MATH. SCAND. 126 (2020), 259–275

LOCAL BOUNDEDNESS FOR MINIMIZERS OF CONVEX INTEGRAL FUNCTIONALS IN METRIC MEASURE SPACES

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Abstract

In this paper we consider the convex integral functional $I := \int_\Omega \Phi(g_u)\, d\mu$ in the metric measure space (X, d, μ), where X is a set, d is a metric, μ is a Borel regular measure satisfying the doubling condition, Ω is a bounded open subset of X, u belongs to the Orlicz-Sobolev space $N^{1,\Phi}(\Omega)$, Φ is an N-function satisfying the Δ_2-condition, g_u is the minimal Φ-weak upper gradient of u. By improving the corresponding method in the Euclidean space to the metric setting, we establish the local boundedness for minimizers of the convex integral functional under the assumption that (X, d, μ) satisfies the $(1, 1)$-Poincaré inequality. The result of this paper can be applied to the Carnot-Carathéodory space spanned by vector fields satisfying Hörmander's condition.

1. Introduction

The integral functional

$$\int_\Omega \Phi(|\nabla u|)\, dx \tag{1.1}$$

has been widely studied in the Euclidean space $\mathbb{R}^n$ (see Breit and Verde [4], Esposito, Leonetti and Mingione [6]), where Ω is a bounded open subset of $\mathbb{R}^n$, $|\nabla u|$ stands for the Euclidean norm of ∇u and Φ is an N-function (see §2). The Euler-Lagrange equation corresponding to (1.1) is

$$\operatorname{div}\left(\frac{\Phi'(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0.$$

The notion of Sobolev spaces was generalized to the metric measure setting by Hajłasz [8], Franchi, Lu and Wheeden [7], Cheeger [5], Shanmugalingam [18]. The class of metric measure spaces contains the Carnot-Carathéodory spaces, in particular the Carnot groups (Hajłasz and Koskela [9]).

This paper is supported by NSFC (Grant No. 11771354) and NSFC (Grant No. 11601427). Huiju Wang also supported by China Postdoctoral Science Foundation (Grant No. 2017M613193); Natural Science Basic Research Plan in Shaanxi Province of China (Grant No. 2017JQ1009). Corresponding author: Pengcheng Niu.

Received 15 July 2018, in final form 7 April 2019. Accepted 14 May 2019.

DOI: https://doi.org/10.7146/math.scand.a-116244

Hereafter, the theory of Sobolev spaces, quasiconformal maps and geometric analysis on metric measure spaces have been extensively studied, see [18], Heinonen and Koskela [10], [5]. The nonlinear potential theory based on Newton-Sobolev spaces $N^{1,p}(X)$ has achieved fruitful results, see Björn and Björn [3]. The generalizations of Orlicz-Sobolev spaces to metric measure spaces were introduced by Tuominen [19], Aïssaoui [2].

For the convex integral functional

$$I := \int_\Omega \Phi(g_u)\, d\mu \tag{1.2}$$

defined on the Orlicz-Sobolev space $N^{1,\Phi}(\Omega)$, Niu and Wang [17] established a Caccioppoli type inequality for minimizers of (1.2), and then the higher order integrability was built. In (1.2), $u \in N^{1,\Phi}(\Omega)$, g_u is the minimal Φ-weak upper gradient of u, Φ is an N-function satisfying the Δ_2-condition. In this paper, we treat the local boundedness for minimizers of (1.2).

For the reader's convenience, we recall the definition of minimizers, see also [17].

Suppose that $\Omega \subseteq X$ is a bounded open set, if a function $u \in N^{1,\Phi}(\Omega)$ satisfies

$$\int_{\Omega'} \Phi(g_u)\, d\mu \le \int_{\Omega'} \Phi(g_{u+\phi})\, d\mu, \quad \text{for any } \phi \in N_0^{1,\Phi}(\Omega'),$$

where Ω' is any open subset of Ω, then u is called a minimizer of I in (1.2).

Mascolo and Papi [14] proved local boundedness for minimizers of (1.1):

Theorem A ([14]). *Let $\Phi \in C^1([0,+\infty))$ be an N-function satisfying the Δ_2-condition. Let $u \in W^1L_\Phi(\Omega)$ be a local minimizer of the functional in* (1.1). *Then there exists a constant C such that for every ρ and R, with $0 < \rho < R < \rho + 1$,*

$$\sup\{\Phi(|u(x)|) : x \in B_\rho\} \le \frac{C}{(R-\rho)^{n\tau}} \int_{B_R} \Phi(|u(x)|)\, dx,$$

where τ is the exponent appearing in (3.2) *below, B_ρ and B_R are balls compactly contained in Ω of radii respectively ρ, R and with the same center.*

Inspired by [14], we establish the local boundedness for minimizers of (1.2):

Theorem 1.1. *Suppose that the metric measure space (X, d, μ) satisfies the* $(1,1)$*-Poincaré inequality* (*see* (2.3) *below*), *μ is a Borel regular measure satisfying the doubling condition, Φ is an N-function satisfying the Δ_2-condition, and Ω is a bounded open subset of X. If $\Phi \in C^1([0,+\infty))$, u is a minimizer* *of I, then for any $z \in \Omega$, $0 < \rho < R < \min\{2\rho, \rho + 1\}$, $B(z, R) \subseteq \Omega$, $R < \operatorname{diam}(X)/4$, there exists a positive constant C only depending on the doubling constant C_μ, Φ and the $(1, 1)$-Poincaré inequality, such that*

$$\sup_{B(z,\rho)} \Phi(|u|) \leq C \frac{R^{1/\theta}}{(R-\rho)^{\tau/\theta}} \frac{1}{\mu(B(z,R))} \int_{B(z,R)} \Phi(|u|)\, d\mu, \tag{1.3}$$

where $\theta = 1 - 1/K$, and K is the constant in (1.4) below.

