A SET OF GENERATORS FOR $\text{Ext}_R(k,k)$

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Introduction.

Let in the following $R$ denote a local ring with maximal ideal $m$ and residue field $k$. Then $E = \text{Ext}_R(k,k)$, with the Yoneda multiplication, is a connected cocommutative Hopf algebra over $k$ (see Gulliksen and Levin [5] or Levin [9]). It was conjectured in [5] page 115 that $E$ is finitely generated. However, as shown by an example of Roos [11] this need not be so. In this paper we construct the algebra structure of $E$ from its definition by projective resolutions of $k$. Then, using the minimal algebra resolution of $k$ (see Tate [14] and Gulliksen [7]), we obtain a set of generators which essentially are the so-called derivations of [7]. This set of generators are then used to study the structure of $E$. In particular we completely characterize those rings $R$ such that $E$ is commutative and finitely generated. We also give an explicit formula for $E$ in the case when $R$ is a local complete intersection. Here, as in the sequel, commutative means strictly commutative i.e.

$$xy = (-1)^{\deg(x) \cdot \deg(y)}yx$$

and $x^2 = 0$ if $\deg(x)$ is odd.

I wish to thank J.-E. Roos who called my attention to the relevance of the Milnor-Moore-André structure theorem to these matters.

1. The Yoneda product.

Let, for the time being, $R$ be any commutative ring and let $A$ be an $R$-module. Then the Yoneda composite provides $\text{Ext}_R(A,A)$ with the structure of a graded algebra over $R$. For details see Mac Lane [10, III 5]. The classical definition exploits the interpretation of $\text{Ext}_R(A,A)$ as the set of equivalence classes of certain long exact sequences. Let instead $\text{Ext}_R(A,A) = H \text{Hom}_R(P_\bullet A,A)$, where $P_\bullet A$ is a projective resolution of $A$. How are we to interpret the Yoneda product?

The answer is given below.

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**Lemma 1.** Let $P_A \xrightarrow{\varepsilon_A} A, P_B \xrightarrow{\varepsilon_B} B$ be projective resolutions of $A$ and $B$ respectively. Then

$$H \text{Hom}_R(1, \varepsilon_B): H \text{Hom}_R(P_A, P_B) \to H \text{Hom}_R(P_A, B)$$

is an isomorphism.

**Proof.** For complexes $X_\bullet, Y_\bullet$ $\text{Hom}_R(X_\bullet, Y_\bullet)$ is given the structure of a complex as in [10, VI 7.6]. We filter $\text{Hom}_R(X_\bullet, Y_\bullet)$ by

$$F^p \text{Hom}_R(X_\bullet, Y_\bullet) = \{ f \mid f(x) = 0 \text{ for } \deg(x) \leq p - 1 \}$$

$$= \text{Hom}_R(\oplus_{n \geq p} X_n, Y_\bullet)$$

It is obvious that $F^* \text{Hom}_R(X_\bullet, Y_\bullet)$ is bicomplete ($P$- and $I$-complete in the terminology of Eilenberg-Mac Lane [4]) if $X_\bullet$ is bounded below. Now, in the corresponding spectral sequence,

$$E_1 \text{Hom}_R(1, \varepsilon_B) = H \text{Hom}_R(1, H\varepsilon_B): H \text{Hom}_R(P_A, HP_A \cdot B) \to$$

$$\to H \text{Hom}_R(P_A, B)$$

is an isomorphism and hence so is $H \text{Hom}_R(1, \varepsilon_B)$.

Note that the lemma above essentially is the classical lifting theorem of homological algebra (it is sufficient for the proof to assume that $P_A$ is projective over $A$ and that $P_B$ is exact over $B$).

Now we define a product

$$\text{Ext}_R(B, C) \otimes_R \text{Ext}_R(A, B) \to \text{Ext}_R(A, C)$$

by

$$R \text{Hom}_R(P_B, C) \otimes R \text{Hom}_R(P_A, B)$$

$$\cong \text{Hom}_R(1, \varepsilon)$$

$$R \text{Hom}_R(P_B, C) \otimes R \text{Hom}_R(P_A, P_B)$$

$$\mu$$

$$\text{Hom}_R(P_B, C) \otimes \text{Hom}_R(P_A, P_B))$$

$$H \zeta$$

$$\text{Hom}_R(P_A, C)$$

where $\zeta$ is the natural morphism of complexes given by $\zeta(g \otimes f) = g \circ f$.

