SOME REMARKS CONCERNING PATHOLOGICAL SUBMEASURES

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Summary.

This paper consists of partly fragmentary results on pathological and almost pathological submeasures as discussed by Christensen and Herer in [1]. Of greatest interest is perhaps the explicit construction of a pathological submeasure, not relying on category arguments or similar methods. This is achieved utilizing a simple construction due to Preiss and Vili'movský of almost pathological submeasures.

Let $X$ be a set and $\mathcal{A}$ an algebra of subsets of $X$. A set function $\varphi : \mathcal{A} \to [0, \infty[ \,$ is called a submeasure if $\varphi$ is monotone and subadditive and $\varphi(\emptyset) = 0$. If further, $\varphi(X) = 1$, $\varphi$ is said to be normalized. For a submeasure $\varphi$, we denote by $\alpha(\varphi)$ the supremum of $\mu(X)$ taken over all finitely additive measures $\mu : \mathcal{A} \to [0, \infty[ \,$ for which $\mu \leq \varphi$. We follow Christensen and Herer [1] and say that $\varphi$ is a pathological submeasure if $\varphi(X) > 0$ and $\alpha(\varphi) = 0$. The submeasure $\varphi$ is called $\varepsilon$-pathological if $\varphi(X) > 0$ and if $\alpha(\varphi) \leq \varepsilon \varphi(X)$.

The interest in these submeasures is due to a well-known conjecture of Dorothy Maharam, cf. [2]; we also refer the reader to the paper by Christensen and Herer for a discussion of this.

Throughout the paper, the algebra $\mathcal{A}$ will simply be $2^X$, the set of all subsets of $X$. Thus all measures and submeasures are assumed without further saying to be defined on $2^X$.

For a submeasure $\varphi : 2^X \to [0, \infty[$ it can be proved that

\begin{equation}
\alpha(\varphi) = \inf \{ \sum c_i \varphi(A_i) : \sum c_i 1_{A_i} \geq 1_X \}.
\end{equation}

Here, we only consider finite sums, the $c_i$'s are positive numbers, the $A_i$'s run over $2^X$ and $1_A$ denotes the indicator-function of $A$. The inequality "$\leq$" in (1), which is in fact the essential one for what follows, is quite trivial, and the reverse inequality can be proved via a Hahn-Banach argument (the reader may wish to consult Lemma 8.5 of [3] or he can, at least for finite $X$, prove (1) by considering the dual problem to

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that of calculating $\alpha(\varphi)$; it turns out that the "inf" in (1) can be replaced by "min".

Let $\mathcal{S}$ be a class of non-empty subsets of $X$ such that $X$ can be covered by finitely many sets in $\mathcal{S}$. By $\varphi_{\mathcal{S}}$, the submeasure generated by $\mathcal{S}$, we understand the submeasure for which $\varphi_{\mathcal{S}}(A)$ is the minimal number of sets in $\mathcal{S}$ needed to cover $A$; $A \in 2^X$. By (1), it is not difficult to show that

\begin{equation}
\alpha(\varphi_{\mathcal{S}}) = \inf \{ \sum c_i : \sum c_i 1_{S_i} \geq 1_X \},
\end{equation}

it being understood that the $S_i$'s run over $\mathcal{S}$.

**Lemma 1.** Consider $\varphi_{\mathcal{S}}$, the submeasure generated by $\mathcal{S}$. Let $\nu$ be a finite finitely additive measure on $2^X$ with $\nu(X) > 0$ and assume that $\nu(S)$ is independent of $S$ for $S \in \mathcal{S}$. If further, for some natural numbers $n, m$ there exist sets $S_1, S_2, \ldots, S_n$ in $\mathcal{S}$ with $\sum_i 1_{S_i} = m 1_X$, then

\[ \alpha(\varphi_{\mathcal{S}}) = \nu(X) / \nu(S) = n / m. \]

**Proof.** If $\sum c_i 1_{S_i} \geq 1_X$, then

\[ \int (\sum c_i 1_{S_i}) d\nu \geq \nu(X), \]

and it follows that

\[ \sum c_i \geq \nu(X) / \nu(S). \]

By (2) this argument shows that $\alpha(\varphi_{\mathcal{S}}) \geq \nu(X) / \nu(S)$.