In order to determine the local boundedness for minimizers of (1.2), we take the following strategy.

(i) According to [9], the self-improvement property of the $(1, 1)$-Poincaré inequality (see (2.3) below) implies a $(K, 1)$-Poincaré inequality for some $K > 1$, so it follows from [3, Theorem 5.51], that

$$\left(\frac{1}{\mu(B)} \int_B |u|^K\, d\mu\right)^{1/K} \leq cr\left(\frac{1}{\mu(B)} \int_B g_{u,1}\, d\mu\right), \tag{1.4}$$

where B is a metric ball of radius $r < \operatorname{diam}(X)/4$, $u \in N_0^{1,1}(B)$, and $g_{u,1}$ is the minimal 1-weak upper gradient of u. Then for a minimizer u of functional (1.2) and any real number k, we demonstrate $\Phi((u-k)_+) \in N^{1,1}(B)$ (see Lemma 3.3 below); next we take a cutoff function η such that $\Phi((u-k)_+)\eta \in N_0^{1,1}(B)$ and insert $\Phi((u-k)_+)\eta$ into (1.4) to get the inequality (3.6) below. Now with results in [3] and Mocanu [16], one can exploit the chain rule to deal with $g_{\Phi((u-k)_+)\eta,1}$. Using Proposition 2.4 below, we claim that in the bounded open set, the minimal 1-weak upper gradient of a function can be controlled by its minimal Φ-weak upper gradient, and obtain the similar result with [14].

(ii) In estimating $g_{\Phi((u-k)_+)\eta,1}$, authors in [14] used the Euler-Lagrange equation corresponding to (1.1). But it is not clear how to define a partial differential equation in the metric measure space, so the method in [14] does not fit here. Luckily, we proved in [17] that minimizers of functional (1.2) satisfy the Caccioppoli type inequality

$$\int_{B(z,\rho)} \Phi(g_{(u-k)_+})\, d\mu \leq C \int_{B(z,R)} \Phi\left(\frac{(u-k)_+}{R-\rho}\right) d\mu \tag{1.5}$$

on the level set

$$\{x \in B(z, R) : u(x) > k\}, \quad (k \in \mathbb{R})$$

here $0 < \rho < R$, which is helpful to estimate $g_{\Phi((u-k)_+)\eta,1}$.

(iii) Finally, we adopt the iteration method similar to that of Kinnunen and Shanmugalingam [11] and obtain the local boundedness for minimizers of I.

REMARK 1.2. According to Tuominen [20], if we let the metric measure space (X, d, μ) be $(\mathbb{R}^n, |\cdot|, dx)$ and let Φ satisfy the Δ_2-condition, then the Orlicz-Sobolev space $N^{1,\Phi}(\Omega)$ coincides with $W^1 L_\Phi(\Omega)$. Assumptions in Theorem 1.1 (such as the doubling condition, the Poincaré inequality) are naturally satisfied in Euclidean space. In the Euclidean space $(\mathbb{R}^n, |\cdot|, dx)$ we may take $\theta = 1/n$ in Theorem 1.1 and so we obtain

$$\begin{aligned}\sup_{B(z,\rho)} \Phi(|u|) &\le C \frac{R^n}{(R-\rho)^{\tau/\theta}} \frac{1}{|B(z,R)|} \int_{B(z,R)} \Phi(|u|)\, dx \\ &\le C \frac{1}{(R-\rho)^{\tau n}} \int_{B(z,R)} \Phi(|u|)\, dx,\end{aligned}$$

which is exactly the result in Theorem A.

REMARK 1.3. Metric measure spaces satisfying the $(1, 1)$-Poincaré inequality include Riemannian manifolds with nonnegative Ricci curvatures, Q-regular orientable topological manifolds satisfying the local linear contractability condition, homogeneous groups, and Carnot-Carathéodory spaces generated by fields satisfying Hörmander's condition [9]. Therefore, local boundedness for convex integral functionals are also true in these settings. In particular, we have the following result.

COROLLARY 1.4. *If the Carnot-Carathéodory space* $(\Omega, |\cdot|_Z, d_Z x)$ *is induced by Hörmander's vector fields* $Z_1, Z_2, \dots, Z_m$, Φ *is an N-function satisfying the* Δ_2*-condition,* Ω' *is a precompact subset of* Ω, $\Phi \in C^1([0, +\infty))$, *and u is a minimizer of*

$$\int_{\Omega'} \Phi(|Zu|)\, d_Z x$$

with the corresponding Euler-Lagrange equation

$$\operatorname{div}_Z\left(\frac{\Phi'(|Zu|)}{|Zu|} Zu\right) = 0,$$

where

$$|Zu| = \left(\sum_{i=1}^m |Z_i u|^2\right)^{1/2}, \quad Zu = (Z_1 u, Z_2 u, \dots, Z_m u),$$

then there exist positive constants c and R_0 *only depending on the Carnot-Carathéodory space and* Φ *such that, for any* $z \in \Omega'$, $0 < \rho < R < \min\{2\rho, \rho + 1\}$, $R < R_0$, $B(z, R) \subseteq \Omega'$,

$$\sup_{B(z,\rho)} \Phi(|u|) \le C \frac{R^{1/\theta}}{(R-\rho)^{\tau/\theta}} \frac{1}{\mu(B(z,R))} \int_{B(z,R)} \Phi(|u|)\, d_Z x,$$

where $\theta = 1-1/K$*, K is the constant in the inequality* (*see* [9, Theorem 11.20] *and* [3, Theorem 5.51])

$$\left(\frac{1}{\mu(B)}\int_B |u|^K\, d_Z x\right)^{1/K} \le cr(B)\left(\frac{1}{\mu(B)}\int_B |Zu|\, d_Z x\right).$$