It is straightforward to check that this product provides $\text{Ext}_R(A, A)$ with the structure of a graded algebra over $R$ with unit given by
\[ \eta : R \xrightarrow{\ast} \text{Hom}_R(A, A) \xrightarrow{H\text{Hom}_R(\text{ev}, 1)} H\text{Hom}_R(P_\ast A, A) \]

where \( \nu(r)(a) = ra \).

The reader may check that the product \( \circ \) differs from the usual Yoneda product \( \hat{\circ} \) in a sign. Precisely:

\[ a \circ b = (-1)^{\text{deg}(a) \cdot \text{deg}(b)} a \hat{\circ} b. \]

This makes no difference since we have an isomorphism of algebras

\[ \chi : (\text{Ext}_R(k, k), \circ) \rightarrow (\text{Ext}_R(k, k), \hat{\circ}) \]

given by \( \chi(a) = (-1)^{\tau(n)} a \), where \( \tau(n) = 0, 1, 1, 0 \) for \( n \equiv 0, 1, 2, 3 \) mod 4.

From now on we assume \( R \) to be local. Let \( P_\ast = P_\ast k \) be a minimal free resolution of \( k \). Then the differential on \( \text{Hom}_R(P_\ast, k) \) is zero, whence

\[ \text{Ext}_R^n(k, k) = \text{Hom}_R(P_n, k) \]

We want to describe

\[ \text{Hom}_R(P_i, k) \otimes \text{Hom}_R(P_j, k) \xrightarrow{R} \text{Hom}_R(P_{i+j}, k) \]

Let \( g \in \text{Hom}_R(P_i, k), f \in \text{Hom}_R(P_j, k) : \) We have

\[ H\text{Hom}_R(1, 1) : H\text{Hom}_R(P_\ast, P_\ast) \xrightarrow{\sim} H\text{Hom}_R(P_\ast, k) = \text{Hom}_R(P_\ast, k) \]

Choose \( F \in \text{Z}^j \text{Hom}_R(P_\ast, P_\ast) \) such that \( e \circ F = f \) that is \( F \) is a chainmap of degree \( j \) \( (F_n : P_n \rightarrow P_{n-j}) \), \( d \circ F = (-1)^j F \circ d \) lifting \( f \). Then by definition

\[ g \circ f = g \circ F_{i+j} : P_{i+j} \rightarrow k. \]

We are going to use the minimal algebra resolution of Tate (cf. [14], [5] and [7]). Thus as in [5, page 50]

\[ P_\ast = R\langle \ldots S_t, \ldots ; dS_t = s_t \rangle \]

with \( e_{i-1} \) variables \( S_t \) of degree \( j \) and with the indexing such that \( i < j \) if \( \text{deg}(S_i) < \text{deg}(S_j) \). Then

\[ \sum_{n=0}^{\infty} \dim_k(\text{Tor}_n^R(k, k))z^n = \prod_{i=0}^{\infty} \frac{(1 + z^{2i+1})^{2i+1}}{(1 - z^{2i+1})^{2i+1}} \]

According to [5, p. 46] there exist so-called derivations \( J_\ast \in \text{Hom}_R(P_\ast, P_\ast) \), associated with the variables \( S_t \), which (with a change of signs) satisfy the following relations, where \( J \) stands for an arbitrary \( J_\ast \):

a. \( d \circ J = (-1)^{\text{deg}(J)} J \circ d \) that is, \( J \in \text{Z}^{\text{deg}(J)} \text{Hom}_R(P_\ast, P_\ast) \)

b. \( J(xy) = J(x) \cdot y + (-1)^{\text{deg}(J) \cdot \text{deg}(x)} x \cdot J(y) \)

c. If \( x \) is of positive even degree then \( J(x^{(n)}) = x^{(n-1)} \cdot J(x) \)

d. \( J_\ast(S_j) = \delta_{i,j} \) if \( \text{deg}(J_\ast) = \text{deg}(S_j) \).
Let, as in [5],
\[ X^{(n)} = R\langle S_1, \ldots, S_{\varepsilon_0 + \ldots + \varepsilon_{n-1}}; dS_i = s_i \rangle. \]
Then it follows from formulas b. and c. that \( J(X^{(n)}) = 0 \) for \( n < \deg(J) \).

