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ON BOUNDS FOR THE DERIVATIVES OF A COMPLEX-VALUED FUNCTION ON A COMPACT INTERVAL

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1. Introduction.

We are here concerned with real or complex-valued functions on a compact interval I. If f is a function of this kind we denote by $\|f\|$ the essential supremum of f on I. If the nth derivative $f^{(n)}$ is mentioned, then we assume that $f^{(n-1)}$ is absolutely continuous on I and that $\|f^{(n)}\|$ is finite. Let m,n be arbitrary integers, $0<m<n$, and let ξ be arbitrary in I. Our aim is to find bounds for $|f^{(m)}(\xi)|$ when bounds for the norms of f and $f^{(n)}$, are given.

The corresponding problem for $I=\mathsf{R}$ was solved completely by Kolmogorov in 1938 (see [5]), and the case where $I=\mathsf{R}_+$ was solved by Schoenberg and Cavaretta in 1970 (see [9]). When I is finite the optimal bounds are not yet known, but various estimates have been given. The estimates for the norm of $f^{(m)}$ given by H. Cartan [1] in 1940, are the best known to the author. In the thesis [4] by the author in 1970, upper bounds for $|f^{(m)}(\xi)|$ are given where ξ is the midpoint of I and $n-m$ is even and in the case where ξ is an endpoint of I and which are more precise than the bounds given by Cartan.

In this article we obtain bounds for $f^{(m)}(\xi)$ which include those in [4]. Moreover our results give, as special cases, refinements of corresponding estimates by Gusev [3] and by Duffin & Schaeffer [2] for polynomials. Our method is based on a representation formula

$$f^{(m)}(\xi) = \sum_{i=1}^{p} a_i f(x_i) + \int_I f^{(n)}(t)K(t)\,dt \tag{a}$$

which holds for suitably chosen points x_i in I, $i=1,2,\ldots,p$. The weights a_i, $i=1,2,\ldots,p$ and the kernel K are given in terms of the points x_i, $i=1,2,\ldots,p$. From (a) we obtain the inequality

$$|f^{(m)}(\xi)| \leqq M_0 \sum_{i=1}^{p} a_i + M_n \int_I |K(t)|\,dt \tag{b}$$

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where $M_0 = \sup|f(x_i)|$, $i=1,2,\ldots,p$, and $M_n = \|f^{(n)}\|$. From (b) we get bounds of the form

$$(c) \qquad |f^{(m)}(\xi)| \leqq M_0 a_{n,m}(\xi) + M_n b_{n,m}(\xi) .$$

The values $a_{n,m}(\xi)$, $b_{n,m}(\xi)$ are given in terms of Cebysev and Zolotarev polynomials and are such that there can be equality in (c) for suitable M_0 and M_n. Our method to find suitable values of the points x_i, to calculate the weights a_i and to estimate the sum and the integral in (b) is based on ideas used in [3], [4] and [9].

Section 2.1 contains the representation formula (a) and some basic lemmas concerning the sign of the numbers a_i, $i=1,2,\ldots,p$ and the sign of the kernel K. Section 2.2 contains two lemmas by Markoff and two theorems by Gusev. Sections 2.3 and 2.4 contain more precise information of the numbers a_i, $i=1,2,\ldots,p$ and of the kernel K in the cases where $p=n$ and $p=n-1$. Section 3 contains our main theorems, including general estimates of the form (c) and refinements of the theorems by Gusev. Section 4, finally, contains some results from [4] which are now established as special cases of our theorems in section 3.

These results in [4] were obtained by similar methods using more general representation formulas but similar to (a). A paper on these representation formulas is published separately in this volume.

2. Preliminaries, requisities and basic lemmas.

2.1. *A representation formula.*

To simplify our formulas we consider the interval $[0,1]$. This restriction is not essential since every compact interval can be transformed to $[0,1]$ by a linear transformation.

Let f be $n-1$ times continuously differentiable in I and be such that $f^{(n)}$ exists a.e. and is bounded on I. Here $I=[0, 1]$. Let ξ be an arbitrary point in I and let $0<m<n$.

We give as a lemma a special case of a theorem by Peano [8].

LEMMA 2.1.1. *Let ξ belong to I and let x_i, $i=1,2,\ldots,p$ be in I, $0 \leqq x_1 \leqq x_2 \leqq \ldots \leqq x_p \leqq 1$. Let L be the linear functional defined by*

$$(1) \qquad L(f) = f^{(m)}(\xi) - \sum_{i=1}^{p} a_i f(x_i) .$$

If the numbers a_i, $i=1,2,\ldots,p$ are such that

$$(2) \qquad L(P) = 0$$

for every polynomial P of degree less than n, we have

$$L(f) = \int_I f^{(n)}(t)K(t)\,dt \tag{3}$$

where

$$K(t) = K_\xi(t) = \frac{1}{(n-1)!}L_x[(x-t)_+^{n-1}]$$

and

$$\begin{cases}(x-t)_+^{n-1} = (x-t)^{n-1}, & x \geqq t\\ (x-t)_+^{n-1} = 0 \quad , & x<t\,.\end{cases}$$

The notation L_x means that the functional is applied to $(x-t)_+^{n-1}$ considered as a function of x.

The kernel K in Lemma 2.1.1 can be expressed more precisely as

$$K(t) = \frac{1}{(n-1)!}\left[\frac{(n-1)!}{(n-1-m)!}(\xi-t)_+^{n-1-m} - \sum_{i=1}^{p} a_i(x_i-t)_+^{n-1}\right]. \tag{4}$$

Since L is linear the condition (2) in Lemma 2.1.1 is equivalent to the system $L(x^k)=0$, $k=0,1,2,\ldots,n-1$. That is,

$$\begin{cases}\sum_{i=1}^{p} a_i x_i^k = 0 & k=0,1,2,\ldots,m-1\\ \sum_{i=1}^{p} a_i x_i^k = \dfrac{k!}{(k-m)!}\xi^{k-m}, & k=m,m+1,\ldots,n-1\,.\end{cases} \tag{5}$$

For all x_i and a_i, $i=1,2,\ldots,p$ such that (5) is satisfied we thus obtain a representation formula

$$f^{(m)}(\xi) = \sum_{i=1}^{p} a_i f(x_i) + \int_I f^{(n)}(t)K(t)\,dt \tag{6}$$

where K is given by (4). From (6) we immediately obtain

$$|f^{(m)}(\xi)| \leqq M_0 \sum_{i=1}^{p} |a_i| + M_n \int_I |K(t)|\,dt \tag{7}$$

where M_0 and M_n are naturally defined.