The paper is organized as follows. In §2, we describe the Φ-weak upper gradient in the metric measure space, the Orlicz-Sobolev space and the (1, 1)-Poincaré inequality. In §3 we prove Theorem 1.1 in three steps: we first obtain (3.4) by choosing the suitable cutoff function, and, combining with properties of Φ-weak upper gradients, the Poincaré inequality and Caccioppoli type inequality; next a weak Harnack type inequality (3.17) is built by iterating from (3.4); and finally we get (1.3) from (3.17).

2. Preliminaries

From now on we always require that the measure μ in (X, d, μ) is Borel regular and doubling, where the doubling condition means that for any metric ball $B(x,r) \subseteq X$, there exists a constant $C_\mu > 0$ such that

$$\mu(B(x,2r)) \le C_\mu \mu(B(x,r)).$$

We also assume the metric balls to be open and have positive and finite measure. According to Lemma 3.3 in [3], for any $B(x,R) \subseteq X$, $y \in B(x,R)$ and $0 < r \le R < +\infty$, there exist positive constants c and Q only depending on the doubling condition such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge c\left(\frac{r}{R}\right)^Q.$$

If ϕ is a real-valued function on $[0,+\infty)$ and satisfies (i) $\phi(0)=0$, $\phi(t) > 0$, for $t > 0$, and $\lim_{t\to+\infty}\phi(t) = +\infty$; (ii) ϕ is monotone increasing; (iii) ϕ is right continuous, then the real-valued function

$$\Phi(t) = \int_0^t \phi(s)\, ds$$

is called an *N-function* on $[0,+\infty)$. See Adams and Fournier [1], Rutickiĭ and Krasnosel'skiĭ [12] for the properties of N-functions.

If there exists a constant $C_1 > 0$ such that

$$\Phi(2t) \le C_1\Phi(t)$$

for any $t \ge 0$, then we say that Φ satisfies the Δ_2*-condition*.

Analogously to the Euclidean space case, the Orlicz space $L_\Phi(X)$ on X is defined by

$$L_\Phi(X) = \left\{ u : \text{there exists } a > 0 \text{ such that } \int_X \Phi(a|u|) < \infty \right\}, \tag{2.1}$$

where u is any measurable extended-real-valued function on X and Φ is some N-function. If Φ satisfies the Δ_2-condition, then the space $L_\Phi(X)$ in (2.1) is a Banach space under the Luxemburg norm

$$\|u\|_\Phi = \inf\left\{ k : \int_X \Phi(|u|/k) \le 1 \right\}.$$

Since the notion of the gradient in Euclidean space is not well-defined in a metric measure space, the upper gradient was introduced as a substitute, see [10]. The definition of an important generalization of upper gradient, called weak upper gradient, involves the modulus of a family of curves.

Suppose that γ is a continuous map from a closed interval I to X; the image of γ in X is called a curve in X. If γ has finite length, then γ is said to be rectifiable. Let γ be a family of these curves and let Φ be an N-function, then

$$\mathrm{Mod}_\Phi(\Gamma) = \inf_\alpha \|\alpha\|_\Phi$$

is called the Φ-modulus of γ, where α is a non-negative Borel function satisfying

$$\int_\gamma \alpha\, ds \ge 1 \quad \text{for any } \gamma \in \Gamma.$$

It follows from [19] that the Φ-modulus is an outer measure defined on γ in X, and if $\Gamma \subseteq \Omega$, for some open subset Ω of X, then the Φ-modulus of Γ with respect to X coincides with one with respect to Ω.

We recall the definitions of the upper gradient [10] and Φ-weak upper gradient [19].

DEFINITION 2.1. Suppose that u is a real-valued measurable function on X. If a non-negative Borel function g satisfies

$$\left|u(\gamma(0)) - u(\gamma(\ell_\gamma))\right| \le \int_\gamma g\, ds, \tag{2.2}$$

for any rectifiable curve $\gamma\colon [0, \ell_\gamma] \to X$, then g is called an *upper gradient* of u. If there exists a non-negative measurable function g satisfying (2.2) for rectifiable curves except for a family with zero Φ-modulus, then g is called a Φ-*weak upper gradient* of u.

Let $\tilde{N}^{1,\Phi}(X)$ be a collection of functions $u \in L_\Phi(X)$ allowing Φ-weak upper gradients in $L_\Phi(X)$. Its semi-norm is defined by

$$\|u\|_{1,\Phi} = \|u\|_\Phi + \inf \|g\|_\Phi,$$

where the infimum is taken over all Φ-weak upper gradients g of u. The *Orlicz-Sobolev space* $N^{1,\Phi}(X)$ is the quotient space of $\tilde{N}^{1,\Phi}(X)$ under the equivalence relation $\|u - v\|_{1,\Phi} = 0$ and the norm in $N^{1,\Phi}(X)$ is

$$\|u\|_{N^{1,\Phi}(X)} = \|u\|_{1,\Phi}.$$

For any open set $\Omega \subseteq X$, one can define the Orlicz-Sobolev space $N^{1,\Phi}(\Omega)$ similarly.