**Definition.** \( Y_i = \{\varepsilon \circ J_i\} \in \text{Ext}_R^{\deg(J)}(k, k). \)

**Theorem 1.** The set \( \{Y_i\}_{i \geq 1} \) (possibly finite) generates \( \text{Ext}_R(k, k) \) as a \( k \)-algebra with the Yoneda product.

**Proof.** We have an \( R \)-basis for \( P_n \) given by elements of type
\[ S^{(r_1, \ldots, r_N)} = S_N^{(r_N)} S_{N-1}^{(r_{N-1})} \ldots S_1^{(r_1)} \]
where \( N = \varepsilon_0 + \ldots + \varepsilon_{n-1} \), \( \Sigma r_i \cdot \deg(S_i) = n \) and \( x^{(0)} = 1, x^{(1)} = x \) when \( x \) is of odd degree. Order the \( N \)-tuples \( (r_1, \ldots, r_N) \) by
\[ (r_1, \ldots, r_N) > (r_1', \ldots, r_N') \]
if the last non-vanishing \( r_1 - r_1' > 0 \). Corresponding to this \( R \)-basis of \( P_n \), there is a dual \( k \)-basis \( \{b^{(r_1, \ldots, r_N)}\} \) of \( \text{Hom}_R(P_n, k) \) given by
\[ b^{(r_1, \ldots, r_N)} S^{(r_1', \ldots, r_N')} = \delta^{(r_1', \ldots, r_N')}_{(r_1, \ldots, r_N)} \quad \text{(Kronecker delta)}. \]
Let
\[ Y^{(r_1, \ldots, r_N)} = Y_1^{r_1} \cdots o Y_N^{r_N} = \varepsilon \circ J_1^{r_1} \cdots o J_N^{r_N}. \]
Then it easily follows that
\[ Y^{(r_1, \ldots, r_N)} S^{(r_1', \ldots, r_N')} = \begin{cases} 1 & \text{for } (r_1, \ldots, r_N) = (r_1', \ldots, r_N') \\ 0 & \text{for } (r_1, \ldots, r_N) > (r_1', \ldots, r_N') \end{cases} \]
i.e. if we express the elements \( Y^{(r_1, \ldots, r_N)} \) in the basis \( \{b^{(r_1, \ldots, r_N)}\} \) then we obtain a triangular matrix with only 1:s in the diagonal. Thus \( \{Y^{(r_1, \ldots, r_N)}\} \) is also a basis for \( \text{Hom}_R(P_n, k) \) and it follows that \( \{Y_i\} \) generates \( \text{Ext}_R(k, k) \) as a \( k \)-algebra.

**Definition.** Let \( \overline{PE} \) denote the graded \( k \)-vector space generated by \( \{Y_i\} \).

Note that \( Y_i \in \text{Hom}_R(P_\ast, k) \) is the element in the dual basis corresponding to \( S_i \).

**Theorem 2.** \( \overline{PE} \) is a graded Lie algebra satisfying \( x^2 \in \overline{PE} \) if \( x \in \overline{PE} \) is of odd degree.

**Proof.** It is sufficient to show that \( [Y_i, Y_j] \in \overline{PE} \) and that, if \( \deg(Y_i) \) is odd, \( Y_i^2 \in \overline{PE} \). Let \( n = i + l \). Note that
\[ P_n = X_n^{(n-1)} \oplus \bigoplus_{M < t \leq N} R S_t, \quad \text{where } M = \varepsilon_0 + \ldots + \varepsilon_{n-2} \]
\[ N = M + \varepsilon_{n-1}, \]
and that a basis of \( X_n^{(n-1)} \) is given by the \( S^{(r_1, \ldots, r_N)} : s \) with \( r_1 + \ldots + r_N > 1 \) (notation as in the proof of theorem 1). Using \( J_i X_u^{(a)} \subset X^{(a - \deg J_i)} \), it is easy to see that the only basis element of \( X_n^{(n-1)} \), which is not annihilated by \( J_i J_i \) and \( J_j J_i \), is:

a. \( S_i S_j \) if \( i \neq j \),

b. \( S_i^{(2)} \) if \( i = j \) and \( \deg(J_i) \) is even,

c. none if \( i = j \) and \( \deg(J_i) \) is odd.

