CONGRUENCES FOR THE FOURIER COEFFICIENTS OF CERTAIN MODULAR FORMS

GUNNAR DIRDAL

1.

Let

\[ \varphi(x) = \prod_{n=1}^{\infty} (1 - x^n), \]

\[ \varphi(x)^k = \sum_{n=0}^{\infty} p_k(n)x^n. \]

Then \( p_{-1}(n) = p(n) \) is the number of unrestricted partitions of \( n \). In this paper we are concerned with congruences to prime moduli, involving \( p(n) \) and the Fourier coefficients of certain modular forms of half-integral dimension.

In particular, application of Theorem 1 for the primes \( q, \ 13 \leq q \leq 23 \), gives congruences of the form

\[ \alpha_1 p\left(qn - \frac{q^2 - 1}{24}\right) + \alpha_2 p\left(\frac{n}{q}\right) \equiv \alpha_3 p_{24k-1}\left(qn - \frac{q^2 - 1}{24} - k\right) \pmod{q}, \]

where \( \alpha_1 \) and \( \alpha_2 \) are not congruent zero simultaneously.

Finally, we briefly mention the results obtained when \( p(n) \) is replaced by \( c(n) \), the Fourier coefficients of the modular invariant \( j(\tau) \).

2.

Put

\[ y = e^{\pi i t/12}, \quad x = y^{24}, \]

\[ \eta(\tau) = y \varphi(x), \quad \text{Im} \tau > 0. \]

The congruence properties of \( p(n) \) modulo \( q \) depend on the residue character of \( 24n - 1 \) modulo \( q \). Therefore we define, as in Kløve [5],

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\[ U(q; \varepsilon) \sum_n a(n)y^n = \sum_{(-n)^e = \varepsilon} a(n)y^n, \]
for any power series \( \sum_n a(n)y^n \). Here and in the following \( q \) always denote a prime \( > 3 \).

We shall examine the case \( \varepsilon = 0 \). Results similar to Theorem 1 below do exist when \( \varepsilon = \pm 1 \), but this will not be considered here.

Let \( \Gamma_0(m) \) denote the subgroup of the full modular group \( \Gamma(1) \), defined by those matrices
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, b, c, d \text{ integers}, \quad ad - bc = 1,
\]
of \( \Gamma(1) \) that satisfy \( c \equiv 0 \pmod{m} \). Further \( C^+(\Gamma_0(m), -k, \chi) \) denotes the space of modular forms of dimension \( -k \), regular in the fundamental domain, \( \Delta(\Gamma_0(m)) \), of \( \Gamma_0(m) \), except possibly at \( \tau = i\infty, \tau = 0 \) and with multiplier system \( \chi \). We denote by \( C_q \) the subspace of \( C^+(\Gamma_0(q), 0, 1) \), consisting of all modular functions which are regular at \( \tau = 0 \). \( g = g(m) \) denotes the genus of the Riemann surface \( H/\Gamma_0(m) \) (compactified) where \( H \) is the upper half plane. At a given point \( \tau_0 \) of the Riemann surface \( H/\Gamma_0(m) \), we say that \( k \) is a gap if no function exists with a pole of order \( k \) at \( \tau_0 \) and regular elsewhere on \( H/\Gamma_0(m) \). Weierstrass Gap Theorem asserts that there are just \( g \) gaps \( k \) at \( \tau_0 \), and that these satisfy \( 1 \leq k \leq 2g - 1 \). Moreover, except for finitely many \( \tau_0 \), the gaps are just the integers \( 1 \) to \( g \). Those exceptional \( \tau_0 \) for which this is not so are called Weierstrass point of \( H/\Gamma_0(m) \) (or, loosely, of \( \Gamma_0(m) \)).

No value of \( q \) is yet found for which \( i\infty \) is a Weierstrass point of \( \Gamma_0(q) \), hence it seems that the contrary is true. In any case this is so for \( q < 100 \) (Atkin [1], [2]).