Let $E \subseteq X$, the *Orlicz-Sobolev capacity* on E is defined by

$$\operatorname{cap}_\Phi(E) = \inf\big\{\|u\|_{N^{1,\Phi}(X)} : u \in N^{1,\Phi}(X),\ u|_E \geq 1\big\}.$$

By [20] we know that cap_Φ is an outer measure.

A functions $u\colon \Omega \to [-\infty, +\infty]$ belongs to $\tilde{N}_0^{1,\Phi}(\Omega)$ if there exists $\tilde{u} \in \tilde{N}^{1,\Phi}(X)$ such that

$$\tilde{u} = \begin{cases} u, \text{ a.e.,} & \text{on } \Omega, \\ 0, \text{ a.e.,} & \text{outside } \Omega. \end{cases}$$

The Orlicz-Sobolev space with zero boundary values, denoted by $N_0^{1,\Phi}(\Omega)$, is a quotient space of $\tilde{N}_0^{1,\Phi}(\Omega)$ under the equivalence relation $u = v$ a.e. The norm on $N_0^{1,\Phi}(\Omega)$ is of the form

$$\|u\|_{N_0^{1,\Phi}(\Omega)} = \|\tilde{u}\|_{N^{1,\Phi}(X)}.$$

It is known from Mocanu [15] that $N_0^{1,\Phi}(\Omega)$ is a closed subspace of $N^{1,\Phi}(X)$ and if $u \in N_0^{1,\Phi}(\Omega)$, then there exists $\hat{u}$ such that

$$\hat{u} = \begin{cases} u, \text{ a.e.,} & \text{in } \Omega, \\ 0, & \text{in } X \setminus \Omega. \end{cases}$$

According to [19], when the N-function Φ satisfies the Δ_2-condition, for a given $u \in \tilde{N}^{1,\Phi}(\Omega)$, its minimal Φ-weak upper gradient g_u exists, that is, there exists a Φ-weak upper gradient $g_u \in L_\Phi(\Omega)$ of u such that

$$g_u \leq g, \quad \mu\text{-a.e.},$$

where g is any Φ-weak upper gradient of u and $g \in L_\Phi(\Omega)$. Note that the minimal Φ-weak upper gradient is unique up to a set of measure zero, and the minimal Φ-weak upper gradient has many properties similar to minimal

p-weak upper gradients. Here we state some properties needed in this paper, see also [17] and references therein. We mention that Ω is a bounded open subset of X.

PROPOSITION 2.2. *If* $u_1, u_2 \in N^{1,\Phi}(\Omega)$, *then*

$$g_{u_1+u_2} \le g_{u_1} + g_{u_2} \text{ a.e. on } \Omega.$$

PROPOSITION 2.3. (1) *If* $u \in N^{1,\Phi}(\Omega)$ *and* c *is a constant, then* g_u *is almost everywhere zero on the set* $\{x : u(x) = c\}$.

(2) *If* $u_1, u_2 \in N^{1,\Phi}(\Omega)$, *then* g_{u_1} *and* g_{u_2} *are equal almost everywhere on the set* $\{x : u_1(x) = u_2(x)\}$.

PROPOSITION 2.4. *If* $u \in N^{1,\Phi}(\Omega)$, *then the minimal* 1*-weak upper gradient* $g_{u,1}$ *exists and*

$$g_{u,1} \le g_u, \text{ a.e. on } \Omega.$$

PROPOSITION 2.5. *If* $u_1, u_2 \in N^{1,\Phi}(\Omega)$, *then* $|u_1|g_{u_2} + |u_2|g_{u_1}$ *is a* Φ*-weak upper gradient of* u_1u_2.

PROPOSITION 2.6. *If* $u \in N^{1,1}(\Omega)$, $F \in C^1((-\infty, +\infty))$, *then* $|F'(u)|g_{u,1}$ *is a* 1*-weak upper gradient of* $F(u)$. *Analogously, if* $u \in N^{1,\Phi}(\Omega)$, *then* $|F'(u)|g_u$ *is a* Φ*-weak upper gradient of* $F(u)$.

PROPOSITION 2.7. *If* $u \in N^{1,\Phi}(X)$, g_u *is the minimal* Φ*-weak upper gradient of* u *on* X, *then* $g_u|_\Omega$ *is the minimal* Φ*-weak upper gradient of* u *on* Ω.

PROPOSITION 2.8. *Let* $u \in N_0^{1,\Phi}(\Omega)$ *and* $p \ge 1$, *if*

$$\lim_{t\to+\infty} \frac{\Phi(t)}{t^p} = +\infty,$$

then $u \in N_0^{1,p}(\Omega)$.

We say that X supports a $(1, 1)$*-Poincaré inequality* if there exist constants $C > 0$, $\lambda \ge 1$ such that for all balls $B \subseteq X$, locally integrable functions u on X and all upper gradients g of u,

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \le Cr \frac{1}{\mu(\lambda B)} \int_{\lambda B} g \, d\mu, \tag{2.3}$$

where r is the radius of B and u_B is the integral average of u over B.