In case a. we get
\[ J_i J_i S_j S_i = J_i ((J_j S_j) S_i + (-1)^{\deg(J_i) \cdot \deg(S_j)} S_j (J_j S_i)) = \]
\[ = J_i S_j S_i S_i + (-1)^{\deg(J_i) \cdot \deg(S_j)} J_i S_j J_j S_i = 1. \]

Similarly we obtain \( J_i J_i S_i S_j = (-1)^{\deg(J_i) \cdot \deg(J_j)} \), which shows that \( [J_i, J_j] X_n^{(n-1)} = 0 \). Case b. is even simpler since then, trivially, \( [J_i, J_j] = 0 \).

In case c. we get \( J_i^2 X_n^{(n-1)} = 0 \). This concludes the proof.

**Remark.** If \( \text{char}(k) \neq 2 \) then the statement about \( x^2 \) obviously follows from \( P E \) being a graded Lie algebra.

2. **Ext\(_R(k,k)\) as a Hopf algebra.**

It is well-known (see [5] page 107) that
\[ \text{Ext}_R(k,k) = \text{Hom}_R(P_*, k) \xrightarrow{\beta} \text{Hom}_k(P_\otimes_R k, k) = \text{Hom}_k(\text{Tor}_R(k, k), k), \]
where \( \beta(f)(x \otimes 1) = f(x) \), is an anti-isomorphism of algebras (for a proof not relying on Yoneda’s interpretation of Ext, and in a situation where this interpretation is not even available, see [12]).

Thus, if we change the usual diagonal in \( \text{Tor}_R(k, k) \) for its opposite we may say that \( \text{Ext}_R(k, k) \) is the dual of a Hopf algebra with divided powers. The diagonal in \( \text{Ext}_R(k, k) \) is the dual of the multiplication in \( P_\otimes_R k \) via \( \beta \) and it is not hard to check that it is given by
\[ \text{Hom}_R(P_*, k) \xrightarrow{\text{Hom}_R(\varphi, 1)} \text{Hom}_R(P_\otimes_R P_*, k) \cong \text{Hom}_R(P_*, k) \otimes_k \text{Hom}_R(P_*, k), \]
where \( \varphi \) is the product of the minimal algebra-resolution \( P_* \) of \( k \). Let \( Q \text{ Tor}_R(k, k) \) be as in André [1, theorem 17]. Let \( P' E \subset \text{Ext}_R(k, k) \) correspond to the dual of \( Q \text{ Tor}_R(k, k) \) via \( \beta \). Note that \( P' E \) equals the set of primitive elements of \( \text{Ext}_R(k, k) \) when \( \text{char}(k) = 0 \) but is strictly contained in this set otherwise. It is easy to check that
\[ P' E = \{ f \in \text{Hom}_R(P_*, k) \mid f(DP_*) = 0 \}. \]
Here $DP_\ast$ denotes the decomposable elements of $P_\ast$ considered as an algebra with divided powers that is $DP_\ast$ is the graded submodule of $P_\ast$ generated by $I^\ast P_\ast$, where $IP_\ast$ is the augmentation ideal of $P_\ast$, as a connected algebra over $R$, that is $IP_\ast = P_1 \oplus P_2 \oplus P_3 \oplus \ldots$, and by divided powers $x^{(n)}$, where $n \geq 2$.

Note that $DP_\ast$ is the graded $R$-module with $\{S^{r_1, \ldots, r_j} \mid r_1 + \ldots + r_j > 1\}$ as a basis. Let $B_\ast$ be the graded $R$-module with $\{S_i\}$ as a basis. Then $P_\ast = B_\ast \oplus DP_\ast$. Thus we have

$$\text{Hom}_R(P_\ast, k) = \text{Hom}_R(B_\ast, k) \oplus \text{Hom}_R(DP_\ast, k)$$

and hence

$$\mathcal{P}E = \text{Hom}_R(B_\ast, k) = \{f \mid f(DP_\ast) = 0\} = P'E.$$

According to [1, theorem 17] we have the following result

**Theorem 3.** If $\text{char}(k) \neq 2$ then, as a Hopf algebra, $\text{Ext}_R(k, k)$ is isomorphic to $U(\mathcal{P}E)$, the universal enveloping algebra of the Lie algebra $\mathcal{P}E$.

Thus, at least when $\text{char}(k) \neq 2$, the Hopf algebra structure of $\text{Ext}_R(k, k)$ is known as soon as we know the Lie algebra structure of $\mathcal{P}E$.