As $\sum_i m^{-1} 1_{S_i} = 1_X$, we get by (2) that $\alpha(\varphi_{\mathcal{S}}) \leq n / m$ and also, it follows that $n / m = \nu(X) / \nu(S)$.

In particular, the lemma applies with $\nu =$ counting measure on a finite set, in which case the essential requirements are that the sets in $\mathcal{S}$ contain the same number of elements and that $\sum_i 1_{S_i} = m 1_X$.

It would be interesting to know how pathological a submeasure a given space $X$ supports. To be more precise, we would like to have information concerning the numbers $\alpha_n$; $n \geq 1$ defined by

\begin{equation}
\alpha_n = \inf \{ \alpha(\varphi) : \varphi \text{ normalized submeasure on } 2^X \text{ with } |X| = n \}.
\end{equation}

Here, and below, $|\cdot|$ indicates cardinality of finite sets. Clearly, $\alpha_n \geq 1/n$. We now derive some upper bounds on $\alpha_n$.

**Example 1 (Herer).** Let $X$ be a set of cardinality $n \geq 2$ and consider the class

\[ \mathcal{S} = \{ S \subseteq X : |S| = n - 1 \}. \]
For \( \varphi = \varphi_{S} \), one has
\[
\varphi(A) = \begin{cases} 
0 & \text{if } A = \emptyset \\
1 & \text{if } A \neq \emptyset \text{ and } A \neq X \\
2 & \text{if } A = X .
\end{cases}
\]

As
\[
\sum_{S \in \mathcal{S}} 1_{S} = (n - 1)1_{X} ,
\]
we get by Lemma 1,
\[
\alpha(\varphi) = n/(n - 1) .
\]

Considering the normalized submeasure \( \frac{1}{2} \varphi \), this implies that
\[
(4) \quad \alpha_{n} \leq \frac{1}{2} + \frac{1}{2(n - 1)} ; \quad n \geq 2 .
\]

I was presented with these details in June 1973 by W. Herer. At the time (4) seemed to be the best bound known, and a main question of Herer was, if the \( \alpha_{n} \) were bounded away from 0.

Clearly, a similar analysis with \( \mathcal{S} \) the class of all subsets of \( X \) of fixed cardinality \( v \) \( (1 \leq v \leq n) \), can be carried out, but this will not decrease the bound in (4) – for reasons which we shall make clear below.

We shall show that the bound in (4) is best possible provided you restrict attention to symmetric submeasures; by a symmetric submeasure we understand a submeasure for which \( \varphi(A) \) only depends on \( |A| \).

**Proposition 1.** For any normalized symmetric submeasure \( \varphi \) on \( X \) with \(|X| = n \geq 2 \), we have
\[
\alpha(\varphi) \geq \frac{1}{2} + \frac{1}{2(n - 1)} .
\]

**Proof.** Let \( \varphi \) be a normalized symmetric submeasure and denote by \( p_{r} \) the value of \( \varphi \) on sets of cardinality \( v \); \( 0 \leq v \leq n \). Then
\[
0 = p_{0} \leq p_{1} \leq \ldots \leq p_{n-1} \leq p_{n} = 1 ,
\]
and
\[
p_{\min(s + t, n)} \leq p_{s} + p_{t} \quad \text{for all } s, t \in \{0, 1, \ldots, n\} .
\]

We first prove that
\[
(7) \quad \alpha(\varphi) = n \cdot \min_{1 \leq r \leq n} p_{r} / v .
\]

Denote the points in \( X \) by \( x_{i} ; i = 1, 2, \ldots, n \) and denote by \( \varepsilon_{x} \) a unit mass at \( x \). The measure
\[
\mu_{0} \sum_{i=1}^{n} \varepsilon_{x_{i}} \quad \text{with } \mu_{0} = \min_{1 \leq r \leq n} p_{r} / v
\]

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is dominated by \( \varphi \), and from this observation the inequality "\( \geq \)" in (7) follows.