It is a conjecture that the elements, \( f_i \), in a (polynomial) basis \( \mathcal{B} \) for \( C_q \) may be taken as
\[
f_i = \Omega_i(W_1, \ldots, W_{(q-1)/2}),
\]
where \( \Omega_i \) is a cyclic \((q,0)\)-isobaric polynomial and
\[
W_k = x^{k(6k-q)/q} \frac{C_{4k}(x)}{C_{2k}(x)}, \quad k \equiv 0 \pmod{q},
\]
\[
C_k(x) = \prod_{n=1}^{\infty} (1 - x^{qn-k})(1 - x^{qn-q+k}),
\]
(see Fine [4]). Hence all elements of \( \mathcal{B} \) have integral Fourier coefficients at the cusps \( \tau = i\infty \) and \( \tau = 0 \). The functions \( W_k \) were introduced by Atkin and Swinnerton-Dyer [3] and studied by Fine [4]. Since \( W_k = W_{-k} = W_{k+q} \), only \((q-1)/2\) of the \( W_k \) are different.

Suppose that \( i\infty \) is not a Weierstrass point of \( \Gamma_0(q) \). Then a basis \( \mathcal{B} \) for \( C_q \),
\[ \mathcal{B} = \{ f_i \mid g + 1 \leq l \leq 2g + 1 \}, \]
is called perfect if all the elements of \( \mathcal{B} \) has a zero at \( \tau = 0 \) and

\begin{align*}
(2.1) \quad f_i(\tau) &= q^{\theta_i(0)} \sum_{i=-l}^{\infty} a_{i,i}x^i, \\
(2.2) \quad f_i\left( -\frac{1}{q\tau} \right) &= q^{\theta_i(0)} \sum_{i=1}^{\infty} a_{i,i}^*x^i, \\
(2.3) \quad \theta(l) &= \theta_1(l) - \psi_1(l) > 0,
\end{align*}

where \( a_{i,i}, a_{i,i}^* \) and \( \theta_1(l), \psi_1(l) \) are integers, and

\[ 0 = \pi_q(a_{-l,l}). \]

\( \pi_q \) is a valuation, defined by

\[ q^{\pi_q(a)} \mid a, \quad q^{\pi_q(a) + 1} \not\mid a, \]

integer \( a \).

To each perfect basis \( \mathcal{B} \) for \( C_q \), we associate an integer \( \xi, 0 \leq \xi \leq g + 1 \). If \( \theta(2g + 1) > 1 \), \( \xi \) is given as the smallest integer such that \( \theta(l) > 1 \) when \( l > \xi + g \). Otherwise we put \( \xi = g + 1 \).

We also need the following definitions to formulate the results;

\begin{align*}
\sigma &= \frac{12}{(12, q-1)}, \\
q &= q - 1, \\
v &= \lfloor \frac{q + 11}{24} \rfloor, \\
\lambda &= \frac{(12, q-1)(q+1)}{24}, \\
\gamma_k &= \begin{cases} 
0 & \text{if } - (\lambda - 1) \leq k \leq \lambda - 1, \\
(2q - 2\sigma(\lambda + k))(-q(\lambda + k)) & \text{if } - (\lambda + (12, q-1)) \leq k \leq - \lambda,
\end{cases} \\
\lambda^* &= 3\lambda - \frac{(12, q-1)}{6} \quad \text{if } q \equiv 1, 7 \pmod{12}, \\
\zeta_k &= \begin{cases} 
(-1)^{q-1/2} & \text{if } 6(\lambda - k) = (12, q - 1), \\
0 & \text{otherwise}.
\end{cases}
\end{align*}

Further we put \( \delta_q = 0, 0, 1, 0, 6, 2, 4, 1 \) if \( q \equiv 1, 5, 7, 11, 13, 17, 19, 23 \pmod{24} \) and
\[ \mu_q = 3, 1, 2, 1 \text{ if } q \equiv 1, 5, 7, 11 \pmod{12} \text{ respectively. A rational number is called } \]
\[ q\text{-integral if the denominator is not divisible by } q. \]