3. Proof of Theorem 1.1

We first give two lemmas after describing two known results and then prove Theorem 1.1.

LEMMA 3.1 (Mascolo and Papi [14]). *An N-function Φ of class C^1 satisfies the Δ_2-condition if and only if there exists $\tau > 1$ such that*

$$\Phi'(t)t \le \tau\Phi(t) \quad \textit{for } t \ge 0. \tag{3.1}$$

By (3.1), we have for any $\lambda > 1$, $t \ge 0$,

$$\Phi(\lambda t) \le \lambda^{\tau}\Phi(t). \tag{3.2}$$

LEMMA 3.2 (Marcellini [13]). *If g and h are nonnegative monotone increasing functions on $[0, +\infty)$, then for any $t_1, t_2 \ge 0$,*

$$g(t_1)h(t_2) \le g(t_1)h(t_1) + g(t_2)h(t_2).$$

LEMMA 3.3. *If an N-function Φ of class C^1 satisfies the Δ_2-condition, and $u \in N^{1,\Phi}(\Omega)$, then $\Phi(|u|) \in N^{1,1}(\Omega)$.*

PROOF. We want to prove $\Phi(|u|) \in L^1(\Omega)$ and $\Phi'(|u|)g_{u,1} \in L^1(\Omega)$. Since $u \in L_\Phi(\Omega)$ and Φ satisfies the Δ_2-condition, it follows that

$$\int_\Omega \Phi(|u|)\, d\mu < +\infty$$

and therefore $\Phi(|u|) \in L^1(\Omega)$. By Proposition 2.4, there exists the minimal 1-weak upper gradient $g_{u,1}$ of u and

$$g_{u,1} \le g_u$$

almost everywhere on Ω. Proposition 2.6 and the fact that any N-function Φ can be extended to an even function with continuous derivatives shows that $\Phi'(|u|)g_{u,1}$ is a 1-weak upper gradient of $\Phi(|u|)$. Moreover, by Lemmas 3.2 and 3.1,

$$\begin{aligned}\int_\Omega \Phi'(|u|)g_{u,1}\, d\mu &\le \int_\Omega \Phi'(|u|)|u|\, d\mu + \int_\Omega \Phi'(g_{u,1})g_{u,1}\, d\mu \\ &\le \tau\int_\Omega \Phi(|u|)\, d\mu + \tau\int_\Omega \Phi(g_{u,1})\, d\mu \qquad (3.3)\\ &\le \tau\int_\Omega \Phi(|u|)\, d\mu + \tau\int_\Omega \Phi(g_u)\, d\mu < +\infty.\end{aligned}$$

By (3.3), it follows that $\Phi'(|u|)g_{u,1} \in L^1(\Omega)$.

In the rest of this paper, we use the following notation

$$u(k,\rho) = \frac{1}{\mu(B(z,\rho))}\int_{B(z,\rho)} \Phi((u-k)_+)\,d\mu,$$

for any $k > 0$, $\rho > 0$.

LEMMA 3.4. *Under the conditions of Theorem 1.1, for any* $0 < \rho < R < \min\{2\rho, \rho+1\}$*,* $B(z,R) \subseteq \Omega$*,* $R < \operatorname{diam}(X)/4$*, and* $0 < h < k$*, there exists a constant* $C > 0$ *such that*

$$u(k,\rho) \le \frac{CR}{(R-\rho)^\tau}\left(\frac{R}{\rho}\right)^Q \frac{1}{\Phi(k-h)^\theta}(u(h,R))^{1+\theta}, \tag{3.4}$$

where $\theta = 1 - \frac{1}{K}$*, the constant* C *does not depend on* u*,* z*,* k*,* h*,* R*, or* ρ*.*

PROOF. First, let $z \in \Omega$ and let $0 < \rho < R < \min\{2\rho, \rho+1\}$. Let the cutoff function η be Lipschitz continuous satisfying

$$0 \le \eta \le 1, \quad \operatorname{supp}\eta \subseteq B\left(z, \frac{R+\rho}{2}\right)$$

and $\eta(x) = 1$ for $x \in B(z,\rho)$, $\operatorname{Lip}\eta \le 4/(R-\rho)$. Write

$$A(k,\rho) = \{x \in B(z,\rho) : u(x) > k\}.$$

By Hölder's inequality and the doubling condition,

$$\begin{aligned}
&\frac{1}{\mu(B(z,\rho))}\int_{B(z,\rho)} \Phi((u-k)_+)\,d\mu \\
&\quad \le \left(\frac{1}{\mu(B(z,\rho))}\int_{B(z,\rho)} \Phi((u-k)_+)^K\,d\mu\right)^{1/K}\left(\frac{\mu(A(k,\rho))}{\mu(B(z,\rho))}\right)^{1-1/K} \\
&\quad \le \left(\frac{1}{\mu(B(z,(\rho+R)/2))}\int_{B(z,(\rho+R)/2)} [\Phi((u-k)_+)\eta]^K\,d\mu\right)^{1/K} \\
&\qquad \cdot \left(\frac{\mu(B(z,(\rho+R)/2))}{\mu(B(z,\rho))}\right)^{1/K}\left(\frac{\mu(A(k,\rho))}{\mu(B(z,\rho))}\right)^{1-1/K} \\
&\quad \le C\left(\frac{1}{\mu(B(z,(\rho+R)/2))}\int_{B(z,(\rho+R)/2)} [\Phi((u-k)_+)\eta]^K\,d\mu\right)^{1/K} \\
&\qquad \cdot \left(\frac{\mu(A(k,\rho))}{\mu(B(z,\rho))}\right)^{1-1/K}\left(\frac{R}{\rho}\right)^{Q/K}.
\end{aligned} \tag{3.5}$$