Our aim is now to find suitable values of x_i, and to calculate the sum and the integral in (7). We start with an example.

EXAMPLE 1. $n=2$, $m=1$, $p=2$.

Take $x_1=0$, $x_2=1$ and let ξ be arbitrary in I. Our system (5) now gives $a_1=-1$, $a_2=1$. The kernel K is easily found. We have

$$\begin{cases} K(t) = t \quad , & t<\xi \\ K(t) = t-1, & t \geqq \xi \,. \end{cases}$$

Then we get that

$$\int_I |K(t)|\,dt = \tfrac{1}{2}(\xi^2+(1-\xi)^2) = (\xi-\tfrac{1}{2})^2+\tfrac{1}{4}\,,$$

and thus

$$|f'(\xi)| \leqq 2M_0+M_2((\xi-\tfrac{1}{2})^2+\tfrac{1}{4})\,,$$

where M_0 is the maximum of the absolute values of $f(0)$ and $f(1)$ and where M_2 is the norm of f'' on I.

In this example the kernel K changed sign once in I. This is a special case of the following lemma.

LEMMA 2.1.2. *Let K_ξ be given by* (4) *and* (5). *Then K_ξ changes sign at most $p-n+1$ times in I for ξ in the interior of I and at most $p-n$ times when ξ is an endpoint of I.*

PROOF. Let $h(t)$ and $g(t)$ denote the two terms in the right-hand side of (4) that is $K(t)=h(t)+g(t)$. Then h is $n-2-m$ times continuously differentiable while $h^{(n-m-1)}$ has a discontinuity at ξ, and g is $n-2$ times continuously differentiable with $g^{(n-2)}$ piecewise linear.

From the definition of K_ξ we get that

$$K_\xi^{(r)}(1) = 0, \qquad r=0,1,2,\ldots,n-2$$

and the system (5) is equivalent to

$$K_\xi^{(r)}(0) = 0, \qquad r=0,1,2,\ldots,n-2\,.$$

Suppose now that there are q points in I where K_ξ changes sign. Let us first consider the case where $0<\xi<1$ and $1<m<n-1$. Since K_ξ is differentiable in I we get that

> $K'_\xi=0$ at at least $q-1$ interior points in I and since K'_ξ is zero at the endpoints of I we get that K'_ξ has at least $q+1$ zeros in I.

Repeating the argument we get that $K_\xi^{(n-m-2)}$ has at least $q+n-m$ zeros in I, say μ zeros in $[0,\xi]$ and ν zeros in $]\xi,1]$.

By the above argument we get that the same holds for $K_\xi^{(n-m-1)}$ and for $K_\xi^{(n-m)}$. Hence $K_\xi^{(n-m)}$ has at least $q+n-m$ zeros in I.

From now on we have no trouble with the point ξ and we get that

$K_\xi^{(n-2)}$ has at least $q+n-2$ zeros in I.

If ξ is zero we get by the same argument that $K_0^{(n-m)}$ has at least $q+n-m$ zeros in $]0,1]$, but since $K_0^{(n-m)}(0)=0$ we get at least $q+n-m+1$ zeros in I and finally that $K_0^{(n-2)}$ has at least $q+n-1$ zeros in I.

The case $\xi=1$ can be handled similarly and we get that $K_1^{(n-2)}$ has at least $q+n-1$ zeros in I.

But $K_\xi^{(n-2)}$ is piecewise linear and continuous. Hence $K_\xi^{(n-2)}$ has at most one zero in every interval $[x_i, x_{i+1}]$, $i=1,2,\ldots,p-1$, and this happens only if the numbers a_i, $i=1,2,\ldots,p$ have alternating signs, or more precisely if and only if the numbers

$$\sum_{i=k}^{p} a_i(x_i-x_k), \qquad k=1,2,\ldots,p-1\ ,$$

have alternating signs. Then $K_\xi^{(n-2)}$ has at most $p-1$ zeros in I and we get that

$$\begin{aligned} p-1 &\geqq q+n-2 && \text{if } \xi \text{ is in the interior of } I \text{ and} \\ p-1 &\geqq q+n-1 && \text{if } \xi=0 \text{ or if } \xi=1\ , \end{aligned}$$

from which the statement in the lemma immediately follows.

The cases $m=1$, $m=n-1$ and the case where K_ξ or some derivative thereof vanishes on an interval can be handled similarly.

By the arguments above we also get as a corollary.

Corollary to Lemma 2.1.2. *If the numbers a_i, $i=1,2,\ldots,p$ do not have alternating signs the number of points where K_ξ changes sign is at most $p-n$ when $0<\xi<1$, and at most $p-n-1$ for $\xi=0$ or $\xi=1$.*

When studying the sign of the numbers a_i, $i=1,2,\ldots,p$ more closely we shall need some lemmas by V. A. Markov.

2.2. *Requisities.*

In this section we cite some lemmas and theorems needed in the future.

Lemma 2.2.1 (V. A. Markov). *Let*

$$G(x) = \prod_{j=1}^{s} (x-x_j), \qquad (x_i \neq x_j,\ i \neq j)$$

and let

$$G_i(x) = \frac{G(x)}{(x-x_i)}, \qquad i=1,2,\ldots,s\,.$$

Then if $G^{(m)}(z)=0$, *all the numbers*

$$G_1^{(m)}(z),\ G_2^{(m)}(z),\ldots,G_s^{(m)}(z) \text{ and } G^{(m+1)}(z)$$

have the same sign.

