac{\varrho^x \int_x^\infty \varrho^{-y} dG(y)}{1-G(x)} .$$

More precisely, (2.1) holds for all η such that $\varrho^{-x}\eta(x)$ is directly Riemann integrable, with (2.2) replaced by

$$\|Q_t^{\langle\cdot\rangle}[\eta]\| = o(\varrho^t) . \tag{2.3}$$

In the standard terminology, A is the *stable age distribution* and V the *reproductive value.*

It is well known that the condition for non-degeneracy of W_∞ becomes

$$\int_0^\infty x \log x \, dF(x) < \infty . \tag{$x \log x$}$$

As $\mathscr{U}$ we can take the set of all a.e. continuous $\xi \in \mathscr{B}$. With the assumption that ($x \log x$) is satisfied, this has been proved in [3]. A proof without ($x \log x$) can be obtained by adapting the proof of Theorem 1 of [1], using (2.3) combined with

$$M := \sup_t \varrho^{-t}\hat{x}_t[1] < \infty \quad \text{a.s.} \tag{2.4}$$

instead of (2.2). While (2.4) is not immediately available for all models covered by [1], there exists a simple argument for Bellman–Harris processes due to Kesten [10]:

Let $n \in \mathsf{N}$, $n \leqq t \leqq n+1$. An individual alive at time t was necessarily present in the population and of age at most 1 at one of the times $s = 1, 2, \ldots, n, t$. Since

$$\varphi(x) = c_1 \cdot V(x) \geqq c_2 > 0, \quad 0 \leqq x \leqq 1 ,$$

there is a constant c_3 such that

$$\hat{x}_t[1] \leqq c_3 \left\{ \sum_{s=1}^n \hat{x}_s[\varphi 1_{[0,1]}] + \hat{x}_t[\varphi 1_{[0,1]}] \right\}, \quad n \leqq t \leqq n+1 . \tag{2.5}$$

In particular

$$\hat{x}_t[1] \leqq c_3 \left\{ \sum_{s=1}^n \hat{x}_s[\varphi] + \hat{x}_t[\varphi] \right\}$$

$$\leqq c_3 \left\{ \sum_{s=1}^n \varrho^s + \varrho^t \right\} \sup_{t \geqq 0} W_t \leqq c_4 \varrho^t \sup_{t \geqq 0} W_t ,$$

and (2.4) follows from $W_\infty < \infty$ a.s..

However, just to eliminate $(x \log x)$, we only have to show

$$\text{(2.6)} \qquad \lim_{t\to\infty} \varrho^{-t}\hat{x}_t[1] = 0 \text{ a.s. if } \int_0^\infty x \log x \, dF(x) = \infty .$$

For $N \in \mathbf{N}$, $t \geqq N$, we have from (2.5)

$$\hat{x}_t[1] \leqq c_3 \left\{ \sum_{s=1}^{N} \hat{x}_s[\varphi] + \sum_{s=N+1}^{[t]} \hat{x}_s[\varphi] + \hat{x}_t[\varphi] \right\} .$$

Hence

$$\limsup_{t\to\infty} \varrho^{-t}\hat{x}_t[1] \leqq c_5 \sup_{t \geqq N} W_t$$

with c_5 independent of N. If $(x \log x)$ is not satisfied, the expression on the right tends to zero a.s. as $N \to \infty$, and (2.6) follows.

A similar idea occurs as part of the proof of the following lemma, to which we return in section 5:

LEMMA 1. *For a supercritical Bellman–Harris process* $\|E^{\langle \cdot \rangle} M\| < \infty$ *if and only if* $(x \log x)$ *holds. Otherwise* $E^{\langle x \rangle} M = \infty$ *for all* $x \in X$.

This fact, which is of interest in itself, was stated in [8] without proof and under the assumption $\inf \varphi > 0$. For Galton–Watson processes it has been known for some time, [9]. For an application different from the one in the present paper see [8].

3. The immigration-branching process as a superposition.

In this section $\{\hat{x}_t, P^{\hat{x}}\}$ may be any of the branching processes admitted in the introduction.

