MATH. SCAND. 38 (1976), 321–324

A NOTE ON INFINITE DIMENSIONAL CONVEX SETS

HEINRICH V. WEIZSÄCKER

The main result of this note is the following

THEOREM. *Let K be an infinite dimensional compact convex set in a locally convex linear space E.*[1] *Then there is a Radon probability measure p on K and a subset C of K whose closure is metrizable such that the following is true:*

a) $\operatorname{supp} p \subseteq C$, *$C$ is convex and a G_δ set in its closure.*
b) $p(L)=0$ *for any compact convex L contained in C.*
c) *C does not contain the barycenter of p.*

This answers in the negative some questions of F. Topsøe ([7], remark on p. 11) and of J. P. R. Christensen ("Is a convex G_δ in a compact convex subset of a locally convex metrizable space measure convex (cf. [2, p. 130]) ?"). At the end of this note we shall see that the set C in the theorem cannot be chosen to be relatively open in its closure. From this it follows that C — though a convex G_δ — is not the insection of relatively open convex subsets of its closure.

The proof of the theorem is based on the following lemma which is of independent interest. It reduces the proof to the construction of a single example of a set K with the properties described in the theorem.

LEMMA. *Let K_1 be a compact metrizable subset of a locally convex space E_1 and let K_2 be a compact convex infinite dimensional subset of a topological linear space E_2. Then there is an affine homeomorphism from the closed convex hull of K_1 to a subset of K_2.*

PROOF OF THE LEMMA. We consider only the case where K_1 in addition is convex. (This is the case we need for the proof of the theorem.) The general case may be deduced from this by Theorem 5.2 in [6].

---

[1] I.e. the linear span $\operatorname{sp} K$ of K in E is infinite dimensional. For the assumption of local convexity cf. the remark after the proof of the lemma.

Received November 24, 1975.

First we embed K_1 into the sequence space l_1 with the norm topology using the ideas of [6]. Since K_1 is compact and metrizable the space $A(K_1)$ of all continuous affine functions on K_1 with the sup-norm is separable. Choose a total sequence $(f_n)_{n\in\mathsf{N}}$ in $A(K_1)$ such that

$$0 \leqq \inf_{\substack{x\in K_1\\ n\in\mathsf{N}}} f_n(x) \leqq \sup_{\substack{x\in K_1\\ n\in\mathsf{N}}} f_n(x) \leqq 1 .$$

Consider the map

$$\psi\colon K_1 \to H = \{(\alpha_n)_{n\in\mathsf{N}} \in l_1 \mid 0 \leqq \alpha_n \leqq 2^{-n} \;\; \forall n \in \mathsf{N}\}$$

sending x to the sequence $(2^{-n}f_n(x))_{n\in\mathsf{N}}$. ψ is affine, continuous (because of dominated convergence) and injective, since $A(K_1)$ and hence $(f_n)_{n\in\mathsf{N}}$ separates points in K_1 by the Hahn–Banach theorem.

Now we embed H into K_2. Without loss of generality we may assume that $0 \in K_2$. Let M be the circled hull of K_2 and let $\|\cdot\|_M$ be the Minkowski functional of M on sp$M$. Since M is compact (and in particular bounded) in E_2, the topology induced by $\|\cdot\|_M$ is finer than the original one and $(\mathrm{sp}M, \|\cdot\|_M)$ is an infinite dimensional Banach space (see e.g. 7C, p. 64 in [4]). Then by Corollary IV.3.10 in [5] there is a closed infinite dimensional subspace F of this Banach space which has a basis $(y_n)_{n\in\mathsf{N}}$, i.e. for any element y of F there is a unique sequence $(\alpha_n)_{n\in\mathsf{N}}$ of scalars such that $y = \sum_{n\in\mathsf{N}} \alpha_n y_n$. (This idea of using the existence of basis sequences in general Banach spaces shortens considerably the author's original argument and is due to H. Pfister.) We may assume $y_n \in K_2$ for all n, multiplying each y_n by a suitable scalar if necessary. Because of

$$\|\textstyle\sum_{n\in\mathsf{N}} \alpha_n y_n\|_M \leqq \sum_{n\in\mathsf{N}} |\alpha_n| \, \|y_n\|_M \leqq \|(\alpha_n)\|_1 ,$$

we can define a linear map $T\colon (\alpha_n) \mapsto \sum_{n\in\mathsf{N}} \alpha_n y_n$ from l_1 to F which is continuous for the norm topology and hence for the topology induced by E_2 on F.

Further, by the uniqueness of the coefficients T is injective. Finally, since $0 \in K_2$ and K_2 is convex and compact we have $T((\alpha_n)) \in K_2$ for all sequences (α_n) in H. Thus $T \circ \psi$ is an affine injective continuous map and hence by the compactness of K_1 an affine homeomorphism from K_1 into K_2.

Remark. In view of the fact that very little is known about compact convex sets in non locally convex spaces it is worthwhile to note that we do not suppose E_2 to be locally convex. Consequently, in the theorem we need the local convexity of E only in order to have an obvious meaning of the term "barycenter" in part c).