3. The generators of degree 1 and their 2-dimensional relations.

It is easy to see that the algebra $\text{Ext}_R(k, k)$ remains unchanged under completion of $R$. Thus, without loss of generality, $R$ is supposed to be complete and we can put $R = \tilde{R}/\mathfrak{M}$, where $\tilde{R}$ is a regular local ring and $\mathfrak{M} \subset \hat{m}^2$. In the following let $n = \varepsilon_0 = \dim_k (m/m)^2$, $r = \varepsilon_1$ and let $X_i = Y_i$, $T_i = S_i$ for $1 \leq i \leq n$, $Y_i =$ the old $Y_{i+n}$, $S_i =$ the old $S_{i+n}$ for $1 \leq i \leq r$. Assume that $s_1, \ldots, s_r$ is a set of cycles inducing a $k$-basis of $H_1X^{(0)}$. Let $x_1, \ldots, x_n$ be a minimal set of generators for $m$. Then we may assume

$$dT_i = x_i \quad \text{and} \quad dS_p = s_p = \sum_{i,j}a_{p,ij}x_iT_j.$$

Now

$$dJ_uS_p = -J_udS_p = -J_u \sum_{i,j}a_{p,ij}x_iT_j = -\sum_i a_{p,iu}x_i = -d\sum_{i}a_{p,iu}T_i$$

i.e. we may assume that

$$J_uS_p = -\sum_{i}a_{p,iu}T_i$$

and then $J_uJ_uS_p = -a_{p,uu}$ which shows that

$$X_iX_uS_p = -\tilde{a}_{p,uu} = -\varepsilon(a_{p,uu}).$$

Let $\mathfrak{M}$ be minimally generated by

$$a_p = \sum_{p=1}^{r'} \tilde{a}_p\bar{x}_j, \quad 1 \leq p \leq r',$$

where the $\bar{x}_j$'s form a minimal set of generators of $\hat{m}$ such that $\bar{x}_j + \mathfrak{M} = x_j$. 
Put $r_{pj} = \tau_{pj} + \mathcal{A}$. Then according to [5, page 43] we can choose
\[ s_p = \sum_{j=1}^{n} r_{pj} T_j \]
and in particular $r = r' = e_1 = \dim_k (\mathcal{A} / \mathcal{m} \mathcal{A})$. Since $\mathcal{A} \subset \mathcal{m}^2$ we have
\[ a_p = \sum_{i,j} \tilde{a}_{p,ij} x_i x_j \]
and consequently
\[ s_p = \sum_{i,j} a_{p,ij} x_i T_j, \]
(i.e. we may choose $a_{p,ij}$ above such that $a_{p,ij} = 0$ for $i > j$) where
\[ a_{p,ij} = \tilde{a}_{p,ij} + \mathcal{A} \quad \text{and} \quad \tilde{a}_{p,ij} = \bar{a}_{p,ij} \in k. \]
It follows that if we let $[X_t, X_u] = X_t^3$ for $t = u$ then
\[ [X_t, X_u] = -\sum_{p=1}^{r} \tilde{a}_{p,tu} Y_p \]
for $t \leq u$. To illustrate we write down the corresponding "matrix of two-dimensional relations" (all empty entries are to be regarded as 0: s) for the case $n = 3, r = 2$:

\[
\begin{array}{cccccccccccc}
\text{X}_2X_1 & \text{X}_2X_1 & \text{X}_2X_2 & [\text{X}_1, \text{X}_3] & [\text{X}_1, \text{X}_3] & [\text{X}_2, \text{X}_3] & \text{X}_1^2 & \text{X}_2^2 & \text{X}_3^2 & \text{X}_4^2 & \text{X}_5^2 & \text{X}_6^2 \\
\hline
T_1T_3 & 1 & & & & & & & & & & \\
T_1T_3 & & 1 & & & & & & & & & \\
T_2T_3 & & & 1 & & & & & & & & \\
-S_1 & & & \tilde{a}_{1,12} & \tilde{a}_{1,13} & \tilde{a}_{1,23} & \tilde{a}_{1,11} & \tilde{a}_{1,12} & \tilde{a}_{1,22} & \tilde{a}_{1,33} & & \\
-S_2 & & & \tilde{a}_{2,12} & \tilde{a}_{2,13} & \tilde{a}_{2,23} & \tilde{a}_{2,11} & \tilde{a}_{2,12} & \tilde{a}_{2,22} & \tilde{a}_{2,33} & & \\
\end{array}
\]

