AN APPLICATION OF A THEOREM OF HIRSBERG AND LAZAR

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Abstract.

We use a theorem of Hirsberg and Lazar to show that complex $E(3)$-spaces are $L_1$-preduals if they are finite dimensional or subspaces of $C_c(X)$-spaces containing the constants.\footnote{1}

1. Preliminaries and notations.

$A$ will be a complex Banach space. $B(a, r)$ denotes the closed ball in $A$ with center $a$ and radius $r$. We write $A_1 = B(0, 1)$. If $J$ is a linear subspace of $A$, we write for $x \in A$

$$d(x, J) = \inf \{d(x, y) : y \in J\} .$$

In the product space $A^n, H^n(A, J)$ denotes the subspace

$$H^n(A, J) = \{ (x_1, \ldots, x_n) \in A^n : \sum_{i=1}^n x_i \in J, \|x_1, \ldots, x_n\| = \sum_{i=1}^n \|x_i\| \}$$

and we write $H^n(A) = H^n(A, (0))$ ($n$ a natural number $\geq 2$). The convex hull of a set $S$ is denoted $\text{co}(S)$ and the set of extreme points of a convex set $C$ is denoted $\partial_e C$. A convex cone $C$ of $A$ is said to be hereditary if for all $x \in C$ and all $y \in A$ such that $\|x\| = \|x - y\| + \|y\|$ we have $y \in C$.

A family $\{B(a_t, r_t)\}_{t \in I}$ of closed balls in $A$ is said to have the weak intersection property if $\bigcap_{t \in I} B(f(a_t), r_t) \neq \emptyset$ for all linear functionals $f$ on $A$ with $\|f\| \leq 1$.

We say that $A$ is an $E(n)$-space for some natural number $n \geq 3$ if every family of $n$ balls in $A$ with the weak intersection property has a non-empty intersection.

The notion of $E(n)$-spaces was introduced by Hustad in [2]. (Actually he used another definition and our definition is a theorem of his). Hustad [2] proved that $E(7)$-spaces are $L_1$-preduals and Lima [4] improved this by showing that $E(4)$-spaces are $L_1$-preduals. The problem whether $E(3)$-spaces are $L_1$-preduals has been open.

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\footnote{1 Since these results were obtained we have shown that every complex $E(3)$-space is an $L_1$-predual, see Appendix.}
A closed subspace $J$ of $A$ is said to be a semi $L$-summand if for all $x \in A$, there exists a unique $y \in J$ such that $\|x - y\| = d(x, J)$ and moreover this element $y$ satisfies $\|x\| = \|y\| + \|x - y\|$. (See [3].)

2. Some finite dimensional results.

In the following we will assume that $A$ is a complex $E(3)$-space. First we will prove a lemma from which it follows that the finite dimensional case is a special case of the case treated in section 3.

**Lemma 1.** If $J$ is a $w^*$-closed hereditary subspace of $A^*$, then $J$ is a semi $L$-summand.

**Proof.** Let $(x, y) \in \partial_e H^2(A^*, J)_1$, and let $z = -(x + y) \in J$. Define $\alpha^{-1} = \|x\| + \|y\| + \|z\|$. Then

$$\alpha(x, y, z) \in H^3(A^*)_1.$$ 

Suppose that there exist $(x_j, y_j, z_j) \in H^3(A^*)_1$ such that

$$\alpha(x, y, z) = \frac{1}{2} \sum_{j=1}^2 (x_j, y_j, z_j).$$

Then we have

$$1 = \alpha(\|x\| + \|y\| + \|z\|)$$

$$= \frac{1}{2}(\|x_1 + x_2\| + \|y_1 + y_2\| + \|z_1 + z_2\|)$$

$$\leq \frac{1}{2}(\|x_1\| + \|x_2\| + \|y_1\| + \|y_2\| + \|z_1\| + \|z_2\|) \leq 1.$$ 

Hence

$$2\alpha z = z_1 + z_2$$

and similar formulas hold for $x$ and $y$. Since $J$ is hereditary, we have $z_1, z_2 \in J$. Hence

$$(x, y) = (1/2\alpha)[(x_1, y_1) + (x_2, y_2)]$$

gives us a convex combination in $H^2(A^*, J)_1$. Since $(x, y)$ is an extreme point, we must have $x_1 = y_1 = z_1 = 0$ or $(x_2, y_2, z_2) = t(x_1, y_1, z_1)$ for some $t > 0$. But this shows that

$$\alpha(x, y, z) \in \partial_e H^3(A^*)_1.$$ 

Hence by [3; Theorem 2.14] there exist $g \in \partial_e A^*_1$ and $(\lambda_1, \lambda_2, \lambda_3) \in \partial_e H^3(C)_1$ such that

$$\alpha(x, y, z) = (\lambda_1 g, \lambda_2 g, \lambda_3 g).$$

Now if $\lambda_3 = 0$ then $x + y = 0$ and if $\lambda_3 \neq 0$ then $g \in J$ and $x, y \in J$. Hence by [3; Corollary 5.13] $J$ is a semi $L$-summand. The proof is complete.

**Corollary 2.** If $e \in \partial_e A^*_1$ and $f \in \partial_e A^{**}$, then $|f(e)| = 1.$
Proof. Let $e \in \partial_e A^*_1$ and let $J = \text{span}(e)$. Then $J$ is a $w^*$-closed hereditary subspace of $A^*$. Hence by Lemma 1, $J$ is a semi $L$-summand. Let $f \in \partial_e A^**$. Since $J^\circ$ is $w^*$-closed in $A^**$, it follows from Theorem 6.11 and Corollary 6.8 in [3] that $d(f, J^\circ) = 1$. Hence $|f(e)| = 1$.