Lemma 2.2.2 (V. A. Markov). *Let*

$$G(x) = A\prod_{i=1}^{s}(x-a_i), \qquad H(x) = B\prod_{i=1}^{s}(x-b_i) \qquad (A>0,\ B>0)$$

and $b_1<a_1<b_2<a_2<\ldots<b_s<a_s$.

If $G^{(m)}(z)=0$, *then*

$$\frac{H^{(m)}(z)}{G^{(m+1)}(z)} > 0\,.$$

Corollary 1 to Lemma 2.2.2 (V. A Markov). *It follows immediately that the zeros of* $G^{(m)}$ *and* $H^{(m)}$ *are interlaced. This is also true when the degrees of G and H differ by unity and the zeros of G and H separate each other.*

Corollary 2 to Lemma 2.2.2. *Lemma 2.2.2 and Corollary 1 are still true when there are equalities among the inequalities in Lemma 2.2.2 but at least one of the inequalities is strict.*

Since we will use notation and arguments from Gusev [3], we prefer to cite some of the results in [3].

We say that a polynomial Q_n of degree less than or equal to n is *extremal* at the point ξ for given m if $|Q_n(x)|\leqq 1$ in I and

$$|Q_n^{(m)}(\xi)| \geqq |P_n^{(m)}(\xi)|$$

for every polynomial P_n of degree less than or equal to n with $|P_n(x)|\leqq 1$ in I.

Let $T_n(x)=\cos(n \arccos x)$.

Theorem 2.2.1 (Gusev). *On the interval I there are* $n-m+1$ *intervals, called Cebysev intervals,*

$$[\alpha_i^{(m)},\beta_i^{(m)}] \qquad (m=1,2,\ldots,n;\ i=1,2,\ldots,n-m+1)$$

at whose points the polynomial T_n *is extremal. The endpoints of the intervals are* *the zeros* $\alpha_2^{(m)},\dots,\alpha_{n-m+1}^{(m)}$ *of* $\Phi_0^{(m)}$ *and* $\alpha_1^{(m)}=0$, *and the zeros* $\beta_1^{(m)},\dots,\beta_{n-m}^{(m)}$ *of* $\Phi_n^{(m)}$ *and* $\beta_{n-m+1}^{(m)}=1$.

Here $\Phi_j(x)=(x-\tau_j)^{-1}R_{n+1}(x)$, $j=0,1,2,\dots,n$ *where* $R_{n+1}(x)=\prod_{i=0}^{n}(x-\tau_i)$ *is the resolvent of* $T_n(x)$ *and* τ_i, $i=0,1,2,\dots,n$ *are its consecutive nodes, that is, its points of maximum deviation from zero in* I.

We denote by $E_T^{n,m}$ the union of the Cebysev intervals.

THEOREM 2.2.2 (Gusev). *Between the Cebysev intervals there are open intervals* $]\beta_i^{(m)},\alpha_{i+1}^{(m)}[$, *called Zolotarev intervals,* $(m=1,2,\dots,n-1;\ i=1,2,\dots n-m)$ *at whose points polynomials of passport* $[n,n,0]$ *(denoted by* $Q_n(x,\theta)$*) are extremal and only these, and indeed each one at that point* ξ *of each interval where* $R_n^{(m)}(\xi)=0$.

Here $R_n(x)=\prod_{i=1}^{n}(x-\sigma)$ *is the resolvent of* $Q_n(x,\theta)$. $(\sigma_i^{\pm})_1^n$ *is its distribution and* θ *is the variable leading coefficient of* Q_n.

We denote by $E_Z^{n,m}$ the union of the Zolotarev intervals.

A polynomial Q_n in I is said to be of passport $\{n,n,0]$ if it is of degree n taking the maximum of its modulus in I at n points σ_i, $i=1,2,\dots,n$ with successive changes of sign.

The sequence $(\sigma_i)_1^n$ is said to be the distribution of Q_n.

For more details about polynomials of passport $[n,n,0]$ we refer to Gusev [3] and Voronovskaya [11].

2.3. $p=n$.

When $p=n$ the system (5) takes the form

$$\text{(8)}\quad \begin{aligned} &\sum_{i=1}^{n} a_i x_i^k = 0 && k=0,1,2,\dots,m-1 \\ &\sum_{i=1}^{n} a_i x_i^k = \frac{k!}{(k-m)!}\xi^{k-m}, && k=m,m+1,\dots,n-1\,. \end{aligned}$$

The lemmas in section 2.1 can now be formulated more precisely.

LEMMA 2.3.1. *The system* (8) *has the solution*

$$a_j=\frac{\Phi_j^{(m)}(\xi)}{\Phi_j(x_j)},\quad j=1,2,\dots,n\,,$$

where

$$\Phi_j(x)=\frac{1}{(x-x_j)}\prod_{i=1}^{n}(x-x_i)\,.$$

Moreover the numbers a_j, $j=1,2,\ldots,n$ *have alternating signs if and only if* ξ *belongs to* $[\alpha_i, \beta_i]$, $i=1,2,\ldots,n-m$, *where* $\alpha_1=0$ *and* α_i, $i=2,3,\ldots,n-m$ *are the consecutive zeros of* $\Phi_1^{(m)}$ *and* $\beta_{n-m}=1$ *and* β_i, $i=1,2,\ldots,n-m-1$ *are the consecutive zeros of* $\Phi_n^{(m)}$.

LEMMA 2.3.2. *Let the numbers* a_i, $i=1,2,\ldots,n$, *be the solution of the system* (8). *Then the corresponding kernel K defined by* (4) *has at most one sign variation in I when* $\xi \in]\alpha_i, \beta_i[$, $i=1,2,\ldots,n-m$, *and no sign variation when* $\xi \in [\alpha_i, \beta_{i+1}]$, $i=1,2,\ldots,n-m-1$, *or when* $\xi=0$ *or* $\xi=1$.