Using $(u-k)_+ \in N^{1,\Phi}(B(x,(R+\rho)/2))$, it follows from Lemma 3.3 that

$$\Phi((u-k)_+) \in N^{1,1}(B(z,(R+\rho)/2)),$$

and by Theorem 5.47 in [3], $\Phi((u-k)_+)\eta \in N_0^{1,1}(B(z,(R+\rho)/2))$, we have by (1.4) that

$$\left(\frac{1}{\mu(B(z,(\rho+R)/2))}\int_{B(z,(\rho+R)/2)}[\Phi((u-k)_+)\eta]^K\,d\mu\right)^{1/K} \leq CR\frac{1}{\mu(B(z,(\rho+R)/2))}\int_{B(z,(\rho+R)/2)} g_{\Phi((u-k)_+)\eta,1}\,d\mu. \quad (3.6)$$

Next we will estimate $g_{\Phi((u-k)_+)\eta,1}$. From properties of minimal 1-weak upper gradients,

$$\begin{aligned} g_{\Phi((u-k)_+)\eta,1} &\leq g_{\Phi((u-k)_+),1}\eta + g_{\eta,1}\Phi((u-k)_+) \\ &\leq \Phi'((u-k)_+)g_{(u-k)_+,1} + \frac{c}{R-\rho}\Phi((u-k)_+). \end{aligned} \quad (3.7)$$

Since $\Phi(t)$ is monotone increasing and convex, it follows that $\Phi'(t)$ is non-negative and monotone increasing and by Lemma 3.2 and Lemma 3.1,

$$\begin{aligned} \Phi'((u-k)_+)g_{(u-k)_+,1} &\leq \Phi'((u-k)_+)(u-k)_+ + \Phi'(g_{(u-k)_+,1})g_{(u-k)_+,1} \\ &\leq \tau\Phi((u-k)_+) + \tau\Phi(g_{(u-k)_+,1}). \end{aligned}$$

Inserting the above inequality into (3.7) we have

$$g_{\Phi((u-k)_+)\eta,1} \leq \tau\Phi((u-k)_+) + \tau\Phi(g_{(u-k)_+,1}) + \frac{c}{R-\rho}\Phi((u-k)_+).$$

By Proposition 2.4, it follows that

$$g_{(u-k)_+,1} \leq g_{(u-k)_+} \quad \text{a.e. on } \Omega,$$

therefore

$$g_{\Phi((u-k)_+)\eta,1} \leq \tau\Phi((u-k)_+) + \tau\Phi(g_{(u-k)_+}) + \frac{c}{R-\rho}\Phi((u-k)_+), \quad (3.8)$$

a.e. on $B(z,R)$. Since $0<\rho<R<\rho+1$, we have

$$1 < \frac{1}{R-\rho} < \frac{1}{(R-\rho)^\tau},$$

and combining this with (3.8) we get

$$g_{\Phi((u-k)_+)\eta,1} \le \tau\Phi(g_{(u-k)_+}) + \frac{C}{(R-\rho)^\tau}\Phi((u-k)_+). \tag{3.9}$$

Inserting (3.9) into (3.6) and using (1.5) and the doubling condition, we obtain that

$$\begin{aligned}
&\left(\frac{1}{\mu(B(z,(\rho+R)/2))}\int_{B(z,(\rho+R)/2)}[\Phi((u-k)_+)\eta]^K\,d\mu\right)^{1/K}\\
&\quad\le \frac{CR}{\mu(B(z,(\rho+R)/2))}\int_{B(z,(\rho+R)/2)}\Phi(g_{(u-k)_+})\,d\mu\\
&\qquad+\frac{CR}{(R-\rho)^\tau}\frac{1}{\mu(B(z,(\rho+R)/2))}\int_{B(z,(\rho+R)/2)}\Phi((u-k)_+)\,d\mu\\
&\quad\le \frac{CR}{\mu(B(z,R))}\int_{B(z,R)}\Phi\left(\frac{(u-k)_+}{R-\rho}\right)d\mu\\
&\qquad+\frac{CR}{(R-\rho)^\tau}\mu(B(z,R))\int_{B(z,R)}\Phi((u-k)_+)\,d\mu.
\end{aligned}$$

Using (3.2), we have

$$\Phi\left(\frac{(u-k)_+}{R-\rho}\right) \le \frac{1}{(R-\rho)^\tau}\Phi((u-k)_+),$$

and so

$$\begin{aligned}
&\left(\frac{1}{\mu(B(z,(\rho+R)/2))}\int_{B(z,(\rho+R)/2)}[\Phi((u-k)_+)\eta]^K\,d\mu\right)^{1/K}\\
&\qquad\le \frac{CR}{(R-\rho)^\tau}\frac{1}{\mu(B(z,R))}\int_{B(z,R)}\Phi((u-k)_+)\,d\mu.
\end{aligned} \tag{3.10}$$

Since $\Phi(t)$ is monotone increasing, it follows that for any $h<k$,

$$\begin{aligned}
\Phi(k-h)\mu(A(k,\rho)) &= \int_{A(k,\rho)}\Phi(k-h)\,d\mu \le \int_{A(k,\rho)}\Phi(u-h)\,d\mu\\
&\le \int_{A(h,R)}\Phi(u-h)\,d\mu = \int_{B(z,R)}\Phi((u-h)_+)\,d\mu.
\end{aligned}$$