Corollary 3. If $\dim A < \infty$, then $|e(x)| = 1$ for all $x \in \partial_e A_1$ and all $e \in \partial_e A^*_1$.

Corollary 4. If $\dim A < \infty$, then $A$ is isometric to a subspace of $C_c(K)$ containing the constants for some compact Hausdorff space $K$.

Proof. Let $u \in \partial_e A_1$ and define

$$K = \{e \in \partial_e A^*_1 : e(u) = 1\}.$$  

From Corollary 3 it follows that $\partial_e A^*_1$ is $w^*$-closed. Hence $K$ is compact. The rest of the proof is obvious.

Remark. In [3] we proved that a real Banach space is an $E(3)$-space if and only if its dual space is an $E(3)$-space. This is not true for complex spaces as the following example show. In $l^2_1(C)$ the balls

$$B_1 = B((1,1), \sqrt{2} - 1), \quad B_2 = B((\frac{1}{2}(1+i), \frac{1}{2}(1-i)), 1)$$

and

$$B_3 = B((\frac{1}{2}(1-i), \frac{1}{2}(1+i)), 1)$$

have the weak intersection property and an empty intersection. In fact, if $(a,b) \in B_2 \cap B_3$ then both $a$ and $b$ are convex combinations of $\frac{1}{2}(1-i)$ and $\frac{1}{2}(1+i)$. Hence it follows that $(a,b) \notin B_1$, so the balls have empty intersection. On the other side the balls have the weak intersection property since if $(x,y) \in \partial_e A^*_1$, then we may assume $x=1$ and $|y|=1$, and a verification shows that $t(x+y) \in \cap_{i=1}^2 B_i$ where

$$t = \frac{1}{2} + \frac{2 - |x-y|}{2|x+y|}$$

if $x+y \neq 0$ and $t = 1$ if $x+y = 0$.

Let $B$ denote $C^3$ with the norm

$$||(z_1, z_2, z_3)|| = \max |z_1 \pm z_2 \pm z_3|.$$  

Let $X = \{1, 2, 3, 4\}$ and let $f_1, f_2, f_3 \in C_c(X)$ be defined as follows:

$$f_1(i) = 1 \quad \text{for all } i,$$

$$f_2(1) = f_2(2) = 1 \quad \text{and} \quad f_2(3) = f_2(4) = -1,$$

$$f_3(1) = f_3(3) = 1 \quad \text{and} \quad f_3(2) = f_3(4) = -1.$$
Let \( E = \text{span}(f_1, f_2, f_3) \), and define a map \( T : B \to E \) by
\[
T(z_1, z_2, z_3) = z_1 f_1 + z_2 f_2 + z_3 f_3.
\]
A verification shows that \( T \) is an isometry of \( B \) onto \( E \).

**Proposition 5.** The space \( E \) has the following properties:

(i) \( E \) contains the constants.
(ii) \( E \) is self-adjoint.
(iii) \( \text{Re} E \) is an \( E(3) \)-space.
(iv) \( E \) is not an \( E(3) \)-space.

**Proof.** (i) and (ii) are trivial. The map \( T \) shows that \( \text{Re} E \) is isometric to \( \ell^1_1(\mathbb{R}) \) which is an \( E(3) \)-space by [5] and [3], so (iii) follows. In order to prove (iv) it suffices by Corollary 3 to find \( e \in \partial_e E_1 \) and \( u \in \partial_e E^*_1 \) such that \( |u(e)| < 1 \). Define \( e = (\lambda_1, \lambda_2, \lambda_3) \in B \) where
\[
\lambda_1 = \frac{1}{2}(1 + i), \quad \lambda_2 = \frac{1}{2}((1 + i)/\sqrt{2} - 1) \quad \text{and} \quad \lambda_3 = \frac{1}{2}(i - (1 + i)/\sqrt{2}).
\]
Then
\[
\begin{align*}
(1) \quad & \lambda_1 + \lambda_2 + \lambda_3 = i, \\
(2) \quad & \lambda_1 + \lambda_2 - \lambda_3 = (1 + i)/\sqrt{2}, \\
(3) \quad & \lambda_1 - \lambda_2 + \lambda_3 = (1 + i)(\sqrt{2} - 1)/\sqrt{2}, \\
(4) \quad & \lambda_1 - \lambda_2 - \lambda_3 = 1.
\end{align*}
\]
Hence \( |e| = 1 \). Suppose \( (\alpha_1, \alpha_2, \alpha_3) \in B_1 \) is such that
\[
||e \pm (\alpha_1, \alpha_2, \alpha_3)|| \leq 1.
\]
Then by (1), (2) and (4):
\[
\begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 &= 0, \\
\alpha_1 + \alpha_2 - \alpha_3 &= 0, \\
\alpha_1 - \alpha_2 - \alpha_3 &= 0.
\end{align*}
\]
so \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \). Hence \( e \in \partial_e B_1 \). Define \( p_i \in E^* \) by \( p_i(f) = f(i), f \in E, i = 1, 2, 3, 4 \). Then clearly every \( u \in \partial_e E^*_1 \) is of the form \( u = zp_i \) for some \( i \) and some \( z \in \mathbb{C} \). An argument by contradiction shows that
\[
p_3 \notin \text{co}(\{zp_i : i = 1, 2, 4 \text{ and } z \in \mathbb{C} \text{ with } |z| = 1\}).
\]
Hence \( p_3 \in \partial_e E^*_1 \). But then by (3):
\[
|p_3(T(e))| = |\lambda_1 - \lambda_2 + \lambda_3| = \sqrt{2} - 1 < 1.
\]
The proof is complete.
3. The structure of $A^\ast$.