In particular the 1-dimensional elements are strictly commutative iff all $\tilde{a}_{p,ij} = 0$ that is iff $\mathcal{A} \subset \mathcal{m}^3$. Since
\[ m^2 / m^3 = \bar{m}^2 / \bar{m}^3 + \mathcal{A} \]
this can also be expressed by
\[ \dim_k (m^2 / m^3) = \dim_k (\bar{m}^2 / \bar{m}^3) = \binom{n+1}{2} \]
which gives a criterion that does not require $R$ to be complete.
A basis of \( P_2 \) is given by \( T_i T_j, i < j \) and the \( S_p : s \). In the dual basis \( T_i ' T_j ' \) corresponds to \( X_i , X_j \). Using this we see that

\[
\dim_k \frac{\mathop{\text{Ext}} R^2(k,k)}{(\mathop{\text{Ext}} R^1(k,k))^2} = r - \text{rank}(\bar{a}_{p,tj})
\]

where \((\bar{a}_{p,tj})\) is regarded as an \( r \times \binom{n+1}{2} \)-matrix. In particular \( \mathop{\text{Ext}} R^1(k,k) \) generates \( \mathop{\text{Ext}} R^2(k,k) \) iff the vectors \((\bar{a}_{p,tj})_{t_j, p = 1, \ldots, r} \) are linearly independent and hence iff \[
\Sigma_{p,tj} t_p \bar{a}_{p,tj} \bar{x}_i \bar{x}_j \in \tilde{m}^3
\]

implies that \( t_p \in \tilde{m}, 1 \leq p \leq r \) that is iff \( \mathcal{U}/\tilde{m} \mathcal{U} \rightarrow \tilde{m}^2/\tilde{m}^3 \) is a monomorphism that is iff \( \tilde{m}^3 \cap \mathcal{U} = \tilde{m} \mathcal{U} \), which is a condition which was first obtained by J.-E. Roos, using different methods. Furthermore, the exact sequence

\[
0 \rightarrow \mathcal{U}/\tilde{m}^3 \rightarrow \tilde{m}^2/\tilde{m}^3 \rightarrow m^2/m^3 \rightarrow 0
\]

shows that \( \tilde{m}^3 \cap \mathcal{U} = \tilde{m} \mathcal{U} \) iff \( \dim_k (m^2/m^3) = \binom{n+1}{2} - r \) and this condition does not require \( R \) to be complete. We summarize in

\section*{Theorem 4. Let the notations be as above. Then}

\[
[X_\omega X_\mu] + \Sigma_{p=1} a_{p,t_j} Y_p = 0 .
\]

The one-dimensional elements are strictly commutative iff \( \mathcal{U} \subset \tilde{m}^3 \) that is, iff \( \dim_k (m^2/m^3) = \binom{n+1}{2} \).

They generate the two-dimensional elements iff \( \mathcal{U} \cap \tilde{m}^3 = \tilde{m} \mathcal{U} \) that is, iff \( \dim_k (m^2/m^3) = \binom{n+1}{2} - r \).

Finally, consider the homogeneous linear system over \( k \)

\[
\Sigma_{1 \leq i \leq j \leq n} a_{p,tj} z_{ij} = 0 \quad 1 \leq p \leq r ,
\]

that is, \( \{z_{ij} \mid 1 \leq i \leq j \leq n\} \) are the "unknown" and we have \( r \) equations. Choose a basis \((t_{ij}^\omega)_{1 \leq i \leq j \leq n}, 1 \leq q \leq N \), for the solutions of this system. Then a basis for the two-dimensional relations of the one-dimensional generators is given by the relations

\[
\Sigma_{1 \leq i < j \leq n} t_{ij}^\omega [X_\omega X_j] = 0 , \quad 1 \leq q \leq N .
\]
4. Local complete intersections.