Then

$$\frac{\mu(A(k,\rho))}{\mu(B(z,\rho))} \le \frac{\dfrac{1}{\mu(B(z,R))}\displaystyle\int_{B(z,R)} \Phi((u-h)_+)\,d\mu}{\Phi(k-h)} \frac{\mu(B(z,R))}{\mu(B(z,\rho))}.$$

And by the doubling condition,

$$\frac{\mu(A(k,\rho))}{\mu(B(z,\rho))} \le \frac{\dfrac{1}{\mu(B(z,R))}\displaystyle\int_{B(z,R)} \Phi((u-h)_+)\,d\mu}{\Phi(k-h)} \left(\frac{R}{\rho}\right)^{Q}. \tag{3.11}$$

Taking (3.10) and (3.11) into (3.5) implies

$$\begin{aligned}
&\frac{1}{\mu(B(z,\rho))}\int_{B(z,\rho)} \Phi((u-k)_+)\,d\mu \\
&\quad\le \frac{CR}{(R-\rho)^{\tau}}\left(\frac{R}{\rho}\right)^{Q/K} \frac{1}{\mu(B(z,R))}\int_{B(z,R)} \Phi((u-k)_+)\,d\mu \\
&\qquad\cdot\left(\frac{1}{\mu(B(z,R))}\left(\frac{R}{\rho}\right)^{Q}\frac{1}{\Phi(k-h)}\int_{B(z,R)} \Phi((u-h)_+)\,d\mu\right)^{1-1/K} \\
&\quad\le \frac{CR^{Q+1}}{(R-\rho)^{\tau}\rho^{Q}\Phi(k-h)^{\theta}}\left(\frac{1}{\mu(B(z,R))}\int_{B(z,R)} \Phi((u-h)_+)\,d\mu\right)^{2-1/K}.
\end{aligned} \tag{3.12}$$

It follows from (3.12) that (3.4) holds.

Proof of Theorem 1.1. Let

$$\rho_\ell = \rho + \frac{R-\rho}{2^\ell}, \quad \ell = 0, 1, 2, \ldots .$$

A direct calculation gives

$$\rho_{\ell+1} < \rho_\ell, \quad \lim_{\ell\to\infty} \rho_\ell = \rho,$$

$$\rho_\ell = \rho_{\ell+1} + \frac{R-\rho}{2^{\ell+1}} < 2\rho_{\ell+1},$$

$$\rho_\ell = \rho_{\ell+1} + \frac{R-\rho}{2^{\ell+1}} < \rho_{\ell+1} + 1,$$

therefore

$$0 < \rho_\ell < \min\{\rho_{\ell+1}+1, 2\rho_{\ell+1}\}, \quad B(z,\rho_\ell) \subseteq B(z,R) \subseteq \Omega.$$

Writing

$$k_\ell = k_0 + d(1 - 2^{-\ell}), \quad \ell = 0, 1, 2, \dots, \ k_0 \in \mathbb{R}, \tag{3.13}$$

$$d = \Phi^{-1}\left(\frac{C^{1/\theta} R^{1/\theta}}{(R-\rho)^{\tau/\theta}} 2^{((1+\theta)\tau + Q + (1+\theta)\tau/\theta)/\theta} u(k_0, R)\right), \tag{3.14}$$

where C in (3.14) is the same as C in (3.4), then

$$k_\ell \le k_{\ell+1}, \quad \lim_{\ell \to +\infty} k_\ell = k_0 + d,$$

and (3.4) yields

$$u(k_{\ell+1}, \rho_{\ell+1}) \le \frac{C\rho_\ell}{(\rho_\ell - \rho_{\ell+1})^\tau}\left(\frac{\rho_\ell}{\rho_{\ell+1}}\right)^Q \frac{1}{\Phi(k_{\ell+1} - k_\ell)^\theta}(u(k_\ell, \rho_\ell))^{1+\theta}. \tag{3.15}$$

We claim that for any non-negative integer ℓ,

$$u(k_\ell, \rho_\ell) \le 2^{-(1+\theta)\tau\ell/\theta} u(k_0, \rho_0). \tag{3.16}$$

In fact, when $\ell = 0$, (3.16) is obvious. Suppose that (3.16) holds for ℓ, we have

$$(\rho_\ell - \rho_{\ell+1})^{-\tau} \le \frac{2^{(\ell+1)\tau}}{(R-\rho)^\tau}, \left(\frac{\rho_\ell}{\rho_{\ell+1}}\right)^Q \le \left(\frac{R}{\rho}\right)^Q \le 2^Q,$$

$$k_{\ell+1} - k_\ell = k_0 + d\big(1 - 2^{-(\ell+1)}\big) - k_0 - d\big(1 - 2^{-\ell}\big) = 2^{-(\ell+1)} d.$$