We will now assume that $A$ is a complex Banach space. We say that $A$ is an almost $E(3)$-space if for every family of three balls $\{B(a_i, r_i)\}_{i=1}^3$ in $A$ with the weak intersection property we have

$$\bigcap_{i=1}^3 B(a_i, r_i + \varepsilon) \neq \emptyset$$

for all $\varepsilon > 0$.

In the study of the properties of dual spaces of $E(3)$-spaces the following theorem will be useful.

**Theorem 6.** If $A$ is a complex Banach space, then the following properties are equivalent:

(i) $A$ is an almost $E(3)$-space.

(ii) $A^{**}$ is an $E(3)$-space.

(iii) $H^3(A^\ast)_1 = \overline{co}(\partial_e A^\ast_1 \cdot H^3(C)_1)$ (w*-closure).

(iv) $H^3(A^\ast)_1 = \overline{co}(A^\ast_1 \cdot H^3(C)_1)$ (norm-closure).

For $S \subseteq A^\ast$, $S \cdot H^3(C)_1$ denotes the set

$$\{(z_1 g, z_2 g, z_3 g) \in H^3(A^\ast)_1 : g \in S \text{ and } (z_1, z_2, z_3) \in H^3(C)_1\}$$

**Proof.** (i) $\iff$ (ii) is Theorem 2.16 in [3] and (i) $\iff$ (iii) is Theorem 2.14 in [3]. (iv) $\implies$ (iii) is trivial and the proof of (ii) $\implies$ (iv) is similar to the proof of (i) $\implies$ (iii). (See [3; Theorem 2.14].)

In [3] we proved that dual spaces of real $E(3)$-spaces were characterized by a kind of weak decomposition property. We will now give a partial extension of this result to complex spaces. First a definition.

**Definition.** A convex cone $C$ in a Banach space is said to be an $R_3$-cone if for all $x, y \in C$ there exist $z, u, v \in C$ such that

$$x = z + u \quad \text{and} \quad ||x|| = ||z|| + ||u||,$$

$$y = z + v \quad \text{and} \quad ||y|| = ||z|| + ||v||$$

and

$$||x - y|| = ||u - v|| = ||u|| + ||v||.$$

In the proof of Lemma 7 and Lemma 9 below we will use the following observation. If $F$ is a convex (nonempty) subset of $A$ such that $||x|| = 1$ for all $x \in F$, then there exists an $f \in A^\ast_1$ such that $f(x) = 1$ for all $x \in F$. In fact, we can choose $f \in A^\ast_1$ such that $||f|| = 1$ and

$$\sup \{\text{Re} f(y) : ||y|| < 1\} \leq \inf \{\text{Re} f(x) : x \in F\}.$$
Then we have
\[
\|f\| = 1 = \sup \{|f(y)| : \|y\| < 1\} = \sup \{\text{Re} f(y) : \|y\| < 1\} \\
\leq \inf \{\text{Re} f(x) : x \in F\} \leq 1.
\]

Hence \(\text{Re} f(x) = f(x)\) for all \(x \in F\).

**Lemma 7.** Let \(F\) be a proper face of \(A^*_1\) and let \(\varepsilon > 0\). If \(A\) is an almost \(E(3)\)-space and \(x, y \in \text{cone}(F) = \bigcup_{\lambda \geq 0} \lambda F\), then there exist \(z, u, v \in A^*\) such that
\[
\|z + u - x\| < \varepsilon \quad \text{and} \quad \|z\| + \|u\| < \|x\| + \varepsilon,
\]
\[
\|z + v - y\| < \varepsilon \quad \text{and} \quad \|z\| + \|v\| < \|y\| + \varepsilon
\]
and
\[
\|u\| + \|v\| < \|x - y\| + \varepsilon.
\]

**Proof.** Let \(x, y \in \text{cone}(F)\). If \(x = 0\) or \(y = 0\) then there is nothing to prove. So assume \(x \neq 0\) and \(y \neq 0\). We may assume that \(\|x\| + \|y\| + \|x - y\| = 1\) and that \(\varepsilon\) is small compared with \(\|x\|\) and \(\|y\|\). Since \((x, -y, y - x) \in H^3(A^*_1)\), there exist by Theorem 6 \(\lambda_j > 0\), \(\sum_{j=1}^m \lambda_j = 1\), \(g_j \in A^*_1\) and \((z_{1j}, z_{2j}, z_{3j}) \in H^3(C)_1\) such that
\[
\|(x, -y, y - x) - \sum_{j=1}^m \lambda_j (z_{1j} g_j, z_{2j} g_j, z_{3j} g_j)\| < \varepsilon.
\]
From (5) we get
\[
\|x - \sum_{j=1}^m \lambda_j z_{1j} g_j\| < \varepsilon, \tag{6}
\]
\[
\|y + \sum_{j=1}^m \lambda_j z_{2j} g_j\| < \varepsilon \tag{7}
\]
and
\[
\|y - x - \sum_{j=1}^m \lambda_j z_{3j} g_j\| < \varepsilon. \tag{8}
\]

Let \(f \in A_{1**}^*\) be such that \(f|_F = 1\) and let \(h \in A_{1**}^*\) be such that \(\|x - y\| = h(x - y)\). Then we get from (6), (7) and (8):
\[
\|\|x\| - \sum_{j=1}^m \lambda_j z_{1j} f(g_j)\| < \varepsilon, \tag{9}
\]
\[
\|\|y\| + \sum_{j=1}^m \lambda_j z_{2j} f(g_j)\| < \varepsilon \tag{10}
\]
and
\[
\|\|y - x\| - \sum_{j=1}^m \lambda_j z_{3j} h(g_j)\| < \varepsilon. \tag{11}
\]