**Theorem 1.** Let \( q \) be a prime \( \geq 3 \) and \( \mathcal{B} \) a perfect basis for \( C_q \). Then there exist integers \( a_k \), not all congruent zero, such that

\[
U(q; 0) \left\{ a_0 \eta^{-1}(\tau) + \eta^{-1}(q^2 \tau) \sum_{i=1}^{\xi} a_i \zeta_i \right\} \\
\equiv U(q; 0) \sum_{i=1}^{\xi} a_k \{ \eta^{24qk_i - 1}(\tau) + \zeta_k \eta^{24q^*}\lambda - 1(\tau) \} \pmod{q},
\]

where \( - (\lambda + (12, q - 1)) \leq k_i \leq \lambda - 1 \).

If, in particular, we choose all the \( k_i \) in Theorem 1 such that

\[ -\delta_q \leq k_i \leq \lambda - \mu_q, \]

then there always exists a congruence similar to the one in Theorem 1. In fact

**Theorem 2.** Let \( q \) be a prime \( \geq 3 \). Then there exist integers \( a_k \), not all congruent zero, such that

\[
a'_0 U(q; 0) \eta^{-1}(\tau) \equiv U(q; 0) \sum_{i=1}^{v} a_k' \eta^{24qk_i - 1}(\tau) \pmod{q}
\]

where

\[ -\delta_q \leq k_i \leq \lambda - \mu_q. \]

In [6], Kolberg proved Theorem 2 \( ^* \) when \( k_i = i \), using an identity of Ramanujan [12]. Our proof of Theorem 2 is just a slight modification of that of his.

3.

In this section we will apply Theorem 1 for the primes \( q, 5 \leq q \leq 23 \). The details in the construction of a perfect basis \( \mathcal{B} \) for \( C_q \) where

\[
\xi = \begin{cases} \\
0 & \text{if } q = 5, 7, 11 \\
1 & \text{if } q = 13, 17, 19, 23
\end{cases}
\]

will not be considered here, but the construction, may easily be carried out.

Hence, for \( q = 5, 7, 11 \) we obtain the well-known result

\[
U(q; 0) \eta^{-1}(\tau) \equiv 0 \pmod{q}.
\]
This was first discovered and proved by Ramanujan [13], [14]. Further, when \( q = 13, 17, 19, 23 \) Theorem 1 and (3.1) gives

\[
U(13; 0)\eta^{-1}(\tau) \equiv U(13; 0)\{\eta^{119}(\tau) + \eta^{455}(\tau)\} \pmod{13}
\]

and

\[
U(q; 0)\{\alpha_{q,k,1}\eta^{-1}(\tau) + \alpha_{q,k,2}\eta^{-1}(q^2\tau)\}
\equiv \alpha_{q,k,3}U(q; 0)\eta^{24qk-1}(\tau) \pmod{q},
\]

where the integers \( \alpha_{q,k,i}, i = 1, 2, 3 \), are given by the tables 1–4.

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Table 3.

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Table 4.

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Some of these results are known. When $q = 13$, $k = 6$ (3.2) was proved by Zuckermann [15]. Kolberg [6], [7] proved (3.2) when $q = 13$, $k = 1$; $q = 17$, $k = 1, -1, -2$; $q = 19$, $k = 1, -2, -3, -4$; and $q = 23$, $k = -1$. The other results seem to be new.
Note that not all values of \( k, -(\lambda + (12, q - 1)) \leq k \leq \lambda - 1, \) are included in the tables above. \( k = 0, -\lambda \) are excluded since (3.2) is trivial in these cases. For the other excluded \( k \)-values we have

\[ \alpha_{q, k, i} = 0, \quad \alpha_{q, k, 3} = 1, \quad i = 1, 2, \]

except if \( q = 19, k = 4 \) where

\[ U(19; 0)\{\eta^{287}(\tau) - \eta^{1007}(\tau)\} \equiv 0 \pmod{19}. \]

When applying Theorem 1 to the primes \( q > 23 \) further results are obtained, but \( \xi \) is probably \( >1 \) in these cases. In particular, there exist a perfect basis \( \mathcal{B} \) for \( C_q, q = 29, 31, \) where \( \xi = 2. \)

4.