THEOREM 1. *If* (1.2) *is satisfied, then almost surely* $\tilde{W}_\infty = \lim_{t\to\infty} \tilde{W}_t$ *exists, is finite, and equals*

$$\tilde{W}^*_\infty := \sum_{v=1}^{\infty} \varrho^{-\tau_v} W_{v,\infty} \, .$$

Conversely, given $(x \log x)$,

$$\text{(3.1)} \qquad \lim_{t\to\infty} \tilde{W}_t = \infty \text{ a.s. } on\ \Gamma := \left\{ \sum_{v=1}^{\infty} \varrho^{-\tau_v} \hat{y}_v[\varphi] = \infty \right\}$$

if $\{\hat{x}_t, P^{\hat{x}}\}$ *is* (i) *positively regular with a finite set of types, or* (ii) *a Bellman–Harris process.*

For the proof, we need the following lemma.

LEMMA 2. *Let $\gamma_1, \gamma_2, \ldots$ be a real constants, $\gamma_n \downarrow 0$, and $U_1, U_2, \ldots$ independent non-negative random variables which are either*

(a) *identically distributed with finite mean, or*
(b) *uniformly bounded, $0 \leqq U_\nu \leqq c < \infty$.*

Then

$$\sum_{\nu=1}^{\infty} \gamma_\nu U_\nu < \infty \quad \text{a.s.} \tag{3.2}$$

if and only if

$$\sum_{\nu=1}^{\infty} \gamma_\nu \boldsymbol{E} U_\nu < \infty , \tag{3.3}$$

while otherwise

$$\boldsymbol{P}\left(\sum_{\nu=1}^{\infty} \gamma_\nu U_\nu = \infty\right) = 1 .$$

PROOF. By the zero-one law it suffices to prove the equivalence of (3.2) and (3.3). Clearly (3.3) implies (3.2), and by Kolmogorov's three series criterion it is necessary for (3.2) that

$$\sum_{\nu=1}^{\infty} \gamma_\nu \boldsymbol{E} U_\nu 1_{\{\gamma_\nu U_\nu \leqq 1\}} < \infty .$$

To get (3.3), note that for ν large

$$\boldsymbol{E} U_\nu 1_{\{\gamma_\nu U_\nu \leqq 1\}} \begin{cases} \geqq \boldsymbol{E} U_\nu / 2 & \text{for (a)} \\ = \boldsymbol{E} U_\nu & \text{for (b)} . \end{cases}$$

PROOF OF THEOREM 1. That (1.2) implies the a.s. existence and finiteness of $\tilde{W}_\infty$ has already been noticed in section 1.

Given (1.2), $\tilde{W}_\infty^*$ is finite a.s.. For $0 < s < t < \infty$ write $\tilde{W}_\infty - \tilde{W}_\infty^*$ in the form

$$\tilde{W}_\infty - \tilde{W}_\infty^* = \tilde{W}_\infty - \tilde{W}_t + \sum_{\tau_\nu \leqq s} \varrho^{-\tau_\nu} (W_{\nu,t} - W_{\nu,\infty}) + \\ + \sum_{s < \tau_\nu \leqq t} \varrho^{-\tau_\nu} W_{\nu,t} - \sum_{\tau_\nu \geqq s} \varrho^{-\tau_\nu} W_{\nu,\infty} .$$

First let $t \to \infty$. Then almost surely the first and the second term tend to zero and the third term to a finite limit $U_s \geqq 0$. Now let $s \to \infty$. Since U_s is nonincreasing in s, $U=\lim_{s\to\infty} U_s \geqq 0$ a.s. exists, and as

$$\tilde{E}(U \mid \mathfrak{I}) \leqq \liminf_{s\to\infty} \liminf_{t\to\infty} \tilde{E}\Big(\sum_{s<\tau_\nu \leqq t} \varrho^{-\tau_\nu} W_{\nu,t} \mid \mathfrak{I}\Big)$$

$$= \liminf_{s\to\infty} \sum_{s<\tau_\nu} \varrho^{-\tau_\nu} \hat{y}_\nu[\varphi] = 0 \quad \text{a.s.},$$

$U=0$ a.s.. The remaining term tends to zero a.s., since $\tilde{W}^*_\infty < \infty$ a.s..