Here α_i *and* β_i, $i=1,2,\ldots,n-m$ *are defined in Lemma* 2.3.1.

PROOF OF LEMMA 2.3.1. The determinant of the system (8) is a Vandermonde determinant and thus (8) has a unique solution.

Let

$$R_n(x) = \prod_{i=1}^{n} (x-x_i)$$

and

$$\Phi_j(x) = \frac{1}{(x-x_j)} R_n(x), \quad j=1,2,\ldots,n\,.$$

Since Φ_j is a polynomial of degree $n-1$ we can write

$$\Phi_j(x) = \sum_{k=0}^{n-1} b_k x^k, \quad \text{for some } b_k,\ k=0,1,2,\ldots,n-1\,.$$

Hence we get by (8)

$$\Phi_j^{(m)}(\xi) = \sum_{k=m}^{n-1} b_k \frac{k!}{(k-m)!} \xi^{k-m} = \sum_{k=m}^{n-1} b_k \sum_{i=1}^{n} a_i x_i^k = \sum_{k=0}^{n-1} b_k \sum_{i=1}^{n} a_i x_i^k$$

$$= \sum_{i=1}^{n} a_i \sum_{k=0}^{n-1} b_k x_i^k = \sum_{i=1}^{n} a_i \Phi_j(x_i) = a_j \Phi_j(x_j)\,.$$

Thus the formula

$$a_j = a_j(\xi) = \frac{\Phi_j^{(m)}(\xi)}{\Phi_j(x_j)}, \quad j=1,2,\ldots,n \tag{9}$$

is proved.

From (9) we get that the numbers a_j, $j=1,2,\ldots,n$ have alternating signs if and only if the values $\Phi_j^{(m)}(\xi)$, $j=1,2,\ldots,n$ have the same sign for given ξ.

Let $q_1, q_2, \ldots, q_{n-m}$ be the succesive zeros of $R_n^{(m)}$ and let $z_{i,j}$ be the successive zeros of $\Phi_j^{(m)}$, $i=1,2,\ldots,n-m-1$; $j=1,2,\ldots,n$. Repeated use of Lemma 2.2.2 and its corollary 2 gives the following relation

$$0 < q_1 < z_{1,n} < z_{1,n-1} < \dots < z_{1,1} < q_2 < z_{2,n} < \dots < z_{2,1} < q_3 < z_{3,n} < \dots < q_{n-m-1} < z_{n-m-1,n} < \dots < z_{n-m-1,1} < q_{n-m} \leq 1 .$$

Let $z_{0,1}=0$ and $z_{n-m,n}=1$. Then in each interval $[z_{i,1}, z_{i+1,n}]$, $i=0,1,2,\dots,n-m-1$, there is a single zero of $R_n^{(m)}$ and each of the derivatives $\Phi_1^{(m)},\dots,\Phi_n^{(m)}$ preserves its sign. Applying Lemma 2.2.1 we reach the conclusion that all the numbers $\Phi_1^{(m)}(q_i),\dots,\Phi_n^{(m)}(q_i)$ have the same sign. Thus it follows that if $\xi \in [z_{i,1}, z_{i+1,n}]$ all the numbers $\Phi_j^{(m)}(\xi)$, $j=1,2,\dots,n$, have the same sign. Denoting $\alpha_i = z_{i,1}$ and $\beta_i = z_{i,n}$, $i=1,2,\dots,n-m$, we get Lemma 2.3.1.

Proof of Lemma 2.3.2. When $\xi \in]\alpha_i, \beta_i[$, $i=1,2,\dots,n-m$, the numbers $a_j(\xi)$, $j=1,2,\dots,n$, have alternating signs and hence it follows from Lemma 2.1.2 that K_ξ has at most one sign variation in I.

When $\xi \in]\beta_i, \alpha_{i+1}[$ it follows from Lemma 2.3.1 and from the corollary of Lemma 2.1.2 that K_ξ has no sign variation in I.

Since we have $a_1(\alpha_i)=0$ and $a_n(\beta_i)=0$, $i=1,2,\dots,n-m$ we can drop the point x_1 from the system (8) when $\xi=\alpha_i$, $i=1,2,\dots,n-m$, and we can drop the point x_n when $\xi=\beta_i$, $i=1,2,\dots,n-m$. Then it follows from Lemma 2.1.2 with p as $n-1$ that K_ξ has no sign variation in I when $\xi=\alpha_i$ or $\xi=\beta_i$, $i=1,2,\dots,n-m$. Thus Lemma 2.3.2 is proved.

2.4. $p=n-1$.

If we take $p=n-1$ in the system (5) we get the system

$$(10) \qquad \begin{cases} \sum_{i=1}^{n-1} a_i x_i^k = 0 & k=0,1,2,\dots,m-1 \\ \sum_{i=1}^{n-1} a_i x_i^k = \dfrac{k!}{(k-m)!}\xi^{k-m}, & k=m,m+1,\dots,n-1 \end{cases}$$

which is not always solvable.

If we drop the equation in (10) corresponding to $k=n-1$, we get a system with the unique solution

$$(11) \qquad a_j = \frac{\Phi_j^{(m)}(\xi)}{\Phi_j(x_j)}, \quad j=1,2,\dots,n-1$$

where

$$(12) \quad \Phi_j(x) = \frac{1}{(x-x_j)} \prod_{i=1}^{n-1} (x-x_i) = \frac{1}{(x-x_j)} R_{n-1}(x), \quad j=1,2,\dots,n-1 .$$

By the arguments used in the proof of formula (10) in section 2.3 it follows that the equation

$$\sum_{i-1}^{n-1} a_i x_i^{n-1} = \frac{(n-1)!}{(n-m-1)!}\xi^{n-m-1}$$

is equivalent to the equation

$$R_{n-1}^{(m)}(\xi) = 0 .$$

Hence we can state the following lemma.