If \( h(\tau) \) is any function in the complex variable \( \tau, \) the operator \( L_m, \) introduced by Lehner \([9]\) for each positive integer \( m, \) is given by

\[ L_m h(\tau) = m^{-1} \sum_{k=0}^{m-1} h\left(\frac{\tau + k}{m}\right). \]

It is immediately seen that \( L_m \) is linear and

\[ L_m \sum_n a(n)x^n = \sum_n a(mn)x^n. \]

Hence we easily obtain

**Lemma 1.** If \( f(\tau) \in C^+(\Gamma_0(q^2), 0, 1) \) then \( L_q f(\tau) \in C^+(\Gamma_0(q), 0, 1). \)

From Newman \([10], [11]\) we have

**Lemma 2.** If

\[ \Phi_{q^i}(\tau) = \left(\frac{\eta(q^i\tau)}{\eta(\tau)}\right)^{24/(12i, q^i - 1)}, \quad i = 1, 2, \]

then

\[ \Phi_{q^i}(\tau) \in C^+(\Gamma_0(q^i), 0, 1), \]

and

\[ \Phi_{q^i}(-1/q^i\tau) = q^{-12/(12, q^i - 1)}\Phi_{q^i}^{-1}(\tau). \]
Put

\[(4.2) \quad h_{u,v}(\tau) = \Phi_q^u(\tau)L_q \Phi_q^{u^2}(\tau), \quad u \geq 0,\]

then

\[\text{Lemma 3.}\]

\[h_{u,v}(\tau) \in C^+ (\Gamma_0(q), 0, 1),\]

and \(h_{u,v}(\tau)\) has a zero of order \(q\mu - [ -q\lambda v / q] \) at \(\tau = i\infty\) and a pole of order \(q\chi\) at \(\tau = 0\) where

\[\chi = \begin{cases} u + \lambda v & \text{if } v \geq 0 \\ u & \text{if } v < 0 \end{cases}.\]

\[\text{Proof.} \] From the definition, \(h_{u,v}(\tau)\) is seen to be regular for \(\text{Im} \, \tau > 0\), and from Lemma 1 and 2 we conclude that \(h_{u,v}(\tau)\) is invariant on \(\Gamma_0(q)\).

Now, \(\Delta (\Gamma_0(q))\) has only two inequivalent cusps, viz. \(\tau = i\infty\) and \(\tau = 0\) with the uniformizing variables \(x\) and \(e^{2\pi i (1/q\tau)}\) respectively.

\[(4.2) \quad \text{gives}\]

\[h_{u,v}(\tau) = \phi(x)^{-2\mu} \phi(x^q)^{2\mu + v} \sum_n p_{-\nu}(qn - q\mu - q\lambda v) x^n,\]

which shows that \(h_{u,v}(\tau)\) has a zero of order \(q\mu - [ -q\lambda v / q] \) at \(\tau = i\infty\).

Subjecting \(4.2\) to the transformation \(\tau \to -1/q\tau\), and applying Lemma 2 and \(4.1\) we obtain after some calculation

\[(4.3) \quad h_{u,v}(-1/q\tau) = q^{-\mu} x^{-\mu} \phi(x)^{2\mu + v} \phi(x^q)^{-2\mu} \cdot \\{ q^{-(v+1)} x^{-\mu} \phi(x^{q^2})^{-v} + d(x) \}\]

where

\[d(x) = \begin{cases} (-1)^{(v+3)(q-1)/4} + [q/4] q^{-(v+1)/2} \sum_{n=0}^{\infty} \left( n + \frac{q\lambda v}{q} \right) p_{-\nu}(n) x^n & \text{if } v \text{ is odd} \\ (-1)^{v(q-1)/4} q^{-v/2 - 1} \left( -\phi(x)^{-v} + q \sum_{n=0}^{\infty} \left( 1 - \left( n + \frac{q\lambda v}{q} \right)^2 \right) p_{-\nu}(n) x^n \right) & \text{if } v \text{ is even} \end{cases},\]