To verify (3.1), we may assume that $\hat{y}_\nu[\mathbf{1}]=1$, $\nu=1,2,\ldots$. This is simply a matter of labelling. Obviously $\liminf_{t\to\infty} \tilde{W}_t \geqq \tilde{W}^*_\infty$ a.s., and it suffices to show that $\tilde{W}^*_\infty = \infty$ a.s. on Γ.

For case (i), define

$$\Gamma_x := \Big\{ \sum_{\nu=1}^{\infty} \varrho^{-\tau_\nu} 1_{\{\hat{y}_\nu = \langle x \rangle\}} = \infty \Big\}, \qquad x \in X .$$

Then $\Gamma = \bigcup_{x\in X} \Gamma_x$ and from part (a) of Lemma 2,

$$\sum_{\nu=1}^{\infty} \varrho^{-\tau_\nu} 1_{\{\hat{y}_\nu = \langle x \rangle\}} W_{\nu,\infty} = \infty \qquad \text{on } \Gamma_x ,$$

from which (3.1) follows. For case (ii) we may write $W_{\nu,\infty} = \varrho^{-\lambda_\nu} U_\nu$, where λ_ν is the residual lifetime of νth individual immigrating and $U_1, U_2, \ldots$ satisfy the assumptions of part (a) of Lemma 2 conditioned upon $\mathfrak{I}$. From the definition of V, φ we have $\tilde{E}(\varrho^{-\lambda_\nu} \mid \mathfrak{I}) = c_6 \hat{y}_\nu[\varphi]$ a.s., and thus

$$\{ \tilde{W}^*_\infty = \infty \} = \Big\{ \sum_{\nu=1}^{\infty} \varrho^{-\tau_\nu} W_{\nu,\infty} = \infty \Big\} = \Big\{ \sum_{\nu=1}^{\infty} \varrho^{-\tau_\nu} \varrho^{-\lambda_\nu} = \infty \Big\} = \Gamma \qquad \text{a.s.},$$

where we have used part (a) of Lemma 2 (conditioning upon $\mathfrak{I}$ and $\{\lambda_\nu\}$) for the second equality and part (b) for the last (conditioning upon $\mathfrak{I}$).

Let us briefly examine the conditions (1.2) and (1.3) in some examples.

PROPOSITION. *Let $\tau_1, \tau_2, \ldots$ be the epochs of a renewal process, and let the $\hat{y}_\nu$ be* i.i.d. *and independent of $\{\tau_\nu\}$. Then for any $\eta \in \mathscr{B}_+$ the condition*

$$E \log^+ \hat{y}_1[\eta] < \infty \tag{3.4}$$

is sufficient for

$$\sum_{\nu=1}^{\infty} \varrho^{-\tau_\nu} \hat{y}_\nu[\eta] < \infty \qquad \text{a.s.}.$$

It is necessary if the mean interarrival time λ is finite.

PROOF. *Let*

$$\beta > 1, \quad K(y) = P(\hat{y}_\nu[\eta] \leqq y), \quad A_\nu = \{\beta^{-\nu}\hat{y}_\nu[\eta] \leqq 1\} .$$

Then

$$\sum_{\nu=1}^{\infty} PA_\nu = \sum_{\nu=1}^{\infty} \int_{\beta^\nu}^{\infty} dK(y) = \int_0^\infty \sum_{\nu=1}^{\infty} 1_{\{y \geqq \beta^\nu\}} dK(y) = \int_0^\infty O(\log^+ y)\, dK(y)$$

and similarly

$$\sum_{\nu=1}^{\infty} E\beta^{-\nu}\hat{y}[\eta]1_{A_\nu} = \int_0^\infty O(1)\, dK(y) ,$$

$$\sum_{\nu=1}^{\infty} E(\beta^{-\nu}\hat{y}_\nu[\eta]1_{A_\nu})^2 = \int_0^\infty O(1)\, dK(y) .$$

