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A SIMPLE PROOF OF THE STONE-WEIERSTRASS THEOREM FOR CCR-ALGEBRAS WITH HAUSDORFF SPECTRUM

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The purpose of this note is to give a simple proof of the following.

THEOREM. *Let A be a* CCR-*algebra with Hausdorff spectrum and $P(A)$ the set of pure states of A. If B is a* C*-*subalgebra of A which separates $P(A)\cup\{0\}$, then $B=A$.*

This was first proved by Kaplansky [5] in a different form. Our proof is based on Dauns-Hofmann's theorem, which we state as Lemma 1, and Akemann's result (see Lemma 2) which is essentially contained in the proof of Theorem III. 8 in [1]. In [4], Elliott and Olesen gave a simple proof of the Dauns-Hofmann theorem. We now give a simple proof of Akemann's result and thus obtain a simpler proof of the main theorem.

LEMMA 1 (Dauns-Hofmann [2]). *Let A be a* C*-*algebra and* $\mathrm{Prim}A$ *its structure space. Let f be a bounded continuous complex-valued function on* $\mathrm{Prim}A$ *and $x \in A$. Then there exists an element $f\cdot x$ in A such that $f\cdot x = f(P)x \bmod P$ for all $P \in \mathrm{Prim}A$.*

LEMMA 2 (Akemann [1]). *Let A be a* C*-*algebra and B a* C*-*subalgebra of A which separates $P(A)\cup\{0\}$. Let* $\mathrm{Prim}A$ *and* $\mathrm{Prim}B$ *be the structure spaces of A and B, respectively. Then there exists a homeomorphism between* $\mathrm{Prim}A$ *and* $\mathrm{Prim}B$.

PROOF. Let φ be a map of $\mathrm{Prim}A$ into $\mathrm{Prim}B$ defined by $\varphi(P)=P\cap B$. The definition of the hull-kernel topology on the structure space implies easily that φ is continuous. Since B is a rich subalgabra of A by Lemme 11.1.7 in [3], φ is an onto correspondence. We assert that $I\cap B\subset P\cap B$ implies $I\subset P$ for any closed two-sided ideal I of A and any primitive ideal P of A. To see this, let π be a non-zero irreducible representation of A with $P=\mathrm{Ker}\,\pi$. Suppose, on the contrary, that $\pi(I)\neq 0$. Then $\pi_{|I}$ is

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non-zero irreducible representation of I. Since $I \cap B$ is a rich subalgebra of I by Lemme 11.1.3 (ii) in [3], we see that $\pi_{|I\cap B} \neq 0$. This contradicts our assumption $I \cap B \subset P \cap B$, as was to be proved. It follows easily from the above assertion that φ is the one-to-one correspondence. Finally, we show that φ^{-1} is continuous. Let $K \subset \mathrm{Prim}\, B$ and any point $Q \in \mathrm{Cl}(K)$, where $\mathrm{Cl}(K)$ denotes the closure of K. Then

$$B \cap \varphi^{-1}(Q) \supset \cap \{Q_\alpha :\ Q_\alpha \in K\} = \cap \{P_\alpha \cap B :\ P_\alpha \in \varphi^{-1}(K)\} .$$

By the assertion mentioned above, we have

$$\varphi^{-1}(Q) \supset \cap \{P_\alpha :\ P_\alpha \in \varphi^{-1}(K)\} ,$$

so that $\varphi^{-1}(Q) \in \mathrm{Cl}(\varphi^{-1}(K))$. Thus φ^{-1} is continuous and the lemma is proved.

Proof of Theorem. Let $\hat{A}$ be the spectrum of A and let $x \in A$ and $\varepsilon > 0$ be chosen arbitrarily. Set

$$K_\varepsilon = \{\varrho \in \hat{A} :\ \|\varrho(x)\| \geqq \varepsilon\} \quad \text{and} \quad G_\varepsilon = \hat{A} \setminus K_\varepsilon .$$