in observing that

\[\Phi_q^{2u}(-1/q^2\tau + k/q) = \Phi_q^{2u} \left( \frac{k(\tau + k'/q) - (1+k'/k)/q}{q(\tau + k'/q) - k'} \right)\]

\[= q^{-\frac{1}{4}} \left( -\frac{k'}{q} \right)^{(q-1)/2} \omega^k \phi(x) \phi(\omega x)^{-1},\]
where \( k'k \equiv -1 \pmod{q}, 1 \leq k' \leq q - 1 \) and \( \omega = e^{2\pi i/q} \). Hence (4.3) completes the proof of Lemma 3.

Now, suppose that

\[
\mathcal{B} = \{ f_l(\tau) \mid g + 1 \leq l \leq 2g + 1 \} ,
\]

is a perfect basis for \( C_q \) where \( f_l(\tau) = \Phi_q^{-1}(\tau) \) if \( l = q \). Let \( t = k(g + 1) + l, k \geq 0, \) and

(4.4) \[
F_t(\tau) = f_{g+1}^{k}(\tau)f_l(\tau) .
\]

Hence from this and (2.2)

\[
F_t(-1/q\tau) = q^{\theta_2(t)}F_t^{*}(\tau) ,
\]

where

\[
\theta_2(t) = k\theta_1(g+1) + \theta_1(l) .
\]

Further, if \( j = kg + r \geq g + 1, r = 0, \ldots, q - 1 \), put

(4.5) \[
S_j(\tau) = \begin{cases} 
\Phi_q^{-k}(\tau) & \text{if } r = 0, k \geq 1 \\
F_{r+q}(\tau)\Phi_q^{-k+1}(\tau) & \text{if } 0 < r \leq g, k \geq 1 \\
F_{r}(\tau)\Phi_q^{-k}(\tau) & \text{if } g < r < q, k \geq 0
\end{cases}
\]

and

(4.6) \[
S_j(-1/q\tau) = q^{\theta_3(j)}T_j(\tau) ,
\]

where

\[
\theta_3(j) = \begin{cases} 
k\sigma & \text{if } r = 0, k \geq 1 \\
\theta_2(r+q) + (k-1)\sigma & \text{if } 0 < r \leq g, k \geq 1 \\
\theta_2(r) + k\sigma & \text{if } g < r < q, k \geq 0 .
\end{cases}
\]

In particular we notice that \( \theta_3 \) satisfies the recursion formula

\[
\theta_3(j+q) = \theta_3(j) + \sigma .
\]

From Lemma 2, (2.1), (2.2), (4.4)–(4.6) we obtain

**Lemma 4.**

\[
T_j(\tau) \in C^{+}(\Gamma_0(q),0,1)
\]

and \( T_j(\tau) \) has a pole of order \( j \) at \( \tau = 0 \) and is otherwise regular.

Now, Lemma 3, 4 and Weierstrass Gap Theorem assert that there exist constant \( b_j \) such that
(4.7) \[ h_{0,1}(\tau) = \sum_{j=g+1}^{g^2} b_j T_j(\tau). \]

Subjecting this to the transformation $\tau \to -1/q\tau$, we obtain

(4.8) \[ q^2 h_{0,1}(-1/q\tau) = \sum_{j=g+1}^{g^2} b_j q^{2-\theta_3(j)} S_j(\tau). \]

From (4.3) we see that $q^2 h_{0,1}(-1/q\tau)$ has integral coefficients in the Fourier expansion at $\tau = i\infty$. In particular

\[ b_{g\lambda} q^{2-\theta_3(g\lambda)} = 1, \]

so that

(4.9) \[ b_{g\lambda} = q^{(q-3)/2}. \]