In this section we assume that $R$ is a local complete intersection. We keep the previous notations. Thus we may suppose that $\mathfrak{U} \subset \mathfrak{m}^2$ is generated by an $R$-sequence and the length of this must then equal $r = \varepsilon_1 = n - \dim(R)$. We know from [14] that $P_* = X^{(2)}$. Hence, using theorem 1, we obtain the result of [5] that $\text{Ext}_R(k,k)$ is generated by its 1- and 2-dimensional elements. We have more precisely (k[...]) means non-commutative free algebras and k[...] commutative free algebras. The sufficiency of $\mathfrak{U} \subset \mathfrak{m}^3$ for commutativity was first shown in [5, page 114]):

**Theorem 5.** Let $R$ be a local complete intersection and assume that $\text{char}(k) \neq 2$. Then, as a Hopf algebra

$$\text{Ext}_R(k,k) = k\langle X_1, \ldots, X_n, Y_1, \ldots, Y_r \rangle / ([X_i, X_j] + \sum_{p=1}^r a_{p,ij} Y_p, [X_1, Y_p], [Y_p, Y_q])$$

In particular, $\text{Ext}_R^1(k,k)$ generates $\text{Ext}_R(k,k)$ iff $\mathfrak{U} \cap \tilde{\mathfrak{m}}^3 = \tilde{\mathfrak{m}} \mathfrak{U}$ and $\text{Ext}_R(k,k)$ is commutative iff $\mathfrak{U} \subset \mathfrak{m}^3$. The subalgebra generated by $Y_1, \ldots, Y_r$ is the polynomial algebra $k[Y_1, \ldots, Y_r]$. The product

$$[,] : \tilde{P}_1 E \times \tilde{P}_1 E \to \tilde{P}_2 E$$

may be chosen at will. Precisely, given any Lie algebra $L = L_1 \oplus L_2$ with $\dim_k L_1 \geq \dim_k L_2$ there is a local complete intersection with $\tilde{P} E = L$.

**Proof.** The Hopf algebra $\text{Ext}_R(k,k)$ is isomorphic to the free algebra $k\langle X_1, \ldots, X_n, Y_1, \ldots, Y_r \rangle$ divided by the ideal generated by the elements describing the Lie product of $\tilde{P} E = \tilde{P}_1 E \oplus \tilde{P}_2 E$ and from this the first formula follows with the aid of theorem 4. The statements about generation and commutativity follow from theorem 4. Suppose that $f(Y_1, \ldots, Y_r)$ is a polynomial in the now commuting variables $Y_t$. We can take $f$ to be homogeneous. Let $Y_1^{n_1} \ldots Y_r^{n_r}$ have non-vanishing coefficient in $f$. Then

$$Y_1^{n_1} \ldots Y_r^{n_r} S_r^{(n_r)} \cdots S_1^{(n_1)} = 1 \quad \text{and} \quad Y_1^{l_1} \ldots Y_r^{l_r} S_r^{(n_r)} \cdots S_1^{(n_1)} = 0$$

when $(l_1, \ldots, l_r) \neq (n_1, \ldots, n_r)$ and hence

$$f(Y_1, \ldots, Y_r) S_r^{(n_r)} \cdots S_1^{(n_1)} \neq 0,$$

which shows that $f(Y_1, \ldots, Y_r) \neq 0$. It follows that the subalgebra generated by $Y_1, \ldots, Y_r$ is the polynomial algebra $k[Y_1, \ldots, Y_r]$. The arbitrariness of the Lie product follows from the following lemma (cf. Kaplansky [8, theorem 124]) applied to $y_1, \ldots, y_r \in \mathfrak{m}^2$ chosen at will and $s = 3$. 


**Lemma 2.** Let $R$ be a Cohen-Macaulay ring of dimension $n$ and let $y_1, \ldots, y_r \in m$, where $r \leq n$. Then for any $s \geq 1$ there is an $R$-sequence $z_1, \ldots, z_s$ such that $z_i - y_i \in m^s$.

**Proof.** By induction it may be assumed that $r = 1$. Then we have to show that if $y \in m$ then

$$y + m^s \ni \{p_t \mid p_t \in \text{Ass}(R)\}.$$  

Let $y \in p_1, \ldots, p_t$ and $y \notin p_{t+1}, \ldots, p_u$. Now $m^s \cap p_{t+1} \cap \ldots \cap p_u \nsubseteq p_t$ for $i \leq t$ and hence there is a

$$z \in m^s \cap p_{t+1} \cap \ldots \cap p_u - p_t \cup \ldots \cup p_t.$$  

Obviously

$$y + z \in (y + m^s) - \{p_t \mid p_t \in \text{Ass}(R)\}.$$  

**Remarks.** 1. Theorem 5 remains true when $\text{char}(k) = 2$ (recall our convention that $[X_i, X_j] = X_i^2$). This follows from the results of [13].