To prove the reverse inequality, assume that \( \mu \leq \varphi \) with

\[
\mu = \sum_{i}^{n} \mu_{i} \varepsilon_{i}.
\]

Assume, as we may, that

\[
\mu_{1} \geq \mu_{2} \geq \ldots \geq \mu_{n}.
\]

Then

\[
\sum_{i}^{v} \mu_{i} \leq p_{v}, \quad v = 1, 2, \ldots, n.
\]

For \( 1 \leq v \leq n \) we have

\[
\mu(X) = \sum_{i}^{v} \mu_{i} + \sum_{v+1}^{n} \mu_{i} \leq p_{v} + \sum_{v+1}^{n} v^{-1} \sum_{j} \mu_{j} = p_{v} + (n-v)v^{-1} \sum_{j} \mu_{j} \leq nv^{-1} p_{v}.
\]

Hence

\[
\mu(X) \leq n \cdot \min_{1 \leq v \leq n} p_{v} / v.
\]

This shows that "\( \leq \)" holds in (7). (7) is thus fully proved.

To finish the proof, we shall show that for \( 1 \leq v \leq n \),

\[
(8) \quad n \cdot p_{v} / v \geq \frac{1}{2} + 1/2(n-1).
\]

This is clear if \( v = n \). If \( v = n-1 \), (8) is equivalent to \( p_{n-1} \geq \frac{1}{2} \), and this inequality holds since

\[
2p_{n-1} \geq p_{n-1} + p_{1} \geq 1.
\]

Now assume that \( 1 \leq v \leq n-2 \). Let \( k \) be the integer determined by

\[
n / v - 1 \leq k < n / v.
\]

From (5) and (6) we deduce the validity of the following \( k + 1 \) inequalities:

\[
p_{v} \geq p_{v} - p_{(i-1)v}; \quad i = 1, 2, \ldots, k,
p_{v} \geq 1 - p_{kv}.
\]

The sum of the right hand sides is 1. Hence at least one of the right hand sides is \( \geq (k+1)^{-1} \). We conclude that

\[
p_{v} \geq \frac{1}{k+1} \geq \frac{1}{n / v + 1} = \frac{v}{n + v} \geq \frac{v}{n + (n-2)} = \frac{v}{2(n-1)},
\]

and it follows that

\[
n \frac{p_{v}}{v} \geq \frac{n}{2(n-1)} = \frac{1}{2} + \frac{1}{2(n-1)},
\]

which proves (8).
**Remark.** According to [1], A. H. Stone also observed that \( \alpha(\varphi) \geq \frac{1}{3} \) when \( \varphi \) is a normalized symmetric submeasure.

The restriction to symmetric submeasures in Proposition 1 is essential; without this restriction the result fails, indeed, it can be proved that \( \alpha_n \to 0 \). The example needed to show that the \( \alpha_n \) are 'small', can either be taken from [1] or we can use a construction by Preiss and Vili'movsky which was found independently of the research in [1] and at about the same time. The latter construction, which seems simpler than the one in [1], was communicated to the author by Preiss in February 1975, and we shall now give the details.

**Example 2 (Preiss and Vili'movsky').** Let \( \Delta \) denote a set consisting of \( n \) elements, let \( 1 \leq k \leq n \), and denote by \( X \) the set of all subsets of \( \Delta \) of cardinality \( k \). Thus \( |X| = \binom{n}{k} \). For each \( i \in \Delta \) define \( S_i \subseteq X \) by

\[
S_i = \{ E \in X : i \in E \}.
\]

Let \( \mathcal{S} \) denote the class of all \( S_i; i \in \Delta \) and consider the submeasure \( \varphi = \varphi_\mathcal{S} \).