By rotating all \(g_j\) and \(z_{kj}\), we may assume that \(f(g_j) \geq 0\) for all \(j\). Then we get from (9), (10) and (11):
\[
\|x\| < \varepsilon + \sum_{j=1}^m \lambda_j |z_{1j}| f(g_j) \leq \varepsilon + \sum_{j=1}^m \lambda_j |z_{1j}|, \tag{12}
\]
\[
\|y\| < \varepsilon + \sum_{j=1}^m \lambda_j |z_{2j}| f(g_j) \leq \varepsilon + \sum_{j=1}^m \lambda_j |z_{2j}|. \tag{13}
\]
(14) \[ ||x - y|| < \varepsilon + \sum_{j=1}^{m} \lambda_j |z_{3j}| \leq \varepsilon + \sum_{j=1}^{m} \lambda_j |z_{3j}|. \]

This now gives
\[
\sum_{k=1}^{3} \sum_{j=1}^{m} \lambda_j |z_{kj}| \leq 1 = ||x|| + ||y|| + ||x - y||
\]
\[
< 3\varepsilon + \sum_{k=2}^{3} \sum_{j=1}^{m} \lambda_j |z_{kj}| + \sum_{j=1}^{m} \lambda_j |z_{1j}| f(g_j)
\]
so
\[
\sum_{j=1}^{m} \lambda_j |z_{1j}| < 3\varepsilon + \sum_{j=1}^{m} \lambda_j |z_{1j}| f(g_j)
\]
and we get
\[
\sum_{j=1}^{m} \lambda_j |z_{1j}| (1 - f(g_j)) < 3\varepsilon .
\]
Hence we get
\[
|\sum_{j=1}^{m} \lambda_j z_{1j} - \sum_{j=1}^{m} \lambda_j z_{1j} f(g_j)| < 3\varepsilon
\]
and
\[
|||x|| - \sum_{j=1}^{m} \lambda_j z_{1j}| < 4\varepsilon .
\]
Similarly we get
\[
\sum_{j=1}^{m} \lambda_j |z_{2j}| < 3\varepsilon + \sum_{j=1}^{m} \lambda_j |z_{2j}| f(g_j) ,
\]
\[
|\sum_{j=1}^{m} \lambda_j z_{2j} - \sum_{j=1}^{m} \lambda_j z_{2j} f(g_j)| < 3\varepsilon ,
\]
\[
||y|| + \sum_{j=1}^{m} \lambda_j z_{2j}| < 4\varepsilon .
\]
Since
\[
\sum_{k=1}^{3} \sum_{j=1}^{m} \lambda_j |z_{kj}| \leq 1 = ||x|| + ||y|| + ||y - x||
\]
we get from (12), (13) and (14):
\[
\sum_{j=1}^{m} \lambda_j |z_{1j}| < ||x|| + 2\varepsilon ,
\]
\[
\sum_{j=1}^{m} \lambda_j |z_{2j}| < ||y|| + 2\varepsilon
\]
and
\[
\sum_{j=1}^{m} \lambda_j |z_{3j}| < ||x - y|| + 2\varepsilon .
\]
If \( \text{Im} z_{1j} \geq 0 \), write
\[
\lambda_j z_{1j} = r_j (\cos \varphi_j + i \sin \varphi_j)
\]
and if \( \text{Im} z_{1j} < 0 \), write
\[
\lambda_j z_{1j} = r_j (\cos \varphi_j - i \sin \varphi_j) .
\]
Let \( \delta, \gamma \in [-4\varepsilon, 4\varepsilon] \) be such that
\[
\sum_{j=1}^{m} \lambda_j \text{Re}(z_{1j}) = ||x|| + \delta
\]
and
\[ \sum_{j=1}^{m} \lambda_j |z_{1j}| = ||x|| + \gamma. \]

If we now compute the maximum of
\[ F(r_1, \ldots, r_m, \varphi_1, \ldots, \varphi_m) = \sum_{j=1}^{m} r_j \sin \varphi_j \]
subject to the conditions
\[ G_1(r_1, \ldots, r_m) = \sum_{j=1}^{m} r_j = ||x|| + \gamma \]
and
\[ G_2(r_1, \ldots, r_m) = \sum_{j=1}^{m} r_j \cos \varphi_j = ||x|| + \delta \]
(with \( \delta \) and \( \gamma \) fixed and \( ||x|| \leq \frac{1}{2} \)) we find
\[ F(r_1, \ldots, r_m) \leq (2||x||(\gamma - \delta) + \gamma^2 - \delta^2)^{\frac{1}{2}} < 5\epsilon^4. \]

Hence from (17) and (21) we get
\[ \sum_{j=1}^{m} \lambda_j |\text{Im} z_{1j}| < 5\epsilon^4. \quad (24) \]

Similarly we get from (20) and (22)
\[ \sum_{j=1}^{m} \lambda_j |\text{Im} z_{2j}| < 5\epsilon^4 \quad (25) \]
and from (24) and (25) we get
\[ \sum_{j=1}^{m} \lambda_j |\text{Im} z_{3j}| < 10\epsilon^4. \quad (26) \]

From (21) and (17) we also get
\[ \sum_{j=1}^{m} \lambda_j |z_{1j}| < ||x|| + 2\varepsilon \]
\[ < 6\varepsilon + \sum_{j=1}^{m} \lambda_j \text{Re}(z_{1j}). \]

Hence
\[ \sum_{j=1}^{m} \lambda_j (|z_{1j}| - \text{Re}z_{1j}) < 6\varepsilon \]
so
\[ \sum_{\text{Re}z_{1j} < 0} \lambda_j |z_{1j}| < 6\varepsilon. \quad (27) \]

