REMARKS ON VECTOR SPACE GENERATED BY THE MULTIPLICATIVE COMMUTATORS OF A DIVISION RING

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Abstract

Let D be a division ring with centre F. An element of the form $xyx^{-1}y^{-1} \in D$ is called a multiplicative commutator. Let $T(D)$ be the vector space over F generated by all multiplicative commutators in D. In M. Aghabali et al., J. Algebra Appl. 12 (2013), no. 8, art. 1350043, the authors have conjectured that every division ring is generated as a vector space over its centre by all of its multiplicative commutators. In this note it is shown that if D is centrally finite, then the conjecture holds.

Throughout this paper D is a division ring with centre F. An element of the form $xyx^{-1}y^{-1} \in D$ is called a *multiplicative commutator*, and $[D, D]$ denotes the *additive commutator* subgroup of D. Also we denote by $T(D)$ the vector space generated by the set of all multiplicative commutators of D over F. An element $a \in D$ is said to be *algebraic* over F if a satisfies a non-zero polynomial in $F[x]$. A set $S \subseteq D$ is called algebraic if each of its elements is algebraic over F. When K is a finite-dimensional extension of F, then we denote by $\mathrm{Tr}_{K/F}$, the regular trace of K over F. If $a \in D$, then $F(a)$ denotes the subfield of D generated by F and $\{a\}$.

The division ring generated by additive commutators or multiplicative commutators of D is the whole of D [5, pp. 205 & 211]. In the algebraic and zero characteristic case, it was proved that D is generated as a vector space over the centre by the union of its additive commutators and the unity, see [2]. In [1], the first author and his colleagues study the F-vector space $T(D)$ generated by the set of multiplicative commutators $\{xyx^{-1}y^{-1} \mid x, y \in D^*\}$. They prove that $T(D)$ contains all separable elements of D and if $T(D)$ is radical over F, then $D = F$. Furthermore, if $\dim_F T(D) < \infty$, then $\dim_F D < \infty$. They also prove that if D is an algebraic division ring over its centre with $\mathrm{char}(D) = 0$, then $T(D) = D$. The following theorem is crucial in their studies.

THEOREM 1.1 ([7, p. 156]). *Let D be a division ring and K be a subfield of D. For $a \in D$, if $\dim_K K[a] \geq n$, then for any distinct elements $\alpha_1, \ldots, \alpha_n \in Z(D)$, $(a - \alpha_1)^{-1}, \ldots, (a - \alpha_n)^{-1}$ are linearly independent.*

They then conjecture that a division ring is generated by all multiplicative commutators as a vector space over its centre, i.e., $T(D) = D$ for any arbitrary division ring. Recently, Hazrat has shown that if $D = L((x, \sigma))$ is the formal Laurent series in which L/F is a field extension and $\sigma \in \mathrm{Gal}(L/F)$ is of infinite order, then $\dim_{T(D)} D = \infty$ [3]. Despite this example disproving the conjecture in the general case, one can still consider whether the conjecture holds for algebraic division rings. In fact, the case which still has not received an answer is about algebraic division rings with characteristic $p > 0$. In this note, we show that the conjecture is true for centrally finite division rings, i.e. every division ring finite dimensional over its centre could be recovered as a vector space by all its multiplicative commutators.

We start with the following result.

THEOREM 1.2. *Let D be an algebraic non-commutative division ring over its centre F. Then $T(D)$ is a non-central Lie ideal in D.*

PROOF. First note that if F is finite, then D is commutative, a contradiction. Hence we may assume that F is infinite and let $a \in D$, $x \in T(D)$ be arbitrary. We show that $ax - xa \in T(D)$. If $ax = xa$, then we have nothing to prove. Otherwise, for every $r \in F$ we have

$$0 \neq r(axa^{-1} - x) = raxa^{-1} + ax - ax - rx$$
$$= axa^{-1}(a + r) - (a + r)x$$
$$= (axa^{-1} - (a + r)x(a + r)^{-1})(a + r).$$

Since $T(D)$ is invariant under conjugation, for every $r \in F$ we obtain that

$$(axa^{-1} - x)(a + r)^{-1} = r^{-1}(axa^{-1} - (a + r)x(a + r)^{-1}) \in T(D).$$

Now, since a is algebraic over F we have $\dim_F F(a) = n < \infty$. Thus by Theorem 7, for every $r_1, \ldots, r_n$, we have $(a - r_1)^{-1}, \ldots, (a - r_n)^{-1}$ are linearly independent and consequently form a basis for $F(a)$ over F. Therefore, for every $a \notin C_D(x)$, $(axa^{-1} - x)F(a) \subseteq T(D)$. In particular, $ax - xa = (axa^{-1} - x)a \in T(D)$. This completes the proof.

REMARK 1.3. By a result due to Herstein [4, p. 9, Theorem 1.5], every non-central Lie ideal of a division ring which is not 4-dimensional over a field of characteristic 2, contains the additive commutator subgroup of the division ring. Combining this fact with previous theorem, we find that if D is a division

ring algebraic over its centre F, then $[D, D] \subseteq T(D)$ provided that either char$(D) \neq 2$ or dim$_F D > 4$.

REMARK 1.4. Let D be a non-commutative division ring finite dimensional over its center F, and let $K = F(a)$ be the separable maximal subfield of D [5, p. 244]. It is known that every element of a division ring that commutes with all multiplicative commutators, is central [5, p. 210]. By this fact, $T(D)$ is not commutative. Then since $T(D)$ contains all separable elements of D, one can obtain that $K \subsetneq T(D)$ [1, Theorem 2.3]. Hence, dim$_F T(D) > $ dim$_F K$.