Clearly,

\[
\sum_{i \in \Delta} 1_{S_i} = k1_X,
\]

hence, according to Lemma 1,

\[
\alpha(\varphi) = \frac{n}{k}.
\]

To evaluate \( \varphi(X) \), first observe, that for any subset \( I \) of \( \Delta \) with \( |I| = n - k + 1 \), we have

\[
\bigcup \{ S_i : i \in I \} = X,
\]

hence \( \varphi(X) \leq n - k + 1 \). On the other hand, if \( |I| = n - k \), we have

\[
\Delta \setminus I \subseteq X \setminus \bigcup \{ S_i : i \in I \},
\]

and this shows that \( \varphi(X) > n - k \). Thus

\[
(9) \quad \varphi(X) = n - k + 1.
\]

Normalizing \( \varphi \), it follows that

\[
\alpha_N \leq \frac{n}{k(n - k + 1)} \quad \text{with} \quad N = \binom{n}{k},
\]

and choosing \( k = n/2 \), say, it follows that \( \alpha_N \to 0 \) for \( N \to \infty \).

We mention a generalization of (9) which we need later on. For any \( I \subseteq \Delta \) with \( |I| \leq n - k + 1 \) it can easily be shown that

\[
(10) \quad \varphi(\bigcup_{i \in I} S_i) = |I|.
\]
It seems very difficult to obtain more precise information on the $\alpha_n$'s. Even for small values of $n$, for instance for $n=5$, the value of $\alpha_n$ is unknown.

Denote by $\Phi_n$ the set of normalized submeasures on $X = \{1, 2, \ldots, n\}$. One could also try and characterize $\text{ext}\Phi_n$, the set of extreme points of $\Phi_n$. This is an ambitious program, and even though we are very far from having such a characterization, we do want to give some comments.

It seems plausible, that if $\varphi \in \text{ext}\Phi_n$, then there exists an integer $m$ with $1 \leq m \leq n-1$ such that $\varphi$ assumes all the values $i/m; i = 0, 1, \ldots, m$ and no other values. For $m=1, 2$ we are able to characterize the extremal submeasures of this type. For $m=1$ this is trivial since any submeasure assuming only the values 0 and 1 is extremal, and the $(0,1)$-submeasures are uniquely determined by the maximal 0-set $M_0$ which could be any set with $\emptyset \subseteq M_X \subset X$ ("\subset" denotes strict inclusion).

For $m=2$ we look at $(0, \frac{1}{2}, 1)$-submeasures. Let $M_0$ and $M_i; 1 \leq i \leq r$ (with $1 \leq r < \infty$), be subsets of $X$ such that

$$M_0 \subset M_i \subset X; \quad i = 1, 2, \ldots, r,$$

$$M_i \not\subset M_j; \quad i \neq j, i \geq 1, j \geq 1.$$

Then $\varphi$ defined by

$$\varphi(A) = \begin{cases} 0 & \text{if } A \subseteq M_0, \\ \frac{1}{2} & \text{if } A \subseteq M_i \text{ for some } i \geq 1 \text{ and } A \not\subseteq M_0, \\ 1 & \text{otherwise,} \end{cases}$$

is a $(0, \frac{1}{2}, 1)$-submeasure. Every $(0, \frac{1}{2}, 1)$-submeasure arises in this way. Furthermore, $\varphi$ defined above is extremal if and only if either $r \geq 3$ or $r = 2$ and $M_1 \cap M_2 \neq M_0$. For instance, with the choice $M_0 = \emptyset$ and $M_1, \ldots, M_n = \text{all subsets of } X$ with cardinality $n-1$, we obtain the normalized submeasure from example 1, and this is extremal, except when $n = 2$.