Similarly we get from (20) and (22):
\[ \sum_{\text{Re}z_{2j} > 0} \lambda_j |z_{2j}| < 6\varepsilon. \quad (28) \]

We now define for \( j = 1, \ldots, m \)
\[ u_{1j} = \begin{cases} \text{Re}z_{1j} & \text{if } \text{Re}z_{1j} \geq 0 \\ 0 & \text{if } \text{Re}z_{1j} < 0 \end{cases} \]
\[ u_{2j} = \begin{cases} \text{Re}z_{2j} & \text{if } \text{Re}z_{2j} \leq 0 \\ 0 & \text{if } \text{Re}z_{2j} > 0 \end{cases} \]
and
\[ u_{3j} = - (u_{1j} + u_{2j}) . \]

For \( k = 1, 2 \) we get from (24), (25), (27) and (28)
\[ \| \sum_{j=1}^{m} \lambda_j u_{kj} g_j - \sum_{j=1}^{m} \lambda_j z_{kj} g_j \| < 6 \epsilon + 5 \epsilon^* . \]

This immediately gives
\[ \| \sum_{j=1}^{m} \lambda_j u_{3j} g_j - \sum_{j=1}^{m} \lambda_j z_{3j} g_j \| < 12 \epsilon + 10 \epsilon^* . \]

Define
\[
\begin{align*}
z & = \sum_{j=1}^{m} \lambda_j [\min(u_{1j}, -u_{2j})] g_j \\
u & = \sum_{j=1}^{m} \lambda_j [u_{1j} - \min(u_{1j}, -u_{2j})] g_j \\
v & = \sum_{j=1}^{m} \lambda_j [-u_{2j} - \min(u_{1j}, -u_{2j})] g_j .
\end{align*}
\]

Then we have
\[
\begin{align*}
z + u & = \sum_{j=1}^{m} \lambda_j u_{1j} g_j , \\
z + v & = - \sum_{j=1}^{m} \lambda_j u_{2j} g_j , \\
v - u & = \sum_{j=1}^{m} \lambda_j u_{3j} g_j .
\end{align*}
\]

From (6) and (29) we get
\[ \| z + u - x \| < 8 \epsilon + 5 \epsilon^* . \]

Similarly we get from (7) and (29)
\[ \| z + v - y \| < 8 \epsilon + 5 \epsilon^* . \]

It follows from (21), (24) and (27) that
\[
\begin{align*}
\| z \| + \| u \|
& \leq \sum_{j=1}^{m} \lambda_j [\min(u_{1j}, -u_{2j})] + |u_{1j} - \min(u_{1j}, -u_{2j})|] \\
& = \sum_{j=1}^{m} \lambda_j |u_{1j}| \\
& \leq \sum_{j=1}^{m} \lambda_j |z_{1j}| + \sum_{j=1}^{m} \lambda_j |\text{Im} z_{1j}| + \sum_{\text{Res}_{1j} < 0} \lambda_j |z_{1j}| \\
& \leq \| x \| + 8 \epsilon + 5 \epsilon^* .
\end{align*}
\]

Similarly it follows from (22), (25) and (28) that
\[ \| z \| + \| v \| \leq \| y \| + 8 \epsilon + 5 \epsilon^* \]

and it follows from (23), (26), (27) and (28) that
\[ \| v \| + \| u \| \leq \| x - y \| + 14 \epsilon + 10 \epsilon^* . \]

The proof is complete.

From Lemma 2 by the \( w^* \)-compactness of \( A^* \) and the \( w^* \)-lower semicontinuity of the dual norm we get:
Corollary 8. If $A$ is an almost $E(3)$-space, then $\text{cone}(F)$ is an $R_3$-cone for every proper face $F$ of $A^*_1$.

Let $F$ be a proper face of $A^*_1$. We say that $F$ is a split face of $\text{co}(FU - iF)$ if every element in $\text{co}(FU - iF)$ can be written in a unique way as a convex combination of an element in $F$ and an element in $-iF$. ($i$ denotes the imaginary unit.)

Lemma 9. Suppose $A$ is an almost $E(3)$-space and that $F$ is a proper face of $A^*_1$. Then $F$ is a split face of $\text{co}(FU - iF)$.

Proof. Assume for contradiction that $F$ is not a split face of $\text{co}(FU - iF)$. Then there exist $x_1, x_2, y_1, y_2 \in \text{cone}(F)$ such that $x_1 = x_2$ and

$$x_1 - iy_1 = x_2 - iy_2.$$ 

By Corollary 8 we may assume $\|x_1 - x_2\| = \|x_1\| + \|x_2\|$ and also $\|y_1 - y_2\| = \|y_1\| + \|y_2\|$. Choose $e \in A^{**}_1$ such that $e(x) = 1$ for all $x \in F$. Then we get by applying $e$ that

$$\|x_1\| - i\|y_1\| = \|x_2\| - i\|y_2\|$$

so $\|x_1\| = \|x_2\|$ and $\|y_1\| = \|y_2\|$. Since $x_1 - x_2 = i(y_1 - y_2)$ we get

$$\|x_1\| + \|x_2\| = \|x_1 - x_2\| = \|y_1 - y_2\| = \|y_1\| + \|y_2\|.$$ 

Hence we may assume $x_1, x_2, y_1, y_2 \in F$. The equation

$$\|x_1 - x_2 + iy_1 - iy_2\| = 2\|x_1 - x_2\| = 4$$

shows that there exists an $f \in A^{**}_1$ such that $f(x_1) = 1$, $f(x_2) = -1$, $f(y_1) = -i$ and $f(y_2) = i$. Now consider the following balls in $A^{**}$: $B_1 = B(a_1, \sqrt{2} - 1)$, $B_2 = B(a_2, 1)$ and $B_3 = B(a_3, 1)$ where

$$a_1 = e + f, \quad a_2 = \frac{1}{2}(1 + i)e + \frac{1}{2}(1 - i)f, \quad a_3 = \frac{1}{2}(1 - i)e + \frac{1}{2}(1 + i)f.$$ 

In order to obtain a contradiction we want to show that these three balls have the weak intersection property and an empty intersection. By Theorem 6 this is impossible since $A$ is an almost $E(3)$-space.