Thus it follows by Kolmogorov's three series criterion, that (3.4) is equivalent to

$$\sum_{\nu=1}^{\infty} \beta^{-\nu}\hat{y}_\nu[\eta] < \infty \quad \text{a.s.} .$$

Now condition upon $\{\tau_\nu\}$ and note that $\tau_\nu/\nu \to \lambda$, that is,

$$\varrho^{-\tau_\nu} \leqq \beta_1^{-\nu} \quad \text{for some } \beta_1 > 1,\ \nu \geqq \nu_0, \quad \text{if } 0 < \lambda \leqq \infty ,$$

$$\varrho^{-\tau_\nu} \geqq \beta_2^{-\nu} \quad \text{for some } \beta_2 > 1, \quad \text{if } 0 < \lambda < \infty .$$

In general, (1.2) holds at least if

$$E \sum_{\nu=1}^{\infty} \varrho^{-\tau_\nu}\hat{y}_\nu[\varphi] < \infty . \tag{3.5}$$

If $\{\tau_\nu\}$ is an inhomogeneous Poisson process with density $p(t)$, and the $\hat{y}_\nu$ are independent conditioned upon $\{\tau_\nu\}$, with the distribution of $\hat{y}_\nu$ depending only on τ_ν, then (3.5) reduces to

$$\int_0^\infty \varrho^{-t} p(t) M^t[\varphi]\, dt < \infty ,$$

where $M^t[\cdot] = E(\hat{y}_\nu[\cdot] \mid \tau_\nu = t)$, cf. [5]. For a decomposable branching process with two components (3.5) is automatic if $\varrho = \varrho_1 > \varrho_2$ in the notation of [9].

We now return to the general theory. Assuming (1.2), define

$$\tilde{\mathscr{U}} := \left\{\eta \in \mathscr{U} :\ \lim_{t\to\infty} \varrho^{-t}\hat{z}_t[\eta] = \varphi^*[\eta]\, \tilde{W}_\infty \ \text{a.s.}\right\}$$

$$\mathscr{U}_+ := \mathscr{U} \cap \mathscr{B}_+, \quad \tilde{\mathscr{U}}_+ := \tilde{\mathscr{U}} \cap \mathscr{B}_+ .$$

LEMMA 3. *For* $\eta \in \mathscr{U}_+$

(3.6) $$\liminf_{t\to\infty} \varrho^{-t}\hat{z}_t[\eta] \geqq \varphi^*[\eta]\,\tilde{W}_\infty \quad \text{a.s.}\,.$$

Furthermore

(3.7) *if* $\xi \in \mathscr{U}_+$ *and* $\vartheta \in \tilde{\mathscr{U}}_+$, *then* $\xi\vartheta \in \tilde{\mathscr{U}}_+$.

PROOF. Relation (3.6) is obvious from the structure of the process as a superposition. To prove (3.7) suppose $\mathbf{0} \leqq \vartheta \leqq \mathbf{1}$. Then the way $\mathscr{U}$ has been chosen in section 2 ensures that ξ, $\xi(1-\vartheta) \in \mathscr{U}$. Thus

$$\begin{aligned}\limsup_{t\to\infty} \varrho^{-t}\hat{z}_t[\xi\vartheta] &= \lim_{t\to\infty} \varrho^{-t}\hat{z}_t[\vartheta] - \liminf_{t\to\infty} \varrho^{-t}\hat{z}_t[\vartheta(\mathbf{1}-\xi)] \\ &\leqq \{\varphi^*[\vartheta] - \varphi^*[\vartheta(\mathbf{1}-\xi)]\}\,\tilde{W}_\infty \\ &= \varphi^*[\xi\vartheta]\,\tilde{W}_\infty \quad \text{a.s.}\,.\end{aligned}$$

The inequality for lim inf follows from (3.6) with $\eta = \xi\vartheta$.