(4.3) also gives

\[ q^2 h_{0,1}(-1/q\tau) \equiv \Phi_{q^{-1}}(\tau) \pmod{q}, \]

and since

\[ \varphi(x^q) \equiv \varphi(x)^q \pmod{q}, \]

we obtain

\[ \Phi_{q^{-1}}(\tau) \equiv \Phi_{q^{-1}}(\tau) \pmod{q}. \]

Together with (4.5) this gives

\[ q^2 h_{0,1}(-1/q\tau) \equiv S_{g\lambda}(\tau) \pmod{q}. \]

Hence there exists a function $h(\tau)$ with integral Fourier coefficients at $\tau = i\infty$, such that

\[ q^2 h_{0,1}(-1/q\tau) = S_{g\lambda}(\tau) + qh(\tau). \]

Thus by (4.8) and (4.9)

(4.10) \[ h(\tau) = \sum_{j=g+1}^{g^2-1} b_j q^{1-\theta_3(j)} S_j(\tau). \]

Put

\[ \psi_2(t) = k\psi_1(g+1) + \psi_1(l), \]

\[ \psi_3(j) = \begin{cases} 0 & \text{if } r=0, \ k \geq 1 \\ \psi_2(r+q) & \text{if } 0 < r \leq g, \ k \geq 1 \\ \psi_2(r) & \text{if } g < r < q, \ k \geq 0, \end{cases} \]
\[ \theta(j) = \theta_3(j) - \psi_3(j) , \]

where \( t = k(g+1) + l \) and \( j = kq + r \). Note that the last identity agrees with (2.3) when \( g + 1 \leq j \leq 2g + 1 \).

If
\[ S_j(\tau) = \sum_{i=-j}^{\infty} a_{i,j} e^i , \quad a_{i,j} \text{ integers} , \]

and
\[ a_{-j,j} = q^{\theta(j) - \theta_3(j)} a'_{-j,j} , \]

then (4.10) asserts that all the
\[ b_j q^{1-\theta_3(j)} a_{-j,j} = b_j q^{1-\theta(j)} a'_{-j,j} , \quad j = g + 1, \ldots, \lambda - 1 , \]

are \( q \)-integral.

Now, \( \theta(j) > 0 \) when \( g + 1 \leq j \leq 2g + 1 \), thus \( \theta(j) > 0 \) for all \( j \geq g + 1 \). Hence we conclude that
\[
(4.11) \quad b_j = q^{\theta(j) - 1} b'_j , \quad j = g + 1, \ldots, \lambda - 1 ,
\]

where \( b'_j \) are \( q \)-integral.

Together with (4.7), (4.9), and (4.11) this gives
\[
(4.12) \quad h_{0,1}(\tau) = \sum_{j=g+1}^{q^{\lambda} - 1} b'_j q^{\theta(j) - 1} T_j(\tau) + q^{(q-3)/2} T_{q^{\lambda}}(\tau) .
\]

Let
\[ \zeta^*_u = \begin{cases} 0 & \text{if } q \equiv 1 \pmod{12} \text{ and } u = 1 \\ (-1)^{(q-1)/2} q^{a - 2} & \text{otherwise} , \end{cases} \]

(4.13) \[ H_u(\tau) = h_{u,-2a+2}(\tau) + \zeta^*_u \Phi_u(\tau) , \]

and
\[ A_q = \{ k \mid 1 \leq k \leq 2\lambda + (12, q-1) \} . \]

If \( u \in A_q \), Lemma 3, 4, (4.3) and Weierstrass Gap Theorem assert that there exist constants \( b_{j,u} \) such that
\[ H_u(\tau) = \sum_{j=g+1}^{q^{u}} b_{j,u} T_j(\tau) + \gamma_{\lambda-u} . \]

When subjecting this to the transformation \( \tau \to -1/q \tau \) and observing from (4.3) that the coefficients in the Fourier expansion of \( qH_u(-1/q \tau) \) at \( \tau = i\infty \) are integers, we conclude that
where \( b'_{j,u} \) are \( q \)-integral. Hence

\[
H_u(\tau) - \gamma_{\lambda - u} = \sum_{j=g+1}^{qu} b'_{j,u}q^{\theta(j)-1}T_j(\tau), \quad \text{if } u \in A_q.
\]