For $n=3$, the extremal submeasures we have found so far, yield 12 elements in $\text{ext}\Phi_n$ and probably, there are no more.

We also mention, that for $1 \leq m \leq n-1$

$$\varphi = \min(m^{-1}\sum_i e_i, 1)$$

belongs to $\text{ext}\Phi_n$.

As the above results are only fragments, we shall not mention the proofs. Instead, we turn to an explicit construction of a pathological submeasure based on the $\varepsilon$-pathological submeasures of Preiss and Vili'movský.
Example 3. Let \((\Delta_n)_{n \geq 1}\) be pairwise disjoint sets with \(|\Delta_n| = 2^n; n \geq 1\). Denote by \(X_n\) the set of all subsets of \(\Delta_n\) with cardinality \(2^{n-1}\).

For a subset \(I \subseteq \Delta_n\) we put
\[
A(n, I) = \{x \in X_n : i \in x \text{ for some } i \in I\}.
\]

We define the submeasure \(\varphi_n\) on \(X_n\) by
\[
\varphi_n(E) = 2^{-n+1} \min \{|I| : I \subseteq \Delta_n, E \subseteq A(n, I)\}; \quad E \subseteq X_n.
\]

According to Example 2,
\[
\varphi_n(X_n) = 1 + 2^{-n+1}.
\]

The sets \((X_n)\) are pairwise disjoint and we now consider the set \(X = \bigcup_1^\infty X_n\) provided with the submeasure \(\varphi\) defined by
\[
\varphi(E) = \limsup_{n \to \infty} \varphi_n(E \cap X_n); \quad E \subseteq X.
\]

By (11), \(\varphi\) is a normalized submeasure on \(X\).

We shall prove that \(\varphi\) is pathological. Assume therefore, that \(\mu\) is a finitely additive measure on \(X\) bounded by \(\varphi\). Fix, for some time, \(n\).

For \(m \geq n\) denote by \((I_{m,v})_{v = 1, 2, \ldots, 2^n}\) a decomposition of \(\Delta_m\) into \(2^n\) sets each consisting of \(2^{m-n}\) elements. Then we have
\[
\sum_{v=1}^{2^n} 1_{A(m, I_{m,v})} \geq 2^{n-1} \cdot 1_{X_m}.
\]

Define subsets \(A_v\) of \(X; v = 1, 2, \ldots, 2^n\), by
\[
A_v = \bigcup_{m \geq n} A(m, I_{m,v}).
\]

By (12) we have,
\[
\sum_{v=1}^{2^n} 1_{A_v} \geq 2^{n-1} \cdot 1_{\bigcup_1^\infty X_m}.
\]

We also need the fact, deduced from (10), that for \(m \geq n\) and \(v = 1, 2, \ldots, 2^n\),
\[
\varphi_m(A(m, I_{m,v})) = 2^{-n+1}.
\]

As \(\mu(\bigcup_1^{n-1} X_k) \leq \varphi(\bigcup_1^{n-1} X_k) = 0\), we now get from (13) and (14):
\[
2^{n-1} \mu(X) = 2^{n-1} \mu(\bigcup_1^\infty X_m)
\leq \sum_{v=1}^{2^n} \mu(A_v)
\leq \sum_{v=1}^{2^n} \limsup_{m \to \infty} \varphi_m(A_v \cap X_m)
= \sum_{v=1}^{2^n} \limsup_{m \to \infty} \varphi_m(A(m, I_{m,v}))
= \sum_{v=1}^{2^n} 2^{-n+1}
= 2.
\]

It follows that \(\mu(X) \leq 2^{-n+2}\). As this holds for each \(n, \mu = 0\). Thus \(\varphi\) is pathological.
We mention that the pathological submeasure constructed above possesses none of the desirable continuity properties discussed in [1].

REFERENCES


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