Specializing Theorem ~~1 to~~ positively regular processes with a finite number of types, we are lead to the following important example.

COROLLARY. *Let* $\{\hat{x}_t, P^{\hat{x}}\}$ *be positively regular with a finite set of types,* $X = \{1,\ldots,k\}$, *and suppose* (1.2) *is satisfied. Then* $\tilde{W}_\infty = \lim_{t\to\infty} \tilde{W}_t$ *exists and is finite* a.s., *and for any k-vector* η

(3.8) $$\lim_{t\to\infty} \varrho^{-t}\hat{z}_t[\eta] = \varphi^*[\eta]\,\tilde{W}_\infty \quad \text{a.s.}\,.$$

Conversely, if $(x \log x)$ *is satisfied, but* (1.2) *fails, then* $\tilde{P}(\lim_{t\to\infty} \tilde{W}_t = \infty) > 0$.

(For (3.8), use (3.7) with $\xi = \eta/\varphi$, $\vartheta = \varphi$). This result is not only substantially sharper and more general than those of [6], [11], its proof is also much simpler. In fact, the case $\inf\varphi > 0$ is settled by (3.7) with $\vartheta = \varphi$. If $\inf\varphi = 0$, it remains essentially to prove $\mathbf{1} \in \tilde{\mathscr{U}}_+$, since then (3.7) with $\vartheta = \mathbf{1}$ applies.

4. Limit results for general positively regular processes with immigration.

It is assumed throughout this section that (M) is satisfied with $\varrho > 1$.

THEOREM 2. *Suppose either* (a) $T = \mathsf{N}$, *or* (b) $T = \mathsf{R}_+$ *with* (C) *and* (C*). *If* (1.3) *is satisfied, then* $\mathbf{1} \in \tilde{\mathscr{U}}_+$ *and thus*

$$\lim_{t\to\infty} \varrho^{-t}\hat{z}_t[\eta] = \varphi^*[\eta]\,\tilde{W}_\infty \quad \text{a.s.}$$

for $\eta \in \mathscr{B}$ *in case* (a) *and for* $\eta \in \mathscr{B}$ *continuous* a.e. $[\varphi^*]$ *in case* (b).

LEMMA 4. *Let* $T=\mathbf{N}$, *or* $T=\mathbf{R}_+$, *and let* $\delta, m \in T\setminus\{0\}$. *Let* $Y_{n,i,v}$; $n\delta<\tau_v\leqq (n+m)\delta$; $i=1,\ldots,\hat{y}_v[1]$ *be non-negative random variables with* $\tilde{E}(Y_{n,i,v}\mid\mathfrak{J})\leqq \gamma<\infty$ a.s.. *Then* (1.3) *implies that*

$$\lim_{n\to\infty} \varrho^{-n\delta} \sum_{n\delta<\tau_v\leqq(n+m)\delta} \sum_{i=1}^{\hat{y}_v[1]} Y_{n,i,v} = 0 \quad \text{a.s.}\,.$$

PROOF.

$$\sum_{n=0}^{\infty} \tilde{E}\left(\varrho^{-n\delta} \sum_{n\delta<\tau_v\leqq(n+m)\delta} \sum_{i=1}^{\hat{y}_v[1]} Y_{n,i,v}\mid\mathfrak{J}\right)$$
$$\leqq \gamma m \varrho^{m\delta} \sum_{v=1}^{\infty} \varrho^{-\tau_v}\hat{y}_v[1] < \infty \quad \text{a.s.}\,.$$

LEMMA 5. *Let* $T=\mathbf{N}$, *and let* $Y_{n,i}$; $n=0,1,2,\ldots$; $i=1,\ldots,\hat{z}_n[1]$ *be non-negative random variables, independent conditioned upon* $\mathfrak{F}_n=\sigma(\hat{z}_m;\, m\leqq n)$, *and such that the distribution function* $K_{\langle x_i\rangle}$ *of* $Y_{n,i}$ *depends only on the type* x_i *of particle* i. *Suppose*

$$\mu(\cdot) := \int_0^\infty y\, dK_{\langle\cdot\rangle}(y) \in \mathscr{B}\,.$$