From the definition of \( \theta(j) \) and \( \zeta \), it is clear that

\[
\theta(j) > 1 \quad \text{for all } j > \xi + g.
\]

By (4.12) and (4.14), we thus have

\[
h_{0,1}(\tau) \equiv \sum_{j=g+1}^{\xi + g} b'_{j}T_j(\tau) \pmod{q},
\]

\[
H_u(\tau) - \gamma_{\lambda - u} \equiv \sum_{j=g+1}^{\xi + g} b'_{j,u}T_j(\tau) \pmod{q}, \quad u \in A_q,
\]

where \( b'_{j} \) and \( b'_{j,u} \) are \( q \)-integral. Hence, by suitable choice of the integers \( a_k \) we obtain

\[
a_0 h_{0,1}(\tau) \equiv \sum_{i=1}^{\xi} a_{\lambda - k_i} \{ H_{k_i}(\tau) - \gamma_{\lambda - k_i} \} \pmod{q},
\]

where \( k_i \in A_q \). This together with (4.2) and (4.13) gives

\[
a_0 L_q \Phi_q^2(\tau)
\]

\[
\equiv \sum_{i=1}^{\xi} a_{\lambda - k_i} \{ \Phi_q^{k_i}(\tau)L_q \Phi_q^{-2\sigma k_i} + \zeta_{\lambda - k_i} \Phi_q^{k_i}(\tau) - \gamma_{\lambda - k_i} \} \pmod{q},
\]

so that

\[
a_0 \phi(x)^q \sum_n p(qn - q\lambda)x^n
\]

\[
\equiv \sum_{i=1}^{\xi} a_{\lambda - k_i} \{ \phi(x)^{2q - 2\sigma k_i} \sum_n p_{2\sigma k_i - 2}(qn + qk_i - 2q\lambda)x^n + \zeta_{\lambda - k_i} x^{qk_i} \phi(x)^{2qk_i} - \gamma_{\lambda - k_i} \} \pmod{q}.
\]

Theorem 1 follows immediately if we put \( k_i = \lambda - k_i \) and observe that

\[
\phi(x)^q \sum_n p_k(qn + m)x^n \equiv \sum_n p_{k+qs}(qn + m)x^n \pmod{q},
\]

and

\[
U(q; 0)\eta^k(\tau) = x^{k/24} \sum_n p_k(qn + kq\lambda)x^{qn + kq\lambda}.
\]
Now, we turn to the proof of Theorem 2. Let

\[ E_{a,b}(x) = \sum_{u,v=1}^{\infty} u^a v^b x^{uv}, \]

\[ Q = 1 + 240E_{0,3}, \quad R = 1 - 504E_{0,5}, \]

\[ D = 12^3F = Q^3-R^2, \quad j = 12^3J = Q^3F^{-1}. \]

Kolberg [6] has shown

\[ D^{-[q/12]} \equiv \]

\[ f(J) \quad q \equiv 1 \pmod{12} \]

\[ Qf(J) \quad q \equiv 5 \pmod{12} \]

\[ Rf(J) \quad q \equiv 7 \pmod{12} \]

\[ QRf(J) \quad q \equiv 11 \pmod{12}, \]

where \( f(J) \) is a polynomial in \( J \) of degree \([q/12]\) and with integral coefficients. Further

\[ (4.16) \quad D^{kq-s} \equiv V^q G_k(J) \delta J \pmod{q}, \]

where \( 24s \equiv 1 \pmod{q}, \ 0 < s < q, \)

\[ \delta = \frac{d}{dx}, \]

and for \( q \equiv 1, 5, 7, 11, 13, 17, 19, 23 \pmod{24} \) respectively

\[ -V = \left\{ \begin{array}{l} Q^{-2}R^{-1}D^y \\ Q^{-1}R^{-1}D^y \\ Q^{-2}R^{-1}D^{3y+1}(J-1)f(J) \\ Q^{-1}R^{-1}D^{7y+3}(J-1)^2f(J)^3 \\ Q^{-2}R^{-1}D^y \\ Q^{-1}R^{-1}D^y \\ Q^{-2}R^{-1}D^{3y}(J-1)f(J) \\ Q^{-1}R^{-1}D^{7y}(J-1)^2f(J) \end{array} \right\} \]