PROOF OF THE THEOREM. First let us recall the following geometric fact (see [1]): Let K be a convex set in a linear space E and let t be a fixed element of K. Then the relative complement $C_t := K \setminus \mathrm{face}_t$ of the set

$$\mathrm{face}_t = \bigcup_{\varepsilon>0} \{x \in K : t+\varepsilon(t-x) \in K\}$$

is convex. If in addition K is closed with respect to a linear topology on E, then the equality

$$C_t = \bigcap_{n\in\mathbb{N}} \{x \in K : t+n^{-1}(t-x) \notin K\}$$

shows that C_t is a G_δ subset of K.

The following example of a set K with the properties in the theorem is a slight modification of the example of a convex set which is not measure convex, given in [2, p. 130]. Let E be the linear space of all finite signed Baire measures on the unit interval endowed with the "weak topology" $\sigma(E, C([0,1]))$. Let K be the compact convex metrizable set of all probability measures on $[0,1]$ and let t be the Lebesgue measure. Let p be the (unique) probability supported by the compact set D of all Dirac measures on $[0,1]$ representing t. Then the set C_t defined as above is a convex G_δ in K. Further, for any $n \in \mathbb{N}$ and any Dirac measure $\delta_x (x \in [0,1])$ the measure $t+n^{-1}(t-\delta_x)$ is not in K because of the orthogonality of the measures t and δ_x. So we have $\mathrm{supp}\, p = D \subsetneqq C_t$ which proves property a) in the theorem. Property c) is trivially satisfied by the definition of p and C_t. Finally, for property b) let L be a compact convex subset of C_t and suppose, if possible, that $p(L) > 0$. Then we must have $p(L \cap D) > 0$, that is, there is a compact set $A \subsetneqq [0,1]$ such that $t(A) > 0$ and $\delta_x \in L$ for all $x \in A$. Being compact and convex, L then contains all probability measures supported by A and in particular the measure

$$t_A := t(A)^{-1} t(A \cap .) .$$

But it is easy to check that $t + t(A) \cdot (t - t_A)$ is a probability measure and so t_A is not in C_t in contradiction to $L \subsetneqq C_t$ and $t_A \in L$. Thus K has the properties described in the theorem and by the lemma the proof of the theorem is complete.

In order to see what happens if C is relatively open in its closure we notice the following simple but useful generalization of Milman's theorem. Due to lack of reference we give a proof.

PROPOSITION. *Let Y be a convex locally compact subset of a locally convex linear space E. Then for any compact subset L of Y the closed convex* *hull* $\overline{\text{co}}L$ *of* L *in* E *is compact and contained in* Y (*that is,* Y *is a Krein set in the terminology of* [3]).

PROOF. Every point of Y has in the relative topology of Y a base of compact convex neighbourhoods. If $L \subseteq Y$ is compact there is a finite number $K_1, \ldots, K_n$ of compact convex subsets of Y such that $L \subseteq \bigcup_{i=1}^n K_i$. Then the convex hull M of this union is compact and contained in Y. In particular we have $\overline{\text{co}}L \subseteq M \subseteq Y$ which completes the proof.

REMARK. Note that the proof uses only the local convexity of the relative topology.

Now assume that a convex set C is relatively open in its compact closure and that p is a Radon probability measure such that $\text{supp}\, p \subseteq C$ or just $p(C) = 1$. Then C is locally compact and from the proposition we get

$$p(C) = \sup\{p(L) : L \subseteq C,\ L \text{ compact and convex}\},$$

and C contains the barycenter of p by Theorem 2G in [3].

ACKNOWLEDGEMENT. The author wishes to express his thanks for the interesting discussions on this subject he had with F. Topsøe and J. P. R. Christensen at Oberwolfach in June 1975 and also later with H. Pfister in München.

LITERATURE

1. E. M. Alfsen, *On the geometry of Choquet simplexes*, Math. Scand. 15 (1964), 97–110.
2. E. M. Alfsen, *Compact convex sets and boundary integrals*, Ergebnisse der Mathematik Bd. 57, Springer-Verlag, Berlin-Heidelberg-New York, 1972.
3. D. H. Fremlin and J. D. Pryce, *Semiextremal sets and measure representation*. Proc. London Math. Soc. 28 (1974), 502–520.
4. J. Kelley and I. Namioka, *Linear topological spaces*. Van Nostrand, Princeton, 1968.
5. J. T. Marti, *Introduction the theory of bases* (Springer tracts in Natural Philosophy 18) Springer-Verlag, Berlin-Heidelberg-New York, 1969.
6. M. A. Rieffel, *The Radon-Nikodym theorem for the Bochner integral*, Trans Amer. Math. Soc. 131 (1968) 466–487.
7. F. Topsøe, *Some special results on convergent sequences of Radon measures*, Københavns Universitet, Mathematisk institut. Preprint series 1975 No. 3.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT MÜNCHEN,
WEST-GERMANY