First we want to show that the balls have the weak intersection property. So let $z \in A^{***}_1$. If $z(e) = z(f) = 0$, then there is nothing to prove. Hence we may assume that there exists an $r \in [1, \infty)$ such that

$$r \cdot \max(|z(e)|, |z(f)|) = 1.$$ 

Now define

$$u = i(z(e) + z(f))$$
where
\[ t = \frac{2 + r|z(e) + z(f)| - r|z(e) - z(f)|}{2|z(e) + z(f)|}. \]

(If \( z(e) + z(f) = 0 \), let \( u = 0 \) and \( t = 0 \).) Since
\[ r|z(e) - z(f)| \leq 2 \leq r|z(e) + z(f)| + r|z(e) - z(f)| \]
we get \( \frac{1}{2} r \leq t \leq r \). Hence
\[
|rz(a_2) - u| \\
= |(t - \frac{1}{2} r)(z(e) + z(f)) - \frac{1}{2} ir(z(e) - z(f))| \\
\leq (t - \frac{1}{2} r)|z(e) + z(f)| + \frac{1}{2} r|z(e) - z(f)| = 1.
\]

This shows that
\[ u/r \in B(z(a_2), 1). \]

Similarly we get
\[ u/r \in B(z(a_3), 1). \]

It is easy to see that
\[ r|z(e) + z(f)| + r|z(e) - z(f)| \leq 2\sqrt{2}. \]

Hence
\[
|rz(a_1) - u| = (r - t)|z(e) + z(f)| \\
= \frac{1}{2} r(|z(e) + z(f)| + |z(e) - z(f)|) - 1 \\
\leq \sqrt{2} - 1.
\]

This shows that
\[ u/r \in B(z(a_1), \sqrt{2} - 1). \]

Hence \( \{B_i\}_{i=1}^3 \) have the weak intersection property.

Suppose that there exists \( g \in \mathcal{A}^{**} \) such that \( g \in \cap_{i=1}^3 B_i \). Then \( g \in B_3 \cap B_3 \), \( a_2(x_2) = i \) and \( a_3(x_2) = -i \) implies that \( g(x_2) = 0 \). Similarly \( g \in B_1 \cap B_2 \), \( a_1(y_1) = 1 - i \) and \( a_2(y_1) = 0 \) implies that \( g(y_1) = (1 - i)/\sqrt{2} \), and \( g \in B_1 \cap B_3 \), \( a_1(y_2) = 1 + i \) and \( a_3(y_2) = 0 \) implies that \( g(y_2) = (1 + i)/\sqrt{2} \). But then we have
\[ g(x_1) = g(x_2) + ig(y_1) - ig(y_2) = \sqrt{2}. \]

Hence
\[ a_1(x_1) - g(x_1) = 2 - \sqrt{2} > \sqrt{2} - 1. \]

This contradicts that \( g \in B_1 \). The proof is complete.
4. The application of the Hirsberg-Lazar theorem.

In this section we will assume that $A$ is an $E(3)$-space, and that $A$ is a subspace of $C_c(X)$ for some compact Hausdorff space $X$.

If $1 \in A$, let $S$ denote the state space

$$S = \{p \in A^*: p(1) = 1 = \|p\|\}.$$

If $1 \in A$, then it follows from Lemma 9 that $S$ is a split face of $\text{co}(SU - iS)$. Hence from Lemma 9 and [1; Lemma 3.3] we get:

**Proposition 10.** If $A$ is an $E(3)$-subspace of $C_c(X)$ containing the constants, then $A$ is self-adjoint.

In the next two lemma we need not assume that $A$ is containing the constants. We only assume that $A$ is a self-adjoint $E(3)$-subspace of $C_c(X)$.

**Lemma 11.** $\text{Re} A$ is an $E(3)$-space.

**Proof.** Assume $f_1, f_2, f_3 \in \text{Re} A$ and $r_1, r_2, r_3 > 0$ are such that the balls $\{B(f_i, r_i)\}_{i=1}^3$ have the weak intersection property in $\text{Re} A$. Then for each $x \in X$, $\cap_{i=1}^3 B(f_i(x), r_i) = \emptyset$. Hence by [3; Theorem 1.1]

$$|\sum_{i=1}^3 z_i f_i(x)| \leq \sum_{i=1}^3 r_i |z_i|$$

for all $(z_1, z_2, z_3) \in H^3(C)$. But then by [2; Corollary 1.4] the balls have the weak intersection property in $A$. Let $f \in \cap_{i=1}^3 B(f_i, r_i)$. Then $\text{Re} f \in \cap_{i=1}^3 B(f_i, r_i)$. This completes the proof of the lemma.

**Lemma 12.** $\text{Re} A$ is an $E(n)$-space for all $n \geq 3$.

**Proof.** By Lemma 11 $\text{Re} A$ is an $E(3)$-space. By [5; Theorem 4.1] it suffices to show that $\text{Re} A$ is an $E(4)$-space. Assume for contradiction that $\text{Re} A$ is not an $E(4)$-space. Let $\varepsilon > 0$. By [3; Corollary 4.5] there exist a linear operator $S : l_1^3(R) \to \text{Re} A$ such that

$$\|x\| \leq \|S(x)\| \leq (1 + \varepsilon)\|x\|$$

for all $x \in l_1^3(R)$ and there exist a projection $P$ in $\text{Re} A$ such that $P(\text{Re} A) = S(l_1^3(R))$ and $\|P\| \leq 1 + \varepsilon$.

Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ and define $f_i = S(e_i)$, $i = 1, 2, 3$. Then $1 \leq \|f_i\| \leq 1 + \varepsilon$ for all $i$ and for all sign:

$$3 = \|e_1 \pm e_2 \pm e_3\| \leq \|f_1 \pm f_2 \pm f_3\| \leq (1 + \varepsilon)3.$$
Choose $x_i \in X$ such that

$$
3 \leq |f_1(x_1) + f_2(x_2) + f_3(x_3)| \leq 3(1 + \varepsilon)
$$

$$
3 \leq |f_1(x_2) + f_2(x_2) + f_3(x_2)| \leq 3(1 + \varepsilon)
$$

$$
3 \leq |f_1(x_3) - f_2(x_3) + f_3(x_3)| \leq 3(1 + \varepsilon)
$$

$$
3 \leq |f_1(x_4) - f_2(x_4) - f_3(x_4)| \leq 3(1 + \varepsilon)
$$

Choose a constant $K$ such that

$$
|\lambda_1 + |\lambda_2| + |\lambda_3| \leq K \max |\lambda_1 \pm \lambda_2 \pm \lambda_3|
$$

for all $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$. Then for all $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$

$$
||\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3||
$$

$$
\geq \sup_{i=1,2,3,4} |\lambda_1 f_1(x_i) + \lambda_2 f_2(x_i) + \lambda_3 f_3(x_i)|
$$

$$
\geq \max |\lambda_1 \pm \lambda_2 \pm \lambda_3| - 2\varepsilon(|\lambda_1| + |\lambda_2| + |\lambda_3|)
$$

$$
\geq (1 - 2K\varepsilon) \max |\lambda_1 \pm \lambda_2 \pm \lambda_3|
$$

The function

$$
g(t_1, t_2, t_3) = |\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3|
$$

is for each $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ continuous and convex on $[-1 - \varepsilon, 1 + \varepsilon]^3$. Since continuous convex functions obtain their supremum at extreme points and all $\|f_i\| \leq 1 + \varepsilon$, we get

$$
|\lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x)|
$$

$$
\leq (1 + \varepsilon) \max |\lambda_1 \pm \lambda_2 \pm \lambda_3|
$$

for all $x \in X$. Let $B$ be the space above. (See Proposition 5.) Then we have just shown that the map $\hat{S}: B \to A$ defined by

$$
\hat{S}(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3
$$

satisfies

$$
(1 - 2K\varepsilon)\|u\| \leq \|\hat{S}(u)\| \leq (1 + \varepsilon)\|u\|
$$

for all $u \in B$.

Extend $P$ to a projection $\hat{P}: A \to A$ by

$$
\hat{P}(f) = P(Re f) + iP(Im f).
$$

Clearly, $\hat{P}$ is a linear projection and $\hat{P}(A) = \hat{S}(B)$. Let $f \in A$. Choose $x \in X$ such that $\|\hat{P}(f)\| = |\hat{P}(f)(x)|$ and choose $z = \cos \varphi - i \sin \varphi$ such that $\|\hat{P}(f)\| = z\hat{P}(f)(x)$. Then

$$
\|\hat{P}(f)\| = (\cos \varphi - i \sin \varphi)[P(Re f) + iP(Im f)](x)
$$

$$
= [\cos \varphi P(Re f) + \sin \varphi P(Im f)](x) +
$$

$$
+ i[\cos \varphi P(Im f) - \sin \varphi P(Re f)](x)
$$
\[ P(\cos \phi \text{ Re} f + \sin \varphi \text{ Im} f)(x) + iP(\cos \varphi \text{ Im} f - \sin \varphi \text{ Re} f)(x) \]
\[ = P(\text{Re}(zf))(x) + iP(\text{Im}(zf))(x) \]
\[ = P(\text{Re}(zf))(x) \]
\[ \leq \|P(\text{Re}(zf))\| \]
\[ \leq (1 + \varepsilon)\|\text{Re}(zf)\| \]
\[ \leq (1 + \varepsilon)\|zf\| \]
\[ = (1 + \varepsilon)\|f\|. \]

Hence \[ \|\tilde{P}\| \leq (1 + \varepsilon). \]

Let \( \{B(x_i, r_i)\}_{i=1}^3 \) be three balls in \( B \) with the weak intersection property. Then the balls \( \{B(\tilde{S}(x_i), (1 + \varepsilon)r_i)\}_{i=1}^3 \) have the weak intersection property in \( A. \) ([2; Corollary 1.4]). Since \( A \) is an \( E(3) \)-space, there exists an \( f \in \bigcap_{i=1}^3 B(\tilde{S}(x_i), (1 + \varepsilon)r_i). \)

Hence
\[ \tilde{P}(f) \in \tilde{S}(B) \cap \bigcap_{i=1}^3 B(\tilde{S}(x_i), (1 + \varepsilon)^2 r_i), \]
and
\[ \tilde{S}^{-1}(\tilde{P}(f)) \in \bigcap_{i=1}^3 B(x_i, (1 - 2K\varepsilon)(1 + \varepsilon)^2 r_i). \]
Since \( \varepsilon > 0 \) is arbitrary, \( \bigcap_{i=1}^3 B(x_i, r_i) \neq \emptyset. \) Since \( B \) is not an \( E(3) \)-space (see Proposition 5), this is a contradiction.

