A PRODUCT THEOREM FOR SKOLEM SEQUENCES

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Abstract.

Given a Skolem sequence of order \(n-1\) and one of order \(m-1\) we obtain a construction for a large number of Skolem sequences of order \(2mn+m+n-1\).

Let \(V\) be a set of non-negative integers and \(s\) a positive integer. An \((s, V)\)-sequence is a sequence of length \(s|V|\) consisting of \(s\) copies of each \(v \in V\) with consecutive occurrences of \(v\) occurring in positions whose position numbers differ by \(v\). In the case \(V\) consists of the integers \(1, 2, \ldots, |V|\) and the sequence positions are numbered \(1, 2, \ldots, s|V|\), then the sequence we obtain by subtracting 1 from each entry in an \((s, V)\)-sequence is a Skolem \((s, |V| - 1)\)-sequence in the sense of Roselle [1]. In this paper we will confine ourselves to the consideration of what we refer to as regular \((s, V)\)-sequences. We call an \((s, V)\)-sequence regular if the sequence positions are numbered \(1, 2, \ldots, s|V|\) (as in Roselle's Skolem sequences). Non-regular \((s, V)\)-sequences include the so-called hooked Skolem sequences [1]. For example,

1. For \(s = 2\) and \(V = \{1, 2, 4\}\), the sequence

\[2, 4, 2, 1, 1, 4\]

is an \((s, V)\)-sequence;

2. For \(s = 2\) and \(V = \{1, 2, 3, 4\}\), the two sequences

\[1, 1, 3, 4, 2, 3, 2, 4\] and \[4, 2, 3, 2, 4, 3, 1, 1\]

are distinct \((s, V)\)-sequences. They are connected with the two Skolem \((2, 3)\)-sequences

\[0, 0, 2, 3, 1, 2, 1, 3\] and \[3, 1, 2, 1, 3, 2, 0, 0\].

3. For \(s = 2\) and \(V = \{1, 2\}\), the sequence

\[1, 1, 2, , , , 2\] (with four positions numbered 1, 2, 3, 5)

is a non-regular \((s, V)\)-sequence connected with the "hooked" Skolem sequence \(0, 0, 1, , , 1\). For a definition of hooked Skolem sequence, see [1].

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The Theorems we present below can be generalized to the case of non-
regular \((s, V)\)-sequences, and we leave it to the interested reader to do so.

Let \(U\) denote the set of all ordered \(s\)-tuples of integers. Let \(\delta(a_1, \ldots, a_s)\)
denote the set of differences
\[
\{a_{i+1} - a_i \mid 1 \leq i < s\}.
\]
Let \(\overline{U}\) denote the set \(\{t \mid t \in U \text{ and } |\delta t| = 1\}\). Then an \((s, V)\)-sequence is
easily shown to be equivalent to a function \(F: V \to \overline{U}\) such that \(\delta F(v) = \{v\}\)
and \(F(v) \cap F(w) = \emptyset\) for \(v \neq w\) in \(V\). (Here we mean the intersection of
\(F(v)\) and \(F(w)\) as sets of integers, without regard to order. For instance
\((1, 2) \cap (2, 3) = \{2\}\).) \(F(v)\) can be thought of as the set of positions in which
\(v \in V\) occurs in the sequence. In particular, we have

**Lemma 1.** A \((2, V)\)-sequence is equivalent to a function \(F: V \to \overline{U}\), where
\(\delta F(v) = \{v\}\), and for any \(v, w \in V\) such that \(v \neq w, F(v) \cap F(w) = \emptyset\).

Note that since \(\delta F(v) = \{v\}\), the function \(F\) is completely described by
its range. Thus, in the case of \((2, V)\)-sequences, we might as well construct
sets \(X\) of disjoint ordered pairs \((a, b), a < b\), of elements taken from some
set of integers such that
\[
(1) \quad \{b - a \mid (a, b) \in X\} = V.
\]

**Definition.** A starter for an abelian group \(G\) of odd order is a partition \(X\) of \(G^*\),
the set of non-zero elements of \(G\), into 2-sets which satisfy
\[
(2) \quad \{b - a \mid (a, b) \in X\} = G^*.
\]
By an ordered starter we shall mean an ordered pair \((X, f)\), where \(X\) is a
starter and \(f\) is a function which takes each pair \((a, b) \in X\) onto one of
the pairs \((a, b)\) or \((b, a)\). We write \((a, b) \in (X, f)\) or, alternatively, \((b, a) \in (X, f)\).
Now we have from (1) and (2):

**Lemma 2.** An ordered starter \((X, f)\) for \(Z_{2n+1}\) with the property that for
any \(t \in X, tf = (a, b)\) implies \(a < b\) is equivalent to a regular \((2, V)\)-sequence,
where \(|V| = n\) and
\[
V = \{b - a \mid (a, b) \in X \text{ and } a < b\}.
\]
For example, the sequences in the examples 1, 2 above correspond to
the following ordered starters:

1. \(\{(4, 5), (1, 3), (2, 6)\} \quad \text{for } Z_7.\)
2. \(\{(1, 2), (3, 6), (4, 8), (5, 7)\} \quad \text{and } \{(7, 8), (3, 6), (1, 5), (2, 4)\} \quad \text{for } Z_9.\)

At this point we find it useful to recall a theorem proved in [2]:
THEOREM 1. Let $X$ and $Y$ be starters for $Z_k$ and $Z_l$, respectively. Suppose that for each $t \in Y$ there is a permutation $\pi_t$ of $Z_k$ such that $\pi_t - I$ is also a permutation of $Z_k$. Then

$$W = \{ (lx, ly) \mid \{x, y\} \in X \} \cup \{ (lz + u, l(z\pi_t + v)) \mid z \in Z_k, t = \{u, v\} \in Y, u < v \}$$

is a starter for $Z_{kl}$.

It is easy to derive the following corollary:

COROLLARY 1. Let $X_1$ and $X_2$ be regular $(2, \{1, 2, \ldots, d_i\})$ sequences, $i = 1, 2$. Then a regular $(2, \{1, 2, \ldots, 2d_1d_2 + d_1 + d_2\})$-sequence can be obtained by applying Theorem 1.

PROOF. Let $\pi_t$ be the permutation defined by

$$y \rightarrow x \quad \text{iff} \quad |y - x| \text{ is in positions } x \text{ and } y \text{ in } X_1, \quad 0 \rightarrow 0.$$ 

Form the starter $X$ from $X_1$ and the starter $Y$ from $X_2$ in the manner described in Lemma 2. Then order the starter $W$ as described in Lemma 2. It is easy to show that if $(\alpha, \beta)$ is a pair in this starter, then $1 \leq \beta - \alpha \leq 2d_1d_2 + d_1 + d_2$. Since $W$ is a starter, the differences are all distinct, hence consecutive.

COROLLARY 2. Given a Skolem $(2, n - 1)$-sequence and a Skolem $(2, m - 1)$-sequence, it is possible to use Theorem 1 to construct $3^n$ distinct Skolem $(2, 2mn + m + n - 1)$-sequences provided $3 \nmid 2n + 1$.

PROOF. As well as $\pi_t$ defined as above, one can also define $x\pi_t \equiv 2x$ or $x\pi_t \equiv \frac{1}{2}x \pmod{2n + 1}$. For each $t \in Y$ there are three distinct choices, since $x \rightarrow 2x$ is not its own inverse, while $\pi_t$ as defined in Corollary 1 has this property.

REFERENCES