Then (1.3) *implies that*

$$\limsup_{n\to\infty} \varrho^{-n} \sum_{i=1}^{\hat{z}_n[1]} Y_{n,i} \leqq \limsup_{n\to\infty} \varrho^{-n}\hat{z}_n[\mu] \quad \text{a.s.}\,.$$

PROOF. Write $\hat{z}_n=\hat{z}_n^*+\hat{z}_n^{**}$, where

$$\hat{z}_n^* := \sum_{v=1}^{N_{n-1}} \hat{x}_{v,n}, \quad \hat{z}_n^{**} := \sum_{N_{n-1}<v\leqq N_n} \hat{y}_v\,.$$

By (M) and (1.3)

$$\tilde{E}\hat{z}_n^*[\eta] \leqq c_7\varrho^n\varphi^*[\eta] \quad \forall\eta\geqq 0\,.$$

Therefore the proof of Lemma 2 of [1] goes through verbatim to yield

$$\limsup_{n\to\infty} \varrho^{-n} \sum_{i=1}^{\hat{z}_n^*[1]} Y_{n,i} = \limsup_{n\to\infty} \varrho^{-n} \sum_{i=1}^{\hat{z}_n^*[1]} \tilde{E}(Y_{n,i}1_{\{Y_{n,i}\leqq\varrho^n\}}\mid\mathfrak{F}_n)$$
$$\leqq \limsup_{n\to\infty} \varrho^{-n}\hat{z}_n^*[\mu] \leqq \limsup_{n\to\infty} \varrho^{-n}\hat{z}_n[\mu] \quad \text{a.s.}\,.$$

Finally, by Lemma 4

$$\lim_{n\to\infty} \varrho^{-n} \sum_{i=1}^{\hat{z}_n^{**}[1]} Y_{n,i} = 0 \quad \text{a.s.}\,.$$

PROOF OF THEOREM 2. Appealing to (3.6) and (3.7) with $\vartheta = 1$, it remains to prove

$$\limsup_{t\to\infty} \varrho^{-t}\hat{z}_t[1] \leqq \tilde{W}_\infty \quad \text{a.s.}\,.$$

First take $T = \mathbf{N}$. Let m be fixed, $Y_{n,i}$ the number of descendants at time $n+m$ of the ith particle alive at time n, and $Y_{n,i,v}$ the number of descendants at time $n+m$ of the ith of the particles that immigrated at time τ_v, $n < \tau_v \leqq n+m$. From (M),

$$\mu(x) \leqq \varrho^m c_m^+ \varphi(x) \quad \text{with } c_m^+ \to 1,\ m \to \infty,$$

so that $\mu \in \tilde{\mathscr{U}}_+$ by (3.7) with $\vartheta = \varphi$, $\xi = \mu/\varphi$. Using Lemmata 4 and 5

$$\begin{aligned}
\limsup_{n\to\infty} \varrho^{-(n+m)}\hat{z}_{n+m}[1] &\leqq \limsup_{n\to\infty} \varrho^{-(n+m)} \sum_{i=1}^{\hat{z}_n[1]} Y_{n,i} + \\
&\quad + \limsup_{n\to\infty} \varrho^{-(n+m)} \sum_{n<\tau_v\leqq n+m} \sum_{i=1}^{\hat{y}_v[1]} Y_{n,i,v} \\
&\leqq \limsup_{n\to\infty} \varrho^{-(n+m)}\hat{z}_n[\mu] \\
&= \varrho^{-m}\varphi^*[\mu]\,\tilde{W}_\infty \leqq c_m^+\,\tilde{W}_\infty \quad \text{a.s.}\,.
\end{aligned}$$

Now let $m \to \infty$.