LEMMA 2.4.1. *The system* (10) *has the unique solution* (11) *if and only if the values* x_i, $i=1,2,\ldots,n-1$, *are so determined that* $R_{n-1}^{(m)}(\xi)=0$, *where* R_{n-1} *is defined by* (12). *Moreover the numbers* $a_j, j=1,2,\ldots,n-1$ *have alternating signs and the corresponding kernel* K_ξ *has constant sign in* I.

The last statement follows immediately from Lemma 2.2.1 and Lemma 2.1.2.

3. Main theorems. General estimates.

3.1. $p=n$, x_i, $i=1,2,\ldots,n$, *equal to the nodes of* T_{n-1}.

We are now able to establish our first theorems. This is done with the aid of our representation formula (6) and special choices of the points x_i, $i=1,2,\ldots,p$.

Let t_i, $i=0,1,2,\ldots,n-1$ be the consecutive points of maximum deviation from zero for T_{n-1} in I, that is,

$$t_i = \sin^2\frac{i\pi}{2(n-1)}, \quad i=0,1,2,\ldots,n-1 . \tag{13}$$

Moreover let

$$R_n(x) = \prod_{i=0}^{n-1} (x-t_i) \tag{14}$$

and

$$\Phi_j(x) = \frac{1}{(x-t_j)} R_n(x), \quad j=0,1,2,\ldots,n-1 . \tag{15}$$

Let

$$a_j(\xi) = \frac{\Phi_j^{(m)}(\xi)}{\Phi_j(t_j)}, \quad j=0,1,2,\ldots,n-1 \tag{16}$$

and

$$K_\xi(t) = \frac{1}{(n-1)!}\left\{\frac{(n-1)!}{(n-m-1)!}(\xi-t)_+^{n-m-1} - \sum_{i=0}^{n-1} a_i(\xi)(t_i-t)_+^{n-1}\right\}. \tag{17}$$

Let $\alpha_1=0$ and α_i, $i=2,3,\ldots,n-m$ be the consecutive zeros of $\Phi_0^{(m)}$ and let $\beta_{n-m}=1$ and β_i, $i=1,2,\ldots,n-m-1$ be the consecutive zeros of $\Phi_{n-1}^{(m)}$.

With these notations we can now establish our first theorem.

THEOREM 3.1.1. *For every* $\xi \in [0,1]$ *and every function* f *satisfying* $|f(t_i)| \leqq M_0$, $i=0,1,\ldots n-1$ *and* $|f^{(n)}(x)| \leqq M_n$ a.e. *in* $[0,1]$ *we have*

$$|f^{(m)}(\xi)| \leqq M_0 \sum_{i=0}^{n-1} |a_i(\xi)| + M_n \int_0^1 |K_\xi(t)|\,dt\,. \tag{18}$$

Moreover we have

$$\sum_{i=0}^{n-1} |a_i(\xi)| = |T_{n-1}^{(m)}(\xi)| \tag{19}$$

for $\xi \in [\alpha_i, \beta_i]$, $i=1,2,\ldots,n-m$, *that is* $\xi \in E_T^{n-1,m}$, *and*

$$\int_0^1 |K_\xi(t)|\,dt = \frac{1}{n!}|R_n^{(m)}(\xi)| \tag{20}$$

for $\xi=0$, $\xi=1$ *or* $\xi \in [\beta_i, \alpha_{i+1}]$, $i=1,2,\ldots,n-m-1$.

PROOF. Our representation formula (6) now has the form

$$f^{(m)}(\xi) = \sum_{i=0}^{n-1} a_i(\xi) f(t_i) + \int_0^1 f^{(n)}(t) K_\xi(t)\,dt \tag{21}$$

from which (18) immediately follows.

Inserting $f=T_{n-1}$ in formula (21) we get $f^{(n)}=0$ and then (19) follows from Lemma 2.3.1.

Inserting $f=R_n$ in (21) we get $f(t_i)=0$, $i=0,1,2,\ldots,n-1$ and then (20) follows from Lemma 2.3.2.

If we take f as a polynomial of degree less then n in formula (18) we get a refinement of Theorem 2.2.1 by Gusev, analogous to a theorem by Duffin and Schaeffer [2].

THEOREM 3.1.2. *For every polynomial* P_n *of degree* $\leqq n$, *satisfying* $|P_n(\tau_i)| \leqq 1$, $i=0,1,\ldots,n$ *we have*

$$|P_n^{(m)}(\xi)| \leqq |T_n^{(m)}(\xi)|, \quad \text{for } \xi \in E_T^{n,m}\,.$$

Here τ_i, $i=0,1,2,\ldots,n$ *are the points of maximum deviation from zero for* T_n *in* $[0,1]$.

Formula (18) is especially simple in the cases where $\xi=\alpha_i$ or $\xi=\beta_i$, $i=1,\ldots,n-m$. For further use we write this special case as a corollary.

COROLLARY TO THEOREM 3.1.1. *With the assumptions made in Theorem 3.1.1 we have*

$$|f^{(m)}(\xi)| \leqq M_0|T_{n-1}^{(m)}(\xi)| + \frac{M_n}{n!}|R_n^{(m)}(\xi)|$$

for $\xi = \alpha_i$ or $\xi = \beta_i$, $i = 1, 2, \ldots, n-m$.

By differentiating the relation

$$2^{2n-3}(n-1)R_n(x) = x(x-1)T'_{n-1}(x)$$

m times, the numbers $R_n^{(m)}(\xi)$ can be written in terms of derivatives of T_{n-1}.

Because of the sign variation for K_ξ when $\xi \in]\alpha_i, \beta_i[$ we have no simple general estimates of the integral in (18). But we can at least give an example.

EXAMPLE 2. $n = m-1$.