It is known that if D is a centrally finite division ring, then $[D, D]$ is a hyperspace. Indeed, let D be a division ring of dimension n^2 over its centre F and suppose that K is a splitting field for D, i.e. $D \otimes_F K \simeq M_n(K)$. Now, looking at $[D, D]$ as a right F-module we know that

$$\begin{aligned}
\dim_F[D, D] &= \dim_K([D, D] \otimes_F K) \\
&= \dim_K[D \otimes_F K, D \otimes_F K] \\
&= \dim_K[M_n(K), M_n(K)] = \dim_K Sl_n(K) = n^2 - 1.
\end{aligned}$$

THEOREM 1.5. *Let D be a division ring finite dimensional over its centre F. Then D is generated as a vector space by all of its multiplicative commutators over F, i.e. $D = T(D)$.*

PROOF. Assume that F is finite. Then D is finite and by Wedderburn's Little Theorem must be commutative and there is nothing to prove. So, we may assume that F is infinite. Put $M = \{x \in D \mid \text{Tr}_{F(x)/F}(x) = 0\}$. Clearly, M is a subspace of D that contains $[D, D]$. Since $[D, D]$ is a hyperspace easily it is seen that either $[D, D] = M$ or $M = D$. The latter case cannot occur, because by the Noether-Jacobson Theorem [5, p. 244], D contains some separable elements, say b. Hence, there would be existing an element $c \in F(b)$ whose trace is non-zero [6, p. 86]. Therefore, $x \in [D, D]$ if and only if $\text{Tr}_{F(x)/F}(x) = 0$.

First assume that either dim$_F D \neq 4$ or char$(F) \neq 2$. Combining Remark 1.3 with the fact that the additive group $[D, D]$ is a hyperspace, we obtain either $[D, D] = T(D)$ or $D = T(D)$. Note that $T(D) = [D, D]$ could not be the case, because as mentioned in Remark 1.4, $T(D)$ admits some separable elements with non-zero trace while as the first paragraph of the proof $[D, D]$ is the set of all elements of trace zero.

Now, assume that dim$_F D = 4$ and char$(F) = 2$. By Remark 1.4, dim$_F T(D) = 3$ or dim$_F T(D) = 4$. If dim$_F T(D) = 4$ the proof is completed. Thus suppose that dim$_F T(D) = 3$. Observe that if for every $x \in D$, $x^{-1} \in T(D)$, then $D = T(D)$ and we are done. So, pick $x \in D$ such that $x^{-1} \notin T(D)$. Since $T(D)$ is a Lie ideal, for every $a \in T(D)$ we have

$ax - xa \in T(D) \setminus F$. This shows that $L := [D, D] \cap T(D) \nsubseteq F$. It is not hard to verify that L is invariant under conjugation (in fact, if $x \in [D, D]$, then for every $a \in D^*$ we have $axa^{-1} - x = (ax)a^{-1} - a^{-1}(ax) \in [D, D]$). Let b be a non-central separable element in D. Then since $F(b)$ is a maximal subfield containing some elements with non-zero trace, we find that $F(b) \nsubseteq L$. Therefore, an argument similar to the proof of Theorem 1.2 shows that there exists an element $x \in L$, which does not commute with b, and $(bxb^{-1} - x)F(b) \subseteq L$. Therefore, $L \cap Lb \neq 0$, and in particular $\dim_F(L \cap La) \geq \dim_F F(b) = 2$. Also, $b \in Lb \setminus L$ implies that $L \cap Lb \subsetneq Lb$, and $\dim_F Lb = \dim_F L = 3$. Thus $[D, D] \cap T(D) = L = [D, D]$. Combining this fact with contrary assumption $\dim_F T(D) = 3$ yields that $T(D) = [D, D]$. Now, repeating aforementioned arguments achieves contradiction. Hence, $\dim_F T(D) = 3$ is not the case and this completes the proof.

REMARK 1.6. Note that by [2, Theorem 2] if D is a division ring algebraic over its centre F with characteristic zero, then $\dim_F D/[D, D] \leq 1$. Hence, applying similar argument as above yields that if D is algebraic over its centre with characteristic zero, then one can recover D from $T(D)$.

REFERENCES

1. Aghabali, M., Akbari, S., Ariannejad, M., and Madadi, A., *Vector space generated by the multiplicative commutators of a division ring*, J. Algebra Appl. 12 (2013), no. 8, art. 1350043, 7 pp.
2. Akbari, S., Arian-Nejad, M., and Mehrabadi, M. L., *On additive commutator groups in division rings*, Results Math. 33 (1998), no. 1-2, 9–21.
3. Hazrat, R., *A note on multiplicative commutators of division rings*, J. Algebra Appl. 18 (2019), no. 2, art. 1950031, 2 pp.
4. Herstein, I. N., *Topics in ring theory*, The University of Chicago Press, Chicago, Ill.-London, 1969.
5. Lam, T. Y., *A first course in noncommutative rings*, second ed., Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001.
6. Morandi, P., *Field and Galois theory*, Graduate Texts in Mathematics, vol. 167, Springer-Verlag, New York, 1996.
7. Rowen, L. H., *Ring theory*, student ed., Academic Press, Inc., Boston, MA, 1991.

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