This completes the proof.

The above results together with Theorem 2 of Hirsberg and Lazar [1] give:

**Theorem 13.** Let \( A \) be a complex \( E(3) \)-space. If \( \dim A < \infty \) or \( A \) is a subspace of \( C_c(X) \) containing the constants, then \( A^* \) is isometric to an \( L_1(\mu) \)-space for some measure \( \mu. \)

**Remarks.** An inspection of the proof given above shows that the conclusion of Theorem 13 holds if we only assume that \( A \) is an almost \( E(3) \)-space i.e. if for every family of three balls in \( A \{B(a_i, r_i)\}_{i=1}^3 \) with the weak intersection property we have \( \bigcap_{i=1}^3 B(a_i, r_i + \varepsilon) \neq \emptyset \) for all \( \varepsilon > 0. \)

In the proof of Theorem 13 we used that \( A \) contains the constants to conclude that \( A \) is self-adjoint. It is essential in our argument that \( A \) contains the constants.

The problem whether every complex \( E(3) \)-space is an \( L_1 \) predual space is still open. We know that if \( A \) is an \( E(3) \)-space then \( A^{**} \) is an \( E(3)- \)
space [3]. Corollary 2 indicate that it might be possible to imbed $A^{**}$ into a $C_c(K)$ space such that the image-space contains the constants.

In the case that $A$ is an $E(4)$-space the argument in Lemma 1 shows that every $w^*$-closed hereditary subspace of $A^{***}$ is an $L$-summand (see [3]) from which it follows that $|f(e)| = 1$ for all $e \in \partial_o A_1^{**}$ and all $f \in \partial_o A_1^{***}$. Hence we can apply Theorem 13 and get that $A^{**}$ is an $L_1$-predual space. But then also $A$ is an $L_1$-predual space. This gives a new proof of the result that $A$ is an $E(4)$-space if and only if $A$ is an $L_1$-predual space.

Almost the same results that Hirsberg and Lazar obtained in [1] were independently obtained by Fuhr and Phelps [8]. See also Lacey [7].

If we combine Theorem 13 with the results in [2] and [5] we get:

**Theorem 14.** If $A$ is finite dimensional or $A$ is a subspace of $C_c(X)$ containing the constants then the following statements are equivalent:

(i) Every linear operator $T : H^1(C) \to A$ admits for every $\varepsilon > 0$ an extension $\tilde{T} : H^1(C) \to A$ such that $\|\tilde{T}\| \leq (1 + \varepsilon)\|T\|.$

(ii) For an arbitrary compact linear operator $T$ from a Banach space $X$ into $A$ and for every Banach space $Y \supseteq X$ and every $\varepsilon > 0$, the operator $T$ admits an extension $\tilde{T} : Y \to A$ such that $\|\tilde{T}\| \leq (1 + \varepsilon)\|T\|.$

**Appendix added June 18, 1976.**

We prove that complex $E(3)$ spaces are $L_1$-predual spaces.

**Theorem 15.** Let $A$ be an almost $E(3)$-space and let $J$ be a closed subspace of $A$ such that $J^0$ is a semi $L$-summand in $A^*$. Let $r_i > 0$ and let $x_i \in A$ be such that $d(x_i, J) \leq r_i$ for $i = 1, 2$ and $\|x_1 - x_2\| \leq r_1 + r_2$. Then for every $\varepsilon > 0$ there exists an $a \in B(x_1, r_1) \cap B(x_2, r_2)$ such that $d(a J) \leq \varepsilon$.

**Proof.** Let

$$0 < \theta \leq \min\{(r_i^2 + \varepsilon^2)^{\frac{1}{2}} - r_i : i = 1, 2\}.$$

By [3; Theorem 6.10] there exists an

$$x \in J \cap B(x_1, r_1 + \theta) \cap B(x_2, r_2 + \theta).$$

By [3; Lemma 6.4] the balls $B(x, \varepsilon)$, $B(x_1, r_1)$ and $B(x_2, r_2)$ have the weak intersection property. Now the same argument as in the proof of [4; Proposition 4.4] shows that there exists an

$$a \in B(x, 2\varepsilon) \cap B(x_1, r_1) \cap B(x_2, r_2).$$

The proof is complete.
An inspection of the proof of [3; Corollary 6.8] shows that from Theorem 15 we get the following Corollary.

COROLLARY 16. Let $A$ be an almost $E(3)$-space and let $e \in \partial_e A_1$. If $J$ is a closed subspace of $A$ such that $J^0$ is a semi $L$-summand, then $d(e,J) = 1$.

THEOREM 17. Let $A$ be a complex $E(3)$-space. Then $A^*$ is isometric to an $L_1(\mu)$-space for some measure $\mu$.

PROOF. Suppose first that the unit ball of $A$ contains an extreme point $e$ and let

$$F = \{f \in A^* : \|f\| = f(e) = 1\}.$$ 

As in Corollary 2 it follows from Lemma 1 and Corollary 16 that $|f(e)| = 1$ for every $f \in \partial_e A_1^*$. Hence the map $S : A \to C_0(F)$ defined by $S(x)(f) = f(x)$ is an isometry into and $S(e) = 1$. From Theorem 13 we get that $A$ is an $L_1$-predual space. If $A_1$ does not contain an extreme point, then by Theorem 6 and the argument above, $A^{**}$ is a predual $L_1$-space and hence also $A$ is a predual $L_1$-space. The proof is complete.

REMARKS. Theorem 17 shows that the initial requirement on $A$ in Theorem 14 is superfluous.

Theorem 17 solve problems 2 and 3 of Hustad [2]. In both problems the best possible number is 3.

REFERENCES