If $m = n-1$ we have $E_T^{n,m} = [0, 1]$ and with the notations from Lemma 2.3.2 we have $[\alpha_1, \beta_1] = [0, 1]$. The kernel K_ξ has one sign variation in $[0, 1]$. Using the symmetry of the sum in the formula

$$K_\xi(t) = (\xi - t)_+^0 - \sum_{i=0}^{n-1} \frac{1}{\Phi_i(t_i)}(t_i - t)_+^{n-1}$$

and the sign variation of K''_ξ we can sketch the graph of K_ξ. If n is even and greater than two we have the situation of fig. 1.

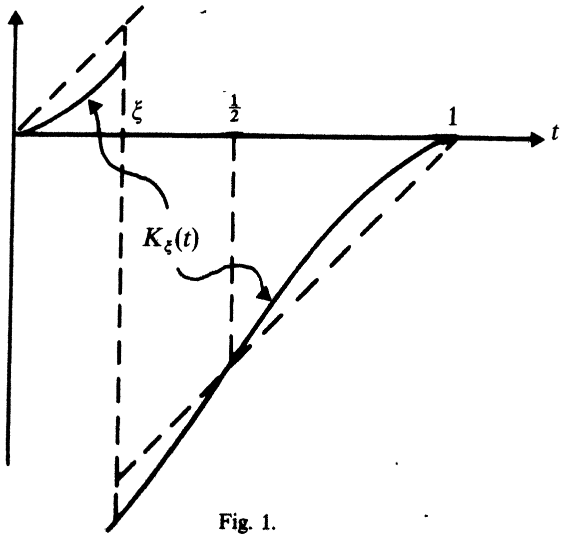

Fig. 1.

We get

$$\int_0^1 |K_\xi(t)|\,dt \leqq \frac{\xi^2+(1-\xi)^2}{2} = (\xi-\tfrac{1}{2})^2+\tfrac{1}{4}\,.$$

Since $T_{n-1}^{(n-1)}(\xi)=2^{2n-3}(n-1)!$, for every ξ we get by (18) and (19)

$$|f^{(n-1)}(\xi)| \leqq M_0 2^{2n-3}(n-1)!+M_n((\xi-\tfrac{1}{2})^2+\tfrac{1}{4})$$

for every function f satisfying the conditions in Theorem 3.1.1, with $m=n-1$. Hence example 1 is a special case of example 2.

3.2. $p=n$, $i=1,2,\ldots,n$, *equal to first* n *nodes of* T_n.

T_n has $n+1$ points of maximum deviation from zero in $[0,1]$. Let τ_i, $i=0,1,2,\ldots,n$ be the consecutive nodes of T_n, that is

$$\tau_i = \sin^2\frac{i\pi}{2n}, \quad i=0,1,2,\ldots,n\,. \tag{22}$$

Let R_n, K_ξ, a_j and Φ_j, $j=0,1,2,\ldots,n-1$, be defined by the formulas (14)–(17), corresponding to the points τ_i, $i=0,1,2,\ldots,n-1$. Then K_ξ has its support on $[0,\tau_{n-1}]$ and our representation formula is of the form

$$f^{(m)}(\xi) = \sum_{i=0}^{n-1} a_i(\xi)f(\tau_i)+\int_0^{\tau_{n-1}} f^{(n)}(t)K_\xi(t)\,dt\,. \tag{23}$$

Thus we get

THEOREM 3.2.1. *For every* ξ *in* $[0,\tau_{n-1}]$ *and every function* f *satisfying* $|f(\tau_i)|\leqq M_0$, $i=0,1,2,\ldots n-1$ *and* $|f^{(n)}(x)|\leqq M_n$ a.e. *on* $[0,\tau_{n-1}]$ *we have*

$$|f^{(m)}(\xi)| \leqq M_0\sum_{i=0}^{n-1}|a_i(\xi)|+M_n\int_0^{\tau_{n-1}}|K_\xi(t)|\,dt\,. \tag{24}$$

Moreover we have

$$\sum_{i=0}^{n-1}|a_i(\xi)| = \frac{1}{n}|(n-m)T_n^{(m)}(\xi)-\xi T_n^{(m+1)}(\xi)| \tag{25}$$

and

$$\int_0^{\tau_{n-1}}|K_\xi(t)|\,dt = \frac{1}{n!}|R_n^{(m)}(\xi)| \tag{26}$$

for $\xi=0$, $\xi=1$, *or* $\xi\in[\beta_i,\alpha_{i+1}]$, $i=1,2,\ldots,n-m-1$.

Here α_i, $i=2,3,\ldots,n-m$ *are the consecutive zeros of* $\Phi_0^{(m)}$ *and* β_i, $i=1,2,\ldots,$ $n-m-1$ *are the consecutive zeros of* $\Phi_{n-1}^{(m)}$.

PROOF. Formula (24) immediately follows from (23). Formula (26) follows from (23) by inserting $f=R_n$ and using the corollary of Lemma 2.1.2. Thus it remains to prove (25).

If we insert $f=T_n$ in (23) we get for ξ in $[\alpha_i,\beta_i]$, $i=1,2,\ldots,n-m-1$

$$\sum_{i=0}^{n-1} |a_i(\xi)| = \left|T_n^{(m)}(\xi)-2^{2n-1}n!\int_0^{\tau_{n-1}} K_\xi(t)\,dt\right| \tag{27}$$

and if we insert $f=R_n$ we get

$$n!\int_0^{\tau_{n-1}} K_\xi(t)\,dt = R_n^{(m)}(\xi)\,. \tag{28}$$

Using the relation

$$2^{2n-1}nR_n(x) = xT_n'(x)$$

we get

$$2^{2n-1}nR_n^{(m)}(\xi) = \xi T_n^{(m+1)}(\xi)+mT_n^{(m)}(\xi)\,. \tag{29}$$

From (27)–(29) now (25) follows.

We will later in section 4 use Theorem 3.2.1 in the special cases $\xi=0$ and $\xi=\frac{1}{2}$ to get precise estimates at the endpoint and at the midpoint of an arbitrary compact interval.