Then K_ε is compact in $\hat{A}$ (Proposition 3.3.7 in [3]) and G_ε is open in $\hat{A}$. Take $\pi \in \hat{A}$. Since A is a CCR-algebra and $\pi_{|B} \in \hat{B}$, we see that both $\pi(A)$ and $\pi(B)$ coincide with the algebra of compact operators on H_π (see 4.3.2 in [3]). So there exists $b_\pi \in B$ such that $\pi(x) = \pi(b_\pi)$. Since the map: $\varrho \to \|\varrho(x - b_\pi)\|$ is continuous on $\hat{A}$ (Corollaire 3.3.9 in [3]), there exists an open neighbourhood U_π of π in $\hat{A}$ such that

$$\|\varrho(x - b_\pi)\| < \varepsilon \quad \text{for all } \varrho \in U_\pi .$$

By the compactness of K_ε, there exists a finite open covering $\{U_{\pi_1}, \ldots, U_{\pi_n}\}$ of K_ε and therefore $\{U_{\pi_1}, \ldots, U_{\pi_n}, G_\varepsilon\}$ is a finite open covering of $\hat{A}$. Thus we can easily construct a partition of the identity $\{h_1, \ldots, h_n, h_\omega\}$ for the covering $\{U_{\pi_1}, \ldots, U_{\pi_n}, G_\varepsilon\}$. Let ψ be a map of $\hat{A}$ into $\hat{B}$ defined by $\psi(\varrho) = \varrho_{|B}$. By Lemma 2, ψ is homeomorphic. Let ψ^* be the dual map of ψ from the algebra of bounded continuous complex-valued functions on $\hat{B}$ onto that on $\hat{A}$. Setting

$$f_1 = (\psi^*)^{-1}(h_1), \ldots, f_n = (\psi^*)^{-1}(h_n)$$

and

$$b_\varepsilon = f_1 \cdot b_{\pi_1} + \ldots + f_n \cdot b_{\pi_n} ,$$

we see that $f_i \cdot b_{\pi_i} \in B$ and $\tau(f_i \cdot b_{\pi_i}) = f_i(\tau)\tau(b_{\pi_i})$ for any $\tau \in \hat{B}$ $(i = 1, \ldots, n)$ by Lemma 1. Then $b_\varepsilon \in B$ and, for any $\varrho \in \hat{A}$,

$$\begin{aligned}\varrho(b_\varepsilon) &= \sum_{i=1}^n f_i(\varrho_{|B})\varrho_{|B}(b_{\pi_i}) \\ &= \sum_{i=1}^n \psi^*(f_i)(\varrho)\varrho(b_{\pi_i}) \\ &= \sum_{i=1}^n h_i(\varrho)\varrho(b_{\pi_i}) .\end{aligned}$$

It follows from the definition of $h_1, \ldots, h_n, h_\omega$ that

$$\begin{aligned}\|\varrho(x-b_\varepsilon)\| &= \|\textstyle\sum_{i=1}^n h_i(\varrho)\varrho(x-b_{\pi_i}) + h_\omega(\varrho)\varrho(x)\| \\ &\leqq \textstyle\sum_{i=1}^n h_i(\varrho)\|\varrho(x-b_{\pi_i})\| + h_\omega(\varrho)\|\varrho(x)\| \\ &< \varepsilon .\end{aligned}$$

for all $\varrho \in \hat{A}$ and therefore $\|x-b_\varepsilon\| < \varepsilon$. As ε is arbitrary, $x \in B$ and the theorem is proved.

REMARK. We can by the same method show: If $B \subset A$ are C*-algebras, Prim$A$ is Hausdorff, $\pi(A) = \pi(B)$ for any irreducible representation π of A, and B separates $P(A) \cup \{0\}$, then $B = A$.

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REFERENCES

1. C. A. Akemann, *The general Stone-Weierstrass problem*, J. Functional Analysis, 4 (1969), 277–294.
2. J. Dauns and K. H. Hofmann, *Representations of rings by continuous sections*, Mem. Amer. Math. Soc., 83 (1968).
3. J. Dixmier, *Les C*-algèbres et leurs représentations*, (Cahiers Scientifique 24), Gauthier-Villars, Paris, 1964.
4. G. Elliott and D. Olesen. *A simple proof of the Dauns-Hofmann theorem*, Math. Scand., 34 (1974), 231–234.
5. I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc., 70 (1951), 219–255.

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