\[ G_k(J) = \left\{ \begin{array}{l} J^{16v}(J-1)^{12v}f(J)^{12v-2-k} \\ J^{12v+1-k}(J-1)^{12v+2}f(J)^{12v-3k} \\ J^{16v+4}(J-1)^{6v-k}f(J)^{12v+1-2k} \\ J^{12v+4-2k}(J-1)^{6v+1-3k}f(J)^{12v+3-6k} \\ J^{16v-8}(J-1)^{12v-6}f(J)^{12v-8-k} \\ J^{12v-5-k}(J-1)^{12v-4}f(J)^{12v-6-3k} \\ J^{16v-4}(J-1)^{6v-3-k}f(J)^{12v-5-2k} \\ J^{12v-2-2k}(J-1)^{6v-2-3k}f(J)^{12v-3-6k} \end{array} \right\} \]
If
\[ B_q = \{ k \mid -\delta_q \leq k \leq \lambda - \mu_q \} \]
and \( k \in B_q \), it is easily seen that the degree of \( G_k(J) \) is less than \((v+1)q - 1\). Hence
\[ G_k(J)\delta J \equiv \delta P_k(J) + \sum_{i=1}^{\nu} c_{i, k} j^{q-1} \delta J \quad \text{(mod q)}, \]
where \( k \in B_q \) and \( P_k(J) \) is a polynomial with integral coefficients. Thus, by suitable choice of \( a''_{ki} \) we obtain
\[ \sum_{i=0}^{\nu} a''_{ki} G_k(J) \delta J \equiv \delta P(J), \quad k_i \in B_q, \]
and by (4.16)
\[ \sum_{i=0}^{\nu} a''_{ki} D^{k_i} \equiv V^q \delta P(J) \quad \text{(mod q)}. \]
Hence with \( a'_k = 12^{3k_0} - 3s a'_k \) and \( k_0 = 0 \) we get
\[ a'_0 x^{-s} \varphi(x)^{-1} \equiv \sum_{i=1}^{\nu} a'_k x^{k_0} \varphi(x)^{24k_0} \varphi(x)^{-1} + \varphi(x)^{24s-1} V^q \delta P(J) \quad \text{(mod q)}, \]
and Theorem 2 follows from this and (4.15) in observing that
\[ U(q; 0) \varphi(x)^{24s-1} V^q \delta P(J) \equiv 0 \quad \text{(mod q)}. \]

5.

In [8] Kolberg proved a result akin to Theorem 2 when \( k_i = i \) involving \( U(q; 0) j(\tau) \) instead of \( U(q; 0) \eta^{-1}(\tau) \). Here we mention the results obtained with this change. Let
\[ j(\tau) = \sum_{i=-1}^{\infty} c_i x^i, \]
\[ M_q = \{ k \mid - (12, q-1) \leq k \leq 2\lambda - 1 \}, \]
\[ N_q = \{ k \mid - (12, q-1) \leq k \leq 2\lambda - \mu_q \}. \]
If \( \mathscr{B} \) is a perfect basis for \( C_q \) we may prove quite similar as for Theorem 1 that there exist integers \( e_k \), not all congruent zero, such that
\begin{equation}
(5.1) \quad e_0 U(q; 0)(j(\tau) - c_0) \\
= U(q; 0) \sum_{i=1}^{\xi} e_{k_i}(\eta^{24qk_i}(\tau) + \zeta_{k_i - \lambda}^{q} \eta^{4q(q-1)}(\tau) - \gamma_{k_i - \lambda}) \pmod{q},
\end{equation}

where $k_i \in M_q$.

In particular, if we choose all $k_i \in N_q$ and $\xi = [q/12]$ then (5.1) is always satisfied.

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