3.3. $p=n-1$, $\xi \in E_Z^{n-1,m}$.

Let ξ belong to $E_Z^{n-1,m}$ and let Q_{n-1}, R_{n-1} and the numbers σ_i, $i=1,2,\ldots,n-1$ be given according to Theorem 2.2.2. Then it follows from Lemma 2.4.1 and from Theorem 2.2.2 that we have a representation formula

$$f^{(m)}(\xi) = \sum_{i=1}^{n-1} a_i(\xi)f(\sigma_i)+\int_0^1 f^{(n)}(t)K_\xi(t)\,dt \tag{30}$$

where

$$a_i(\xi) = \frac{\Phi_i^{(m)}(\xi)}{\Phi_i(\sigma_i)}, \qquad i=1,2,\ldots,n-1 \tag{31}$$

and

$$K_\xi(t) = \frac{1}{(n-1)!}\left\{\frac{(n-1)!}{(n-m-1)!}(\xi-t)_+^{(n-1)}-\sum_{i=1}^{n-1} a_i(\xi)(\sigma_i-t)_+^{n-1}\right\}. \tag{32}$$

Here

$$\Phi_i(x) = \frac{1}{(x-\sigma_i)}R_{n-1}(x), \qquad i=1,2,\ldots,n-1\,.$$

According to Lemma 2.4.1 and Theorem 2.2.2 the numbers $a_i(\xi)$, $i=1,2,\ldots,n-1$ have alternating signs and we get by (30) with $f=Q_{n-1}$ that

$$\sum_{i=1}^{n-1} |a_i(\xi)| = |Q_{n-1}^{(m)}(\xi)| \,. \tag{33}$$

If we take $f(x)=xR_{n-1}(x)$ in (30) and if we use the relation $R_{n-1}^{(m)}(\xi)=0$ we get

$$n!\int_0^1 K_\xi(t)\,dt = mR_{n-1}^{(m-1)}(\xi)$$

and since K_ξ has no sign variation in $[0,1]$ we get

$$\int_0^1 |K_\xi(t)|\,dt = \frac{m}{n!}|R_{n-1}^{(m-1)}(\xi)| \,. \tag{34}$$

We now establish a theorem which in some sense is complementary to Theorem 3.1.1.

THEOREM 3.3.1. *Let $E_Z^{n-1,m}$, Q_{n-1} and R_{n-1} be given according to Theorem 2.2.2. Then we have for every ξ in $E_Z^{n-1,m}$ and for every function f satisfying $|f(\sigma_i)|\leq M_0$, $i=1,2,\ldots,n-1$ and $|f^{(n)}(x)|\leq M_n$ a.e. in $[0,1]$*

$$|f^{(m)}(\xi)| \leq M_0|Q_{n-1}^{(m)}(\xi)|+M_n\frac{m}{n!}|R_{n-1}^{(m-1)}(\xi)| \,. \tag{35}$$

If we take f as a polynomial of degree less than n in (35) we get a refinement of Theorem 2.2.2 analogous to Theorem 3.1.2.

THEOREM 3.3.2. *For every polynomial P_n of degree $\leq n$, satisfying $|P_n(\sigma_i)|\leq 1$, $i=1,2,\ldots,n$ we have*

$$|P_n^{(m)}(\xi)| \leq |Q_n^{(m)}(\xi,\theta)|, \text{ for } \xi \in E_Z^{n,m}.$$

Here $Q_n(x,\theta)$ and σ_i, $i=1,2,\ldots,n$ are defined in Theorem 2.2.2.

Formula (35) is more precise than formula (18) for ξ in $E_Z^{n-1,m}$, which easily follows from (18) by inserting $f=Q_{n-1}$.

We might use polynomials of passport $[n,n,0]$ and drop the largest node to get theorems similar to those in section 3.2 but since the method is already familiar and since we are not going to make any further conclusions we omit these details.

4. Estimates of intermediate derivatives at the endpoints and at the midpoints of an arbitrary interval.

4.1. *The endpoints.*

If we take $\xi=0$ in the formulas (24)–(26) and (29) we get

$$|f^{(m)}(0)| \leqq \frac{1}{n} T_n^{(m)}(0)\left\{(n-m)M_0+\frac{m}{n!2^{2n-1}}M_n\right\} \tag{36}$$

for every function f satisfying the conditions in Theorem 3.2.1.

Let $[0,a]$ be an arbitrary interval and let τ_i, $i=0,1,2,\ldots,n-1$ be defined by (22). If we make a change of scale so that the interval $[0,a]$ is mapped to $[0,\tau_{n-1}]$ we get by (36) estimates which essentially improve those by H. Cartan [1]. We formulate these estimates in a theorem.

THEOREM 4.1. *Let*

$$\tau_i = \sin^2\frac{i\pi}{2n}, \qquad i=0,1,2,\ldots,n-1\,.$$

For every function f on $[0,a]$ satisfying

$$\left|f\left(a\frac{\tau_i}{\tau_{n-1}}\right)\right| \leqq M_0, \qquad i=0,1,2,\ldots,n-1$$

and

$$|f^{(n)}(x)| \leqq M_n \qquad \text{a.e. } on\ [0,a]$$

we have

$$|f^{(m)}(0)| \leqq \frac{1}{n}|T_n^{(m)}(0)|\cdot \cdot\left\{(n-m)\left(\frac{a}{\tau_{n-1}}\right)^{-m}M_0+\frac{m}{n!2^{2n-1}}\left(\frac{a}{\tau_{n-1}}\right)^{n-m}M_n\right\}. \tag{37}$$

The corresponding estimates by H. Cartan are

$$\|f^{(m)}\| \leqq |T_n^{(m)}(0)|\left\{a^{-m}\|f\|+a^{n-m}\frac{\|f^{(n)}\|}{n!}\right\},$$

where the norms are taken over $[0,\mathrm{a}]$.