In case (b), we first remark that for any $\delta > 0$ we can consider $\hat{z}_n^* = \hat{z}_{n\delta}$ as a discrete time immigration-branching process, with immigration times $\tau_v^* = ([\tau_v/\delta]+1)\delta$ and $\hat{y}_v^* = \hat{x}_{v,\tau_v^*}$. From (C*)

$$\tilde{E}(\hat{y}_v^*[1] \mid \mathfrak{J}) \leqq \|E^{\langle\cdot\rangle}H^\delta\| \hat{y}_v[1] \quad \text{a.s.}\,.$$

Therefore (1.3) holds with $\{\tau_v, \hat{y}_v\}$ replaced by $\{\tau_v^*, \hat{y}_v^*\}$. Defining $H_{n,i}^\delta$ for $i = 1, \ldots, z_{n\delta}$ according to (C*), we conclude from Lemma 5 and the first half of this proof that

$$\begin{aligned}
\limsup_{t\to\infty} \varrho^{-t}\hat{z}_t[1] &\leqq \limsup_{n\to\infty} \varrho^{-n\delta} \sum_{i=1}^{\hat{z}_{n\delta}[1]} H_{n,i}^\delta \\
&\leqq \limsup_{n\to\infty} \varrho^{-n\delta}\hat{z}_{n\delta}[E^{\langle\cdot\rangle}H^\delta] \\
&\leqq \|E^{\langle\cdot\rangle}H^\delta\|\,\tilde{W}_\infty \quad \text{a.s.}\,.
\end{aligned}$$

Now let $\delta \downarrow 0$.

5. Limit theorems for Bellman–Harris processes with immigration.

In this last section $\{\hat{x}_t, \boldsymbol{P}^{\hat{x}}\}$ is a supercritical Bellman–Harris process. We first state and prove the main result, assuming Lemma 1, and then prove Lemma 1.

THEOREM 3. *If* (1.3) *is satisfied, and if either* $(x \log x)$ *holds, or* $\hat{y}_\nu[\mathbf{1}] = \hat{y}_\nu[1_{\{0\}}]$, $\nu = 1, 2, \ldots$ (that is, all particles immigrating are of age zero), *then* $\mathbf{1} \in \tilde{\mathscr{U}}_+$ *and thus*

$$\lim_{t \to \infty} \varrho^{-t} \hat{z}_t[\eta] = \varphi^*[\eta] \tilde{W}_\infty$$

for all bounded a.e. *continuous* η.

PROOF. If $\hat{y}_\nu[\mathbf{1}] = \hat{y}_\nu[1_{\{0\}}]$, $\nu = 1, 2, \ldots$, the argument of section 2 shows that

$$\limsup_{t \to \infty} \varrho^{-t} \hat{z}_t[\mathbf{1}] \leqq c_3 \sup_{t \geqq N} \tilde{W}_t$$

for any integer N. If $(x \log x)$ fails to hold, then by Theorem 1 the expression on the right tends to zero a.s. as $N \to \infty$, and $\mathbf{1} \in \tilde{\mathscr{U}}_+$.

Assuming that $(x \log x)$ does hold and recalling (3.6), (3.7), it suffices to prove

$$\limsup_{t \to \infty} \varrho^{-t} \hat{z}_t[\mathbf{1}] \leqq \tilde{W}_\infty \quad \text{a.s.} \,. \tag{5.1}$$

Setting

$$M_\nu := \sup_{t \geqq \tau_\nu} \varrho^{-(t-\tau_\nu)} \hat{x}_{\nu, t}[\mathbf{1}] \,,$$

Lemma 1 implies

$$\tilde{E}(M_\nu \mid \mathfrak{Z}) \leqq c_8 \hat{y}_\nu[\mathbf{1}] \quad \text{a.s.} \,.$$

Hence, using (1.3),

$$\sum_{\nu=1}^{\infty} \varrho^{-\tau_\nu} M_\nu < \infty \quad \text{a.s.} \,.$$

Obviously

$$\limsup_{t \to \infty} \varrho^{-t} \hat{z}_t[\mathbf{1}] \leqq \sum_{\nu=1}^{N} \varrho^{-\tau_\nu} W_{\nu, \infty} + \sum_{\nu=N+1}^{\infty} \varrho^{-\tau_\nu} M_\nu$$

for all N. Now let $N \to \infty$ to obtain (5.1).