According to Lemma 3.1,

$$\begin{aligned}
&\Phi(k_{\ell+1} - k_\ell) \\
&\quad = \Phi\big(2^{-(\ell+1)} d\big) \ge 2^{-(\ell+1)\tau}\Phi(d) \\
&\quad \ge 2^{-(\ell+1)\tau}\Phi\left[\Phi^{-1}\left(\frac{C^{1/\theta}R^{1/\theta}}{(R-\rho)^{\tau/\theta}} 2^{((1+\theta)\tau + Q + (1+\theta)\tau/\theta)/\theta} u(k_0, R)\right)\right] \\
&\quad \ge 2^{-(\ell+1)\tau} \frac{C^{1/\theta}R^{1/\theta}}{(R-\rho)^{\tau/\theta}} 2^{((1+\theta)\tau + Q + (1+\theta)\tau/\theta)/\theta} u(k_0, R),
\end{aligned}$$

which implies

$$\Phi(k_{\ell+1} - k_\ell)^{-\theta} \le \frac{(R-\rho)^\tau}{CR} 2^{(\ell+1)\tau\theta} 2^{-[(1+\theta)\tau + Q + (1+\theta)\tau/\theta]} u(k_0, R)^{-\theta}.$$

From (3.15), the induction assumption and the above calculation, it follows that

$$\begin{aligned} & u(k_{\ell+1}, \rho_{\ell+1}) \\ & \quad \le CR\frac{2^{(\ell+1)\tau}}{(R-\rho)^\tau}2^Q\frac{(R-\rho)^\tau}{CR}2^{-[(1+\theta)\tau+Q]-(1+\theta)\tau/\theta}2^{\theta\tau(\ell+1)} \\ & \qquad \cdot u(k_0, R)^{-\theta}(u(k_\ell, \rho_\ell))^{1+\theta} \\ & \quad \le 2^{(\ell+1)\tau}2^Q2^{\theta\tau(\ell+1)}2^{-[(1+\theta)\tau+Q]-(1+\theta)\tau/\theta}2^{-(1+\theta)\tau\ell(1+\theta)/\theta}u(k_0, R) \\ & \quad = 2^{-(1+\theta)\tau(1+\ell)/\theta}u(k_0, R), \end{aligned}$$

and this proves (3.16) with ℓ replaced by $\ell+1$.

Noting

$$\begin{aligned} u(k_0+d, \rho) &= \frac{1}{\mu(B(z,\rho))}\int_{B(z,\rho)}\Phi((u-k_0-d)_+)\,d\mu \\ &\le \frac{C_\mu}{\mu(B(z,R))}\int_{B(z,\rho)}\Phi((u-k_0-d)_+)\,d\mu \\ &\le \frac{C_\mu}{\mu(B(z,\rho_\ell))}\int_{B(z,\rho_\ell)}\Phi((u-k_\ell)_+)\,d\mu, \end{aligned}$$

we have

$$0 \le u(k_0+d, \rho) \le C_\mu u(k_\ell, \rho_\ell) \le C_\mu 2^{-(1+\theta)\tau\ell/\theta}u(k_0, \rho_0).$$

Letting $\ell \to +\infty$, we obtain

$$u(k_0+d, \rho) = 0,$$

i.e.

$$\frac{C}{\mu(B(z,\rho))}\int_{B(z,\rho)}\Phi((u-k_0-d)_+)\,d\mu = 0,$$

then it holds on $B(z,\rho)$ that

$$u \le k_0 + \Phi^{-1}\left(\frac{C^{1/\theta}R^{1/\theta}}{(R-\rho)^{\tau/\theta}}\frac{2^{((1+\theta)\tau+Q+(1+\theta)\tau/\theta)\theta}}{\mu(B(z,R))}\int_{B(z,R)}\Phi((u-k_0)_+)\,d\mu\right). \tag{3.17}$$

Putting $k_0 = 0$ in (3.17), it follows that

$$u(x) \le \Phi^{-1}\left(\frac{C^{1/\theta}R^{1/\theta}}{(R-\rho)^{\tau/\theta}}\frac{2^{((1+\theta)\tau+Q+(1+\theta)\tau/\theta)/\theta}}{\mu(B(z,R))}\int_{B(z,R)}\Phi(|u|)\,d\mu\right), \quad x \in B(z,\rho). \tag{3.18}$$

Analogously, for $-u$,

$$-u(x) \leq \Phi^{-1}\left(\frac{C^{1/\theta}R^{1/\theta}}{(R-\rho)^{\tau/\theta}}\frac{2^{((1+\theta)\tau+Q+(1+\theta)\tau/\theta)/\theta}}{\mu(B(z,R))}\int_{B(z,R)}\Phi(|u|)\,d\mu\right) \tag{3.19}$$

holds on $B(z,\rho)$. Combining (3.18) and (3.19), we get

$$|u(x)| \leq \Phi^{-1}\left(\frac{C^{1/\theta}R^{1/\theta}}{(R-\rho)^{\tau/\theta}}\frac{2^{((1+\theta)\tau+Q+(1+\theta)\tau/\theta)/\theta}}{\mu(B(z,R))}\int_{B(z,R)}\Phi(|u|)\,d\mu\right),$$

$x \in B(z,\rho)$, and therefore

$$\Phi(|u(x)|) \leq \frac{C^{1/\theta}R^{1/\theta}}{(R-\rho)^{\tau/\theta}}2^{((1+\theta)\tau+Q+(1+\theta)\tau/\theta)/\theta}\frac{1}{\mu(B(z,R))}\int_{B(z,R)}\Phi(|u|)\,d\mu$$

holds on $B(z,\rho)$. Denoting the constant $C^{1/\theta}2^{((1+\theta)\tau+Q+(1+\theta)\tau/\theta)/\theta}$ by C, and taking the supremum over $x \in B(z,\rho)$ we get (1.3).

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