It is of no use to extend the interval $[0,a]$ beyond the interval $[0,\bar{a}]$, where $\bar{a}$ is the value of a giving minimum of the right-hand side in (37) that is

$$\bar{a} = \tau_{n-1}\left(\frac{M_0}{M_n}2^{2n-1}n!\right)^{1/n}. \tag{38}$$

We formulate the estimates for the special case $a=\bar{a}$ as a corollary.

COROLLARY TO THEOREM 4.1. *For every function f satisfying*

$$\left|f\left(\bar{a}\frac{\tau_i}{\tau_{n-1}}\right)\right| \leqq M_0, \quad i=0,1,2,\ldots,n-1$$

and

$$|f^{(n)}(x)| \leqq M_n \quad \text{a.e. } on\ [0,\bar{a}]$$

we have

$$|f^{(m)}(0)| \leqq C_{n,m}M_0^{1-m/n}M_n^{m/n} \tag{39}$$

where

$$C_{n,m} = |T_n^{(m)}(0)|(n!\,2^{2n-1})^{-m/n}.$$

The values in the right hand side of (39) are the same as the bounds found by Matorin [7] for a half line. See also Stechkin [10].

4.2. $\xi=\frac{1}{2}$, $n-m$ *even.*

We know that $T_n^{(k)}(\frac{1}{2})=0$ if $n-k$ is odd.

Using the notations from section 3.2 we get by the relation

$$n2^{2n-1}\Phi_0(x) = T_n'(x)$$

that

$$\Phi_0^{(m)}(\tfrac{1}{2}) = 0 \quad \text{if } n-m \text{ is even}. \tag{40}$$

Hence the formulas (24)–(26) hold for $\xi=\frac{1}{2}$, $n-m$ even. Thus we have

$$|f^{(m)}(\tfrac{1}{2})| \leqq \frac{1}{n}|T_n^{(m)}(\tfrac{1}{2})|\left\{(n-m)M_0+\frac{mM_n}{n!\,2^{2n-1}}\right\} \tag{41}$$

for every function f satisfying the conditions in Theorem 3.2.1.

Moreover we have $a_0(\frac{1}{2})=0$ and hence $K_{\frac{1}{2}}$ has its support on $[\tau_1,\tau_{n-1}]$ and thus (41) is satisfied if

$$|f(\tau_i)| \leqq M_0, \quad i=1,2,\ldots,n-1$$

and

$$|f^{(n)}(x)| \leqq M_n \quad \text{a.e. on } [\tau_1,\tau_{n-1}].$$

The interval $[\tau_1, \tau_{n-1}]$ is symmetric around the point $\frac{1}{2}$ and hence every interval $[-a, a]$ can be linearly mapped to $[\tau_1, \tau_{n-1}]$ so that 0 is mapped to $\frac{1}{2}$. Then we get by (41) estimates for an arbitrary interval $[-a, a]$.

THEOREM 4.2.1. *Let*

$$x_i = \cos\frac{i\pi}{n}, \qquad i = 1, 2, \ldots, n-1 .$$

Then for every function f on $[-a, a]$ satisfying

$$\left|f\left(a\frac{x_i}{x_1}\right)\right| \leqq M_0, \qquad i = 1, 2, \ldots, n-1$$

and

$$|f^{(n)}(x)| \leqq M_n \qquad \text{a.e. } on\ [-a, a]$$

we have when $n-m$ is even

$$|f^{(m)}(0)| \leqq \frac{1}{n}|T_n^{(m)}(\tfrac{1}{2})| \cdot \left\{(n-m)\left(\frac{2a}{\cos \pi/n}\right)^{-m} M_0 + \frac{m}{n!\, 2^{2n-1}}\left(\frac{2a}{\cos \pi/n}\right)^{n-m} M_n\right\}. \tag{42}$$

The estimates (42) essentially improve the corresponding estimates by H. Cartan.

We compare our estimates with those by Cartan in the special case where a is the value giving minimum of the right-hand side of (42).

Let a' be the value of a giving minimum in the right-hand side of (42), that is

$$a' = \cos\frac{\pi}{n}\left(\frac{M_0}{M_n} n!\, 2^{n-1}\right)^{1/n} \tag{43}$$

Then we get that the minimum of the right-hand side of (42) can be written

$$C_{n,m} M_0^{1-m/n} M_n^{m/n}$$

where

$$C_{n,m} = 2^{-2m}|T_n^{(m)}(\tfrac{1}{2})|\left(\frac{2}{n!}\right)^{m/n} .$$

Using the inequality $|T_n^{(m)}(\frac{1}{2})| < (2n)^m$ and the formula

$$n! = c_n\sqrt{n}\left(\frac{n}{e}\right)^n$$

we get

$$C_{n,m} < \left(\frac{e}{2}\right)^m \left(\frac{2}{c_n\sqrt{n}}\right)^{m/n}.$$

The values $c_n\sqrt{n}$, $n=3,4,\ldots$ are increasing and hence we get

$$C_{n,m} < \left(\frac{e}{2}\right)^m.$$

We write these results as a corollary to Theorem 4.2.

COROLLARY TO THEOREM 4.2. *For every function f satisfying*

$$\left|f\left(a'\frac{x_i}{x_1}\right)\right| \leqq M_0, \qquad i=1,2,\ldots,n-1$$

and

$$|f^{(n)}(x)| \leqq M_n \text{ a.e. } on\ [0,a'],$$

where a' is given by (43) *and $x_i=\cos i\pi/n$, $i=1,2,\ldots,n-1$, we have when $n-m$ is even*

$$|f^{(m)}(0)| < \left(\frac{e}{2}\right)^m M_0^{1-m/n} M_n^{m/n}. \tag{44}$$

The estimates by Cartan [1] were of the form (44) with the factor $(e/2)^m$ replaced by $2e^m$. The best possible bounds when the interval is the whole axis are found by Kolmogorov [5] and are of the form (44) with the factor $(e/2)^m$ replaced by the absolute constant $\pi/2$.

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