PROOF OF LEMMA 1. Assume $(x \log x)$ and define

$$\gamma := \inf\{\varphi(x) : 0 \leqq x \leqq 1\} > 0 \,,$$

$$\varkappa_N := \sup_{\hat{x}[\mathbf{1}] \geqq N} \boldsymbol{P}^{\hat{x}}(| W_\infty - \hat{x}[\varphi]| > \hat{x}[\mathbf{1}]\gamma/2) \,.$$

Fix N such that $\varkappa_N < 1$. This is possible by the branching property and the law of large number in the form of Lemma 1 of [8], the problem being the uniformity with respect to the types. From (2.5), as in section 2,

$$M \leqq c_9 \sup_{t \geqq 0} \varrho^{-t} \hat{x}_t[\varphi 1_{[0,1]}] \,. \tag{5.2}$$

Define Δ by $\varrho^{\Delta} = N$, and let

$$t_\alpha := \inf\{t \geqq \Delta : \varrho^{-t}\hat{x}_t[\varphi 1_{[0,1]}] \geqq \alpha\gamma\} \,,$$

$$\mathfrak{F}_{\Delta,\alpha} := \sigma(\hat{x}_t \,;\; \Delta \leqq t \leqq t_\alpha)$$

for $\alpha \in \mathrm{R}_+$ with the usual convention for $t_\alpha = \infty$. If $\varphi(x) \leqq c_{10}$, then $\varrho^{-t}\hat{x}_t[\varphi 1_{[0,1]}] > \alpha c_{10}$ implies that $\hat{x}_t[1_{[0,1]}] \geqq [\alpha\varrho^t] + 1$ and if $\hat{x}$ is chosen such that $\tilde{x}[1] = \hat{x}[1_{[0,1]}] = [\alpha\varrho^t] + 1$, it follows that

$$\begin{aligned} P^{\langle x\rangle}(W_\infty > \alpha\gamma/2 \mid \varrho^{-t}\hat{x}_t[\varphi 1_{[0,1]}] > \alpha c_{10}) &\geqq P^{\hat{x}}(\varrho^{-t} W_\infty > \alpha\gamma/2) \\ \geqq P^{\hat{x}}(W_\infty > \hat{x}[1]\gamma/2) \geqq P^{\hat{x}}(W_\infty - \hat{x}[\varphi] &> -\hat{x}[1]\gamma/2) \geqq 1 - \varkappa_N \,, \end{aligned}$$

$$\begin{aligned} P^{\langle x\rangle}(W_\infty > \alpha\gamma/2) &\geqq E^{\langle x\rangle} P^{\langle x\rangle}(W_\infty > \alpha\gamma/2,\, t_\alpha < \infty \mid \mathfrak{F}_{\Delta,\alpha}) \\ &\geqq (1 - \varkappa_N) P^{\langle x\rangle}(t_\alpha < \infty) \\ &\geqq (1 - \varkappa_N) P^{\langle x\rangle}\left(\sup_{t \geqq \Delta} \varrho^{-t}\hat{x}_t[\varphi 1_{[0,1]}] \geqq \alpha\gamma\right) \end{aligned}$$

for $\alpha \geqq 1$. Integration with respect to $d\alpha$ yields

$$E^{\langle x\rangle} \sup_{t \geqq \Delta} \varrho^{-t}\hat{x}_t[\varphi 1_{[0,1]}] \leqq c_{11} + c_{12} E^{\langle x\rangle} W_\infty < \infty \,,$$

and $\|E^{\langle\cdot\rangle} M\| < \infty$ now follows from (5.2), as

$$\left\| E^{\langle\cdot\rangle} \sup_{0 \leqq t \leqq \Delta} \varrho^{-t}\hat{x}_t[\varphi 1_{[0,1]}] \right\| < \infty$$

is immediate. For the converse, note that $E^{\langle x\rangle} M < \infty$ implies the uniform integrability of $\{W_t\}$ and thus $(x \log x)$.

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