2. With a suitable change of basis of $\mathcal{P} \mathcal{E}$ we can arrange it so that $Y_{s+1}, \ldots, Y_r$ is a basis of the linear space spanned by the $[X_i, X_j]: s$ and then

$$\text{Ext}_R(k, k) = k\langle X_1, \ldots, X_n \rangle / \mathfrak{B} \otimes k[Y_1, \ldots, Y_s]$$

where $\mathfrak{B}$ is the ideal generated by elements of type $[X_i, X_j], X_i$ and by the elements corresponding to the two-dimensional relations between the $X_i$:s.

3. The three-dimensional relations $[X_i, X_j], X_i = 0$, in remark 2 above, may be essential i.e. not a consequence of the two-dimensional relations. An example is provided by

$$R = k\langle x_1, x_2, x_3 \rangle / (x_1 x_3 + x_2^3, x_2 x_3).$$

Then the two-dimensional relations are $X_1^2 = X_2^2 = X_3^2 = [X_1, X_2] = 0$, which shows that $\text{Ext}_R(k, k)$ is a quotient of

$$A = k\langle X_1, X_2, X_3 \rangle / (X_1^2, X_2^2, X_3^2, [X_1, X_2]).$$

But obviously $A = k[X_1, X_2] * k[X_3]$, where $*$ denotes the "free product" of graded algebras. Then, using (7) of Cohn [3, page 5], we get the Hilbert-series

$$H_A(z) = \frac{(1 + z)^3}{1 - z - z^2} = 1 + 3z + 5z^2 + 8z^3 + \ldots$$

whereas
\[ H_{\text{Ext}_R(k,k)}(z) = \frac{(1+z)^3}{(1-z^2)^2} = \frac{(1+z)}{(1-z)^2} = 1 + 3z + 5z^2 + 7z^3 + \ldots \]

which shows that there is exactly one additional relation in dimension 3. It follows that

\[ \text{Ext}_R(k, k) = k[X_1, X_2] * k[X_3]/[[[X_2, X_3], X_1]]. \]

4. Suppose that we are given a graded Lie algebra \( L = L_1 \oplus L_2 \oplus L_3 \oplus \ldots \) over a field \( k \) such that there exists a local ring \( R \) with \( k = R/m \) and \( \varepsilon_{t-1} = \dim_k L_t \). Is it then possible to choose \( R \) such that \( \overline{PE} = L \)?

5. The finitely generated commutative case.

We have

**Theorem 6.** The algebra \( \text{Ext}_R(k,k) \) is finitely generated and commutative iff \( R \) is a local complete intersection with

\[ \dim_k (m^2/m^3) = \binom{n+1}{2}. \]

**Proof.** We only need to prove that if \( \text{Ext}_R(k,k) \) is finitely generated and commutative then \( R \) is a local complete intersection. According to [6] it is sufficient to show that \( \varepsilon_q = 0 \) for \( q \) large. Assume the contrary and let \( \{ Y_t \mid \deg Y_t \leq M_1 \} \) generate \( \text{Ext}_R(k,k) \). Choose \( M_2 \geq M_1 \) such that \( \varepsilon_{M_2} = 0 \). Then there are \( \varepsilon_{M_2} \) variables adjoined in dimension \( M = M_2 + 1 > M_1 \) in the minimal algebra resolution of \( k \). Let \( \deg Y_t = M \). Since the algebra is commutative \( Y_t \) may be written as a linear combination of monomials of type

\[ Y^{(r_1, \ldots, r_N)} = Y_1^{r_1} \cdots Y_N^{r_N}, \quad \text{where} \quad N = \varepsilon_0 + \cdots + \varepsilon_{M_1-1}. \]

Let \( S^{(r_1, \ldots, r_N)} = S_N^{(r_N)} \cdots S_1^{(r_1)} \) and order the \( N \)-tuples \( (r_1, \ldots, r_N) \) as in section 1.

Let \( Y^{(r_1, \ldots, r_N)} \) be the monomial in the expression for \( Y_t \) with the least exponent \( (r_1, \ldots, r_N) \). Then

\[ Y^{(r_1, \ldots, r_N)} S^{(r_1, \ldots, r_N)} = 1 \quad \text{and} \quad Y^{(r_1, \ldots, r_N)} S^{(r_1, \ldots, r_N)} = 0 \]

for the other monomials in the expression for \( Y_t \). Thus \( Y_t S^{(r_1, \ldots, r_N)} \neq 0 \), which is a contradiction. It follows that \( R \) is a local complete intersection.

**References**


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