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STABILITY ANALYSIS OF A DELAY DIFFERENTIAL KALDOR'S MODEL WITH GOVERNMENT POLICIES

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Abstract

This paper is devoted to analysis of the stability of the economy according to an extended version of Kaldor's economic growth model. We consider the role of the government and its simultaneous monetary and fiscal policies and we study whether or not a time delay between the recognition and the implementation of its fiscal policy can affect the economic stability. Numerical simulations provide further conclusions about the long-term behavior of the four variables modeled – namely, national income, capacity of production, bonds value and money supply.

1. Introduction

In James Tobin's words [15], a contemporaneous economist who defended government intervention to stabilize output and avoid recessions, "the question of growth is nothing new but a new disguise for an age-old issue, one which has always intrigued and preoccupied economics: the present versus the future."

Academics [1], [16], [15] and organizations such as World Bank agree that the economic growth and poverty reduction have a positive correlation and hence predicting the former is an important goal to be pursued. As stated in [15], thanks to rising incomes, material standards of living have improved substantially over time for most families in most countries.

Economists have realized throughout the last century how imperative mathematics is for such a goal and often have faced two basic differential equations. We shall denote the national income by Y (sometimes called gross domestic production, GDP), which evolves in time as do the other economic variables, without any further mention. Such a quantity dictates how rich a country is and it grows as long as the nation is capable of accumulating richness, which is possible basically by either increasing the capacity of production or decreasing the amount of money kept under the mattress, and it can be expressed by $Y'(t) = \alpha(I(t) - S(t))$, where $I(t)$ and $S(t)$ denote investment and saving, respectively. And implicitly, it has been set that $K'(t) = I(t)$, where $K(t)$ is

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the capacity of production which is called capital stock and which refers to factories, machines, warehouses, . . .

For instance, the IS-LM [15] model is a classical Keynesian one that considers both I and S as linear functions of Y and K. It is very handy for didactic purposes but its linear formulation inevitably conducts to only two extreme unrealistic situations: complete economic stability or complete economic instability. As in Kaldor's paper [11], the economy is usually assumed to be closed, that is, there is no trade with other nations. In the early forties, Kaldor was one of the first economists to propose a nonlinear formulation for investment I and saving S as functions of Y and K in order to explain the natural fluctuations of the economy. Kaldor's idea is a tremendous improvement over the IS-LM model and it is invariably present over the last decades: [2], [3], [4], [6], [9], [13], [21], [20], [22], [23]. In the sixties, Goodwin [7] proposed a model inspired on Lotka-Volterra equations with the same aim, which also has been studied over the years, as in [5], [18], [19]. As pointed out by Matsumoto and Szidarovszky, "nonlinearities and the presence of delay time are the main ingredients for endogenous cycles". This means that these two ingredients should be added in order to obtain a more realistic model.

We propose to continue the analysis started out by Takeuchi and Yamamura [23] who consider extreme cases of an extended version of Kaldor's model that aggregates the government role and its fiscal and monetary policies. To do so, we improve the treatment given by these authors by considering the full version of the model, that is, with four variables instead of three. In §2, after a brief discussion about this extended version, its improvements, its restrictions and some assumptions, we prove the existence and uniqueness of a positive equilibrium point. We then establish sufficient conditions so that such a point is asymptotically stable, which can be done with or without delay time on the fiscal policy implementation, as we shall see in §3. For further conclusions, we run simulations and analyze the effect of the fiscal policy strength and the delay time size over the long-term stability in the model. See §4. All results, some conclusions and future considerations are included in §5.

2. An extended Kaldor's business cycle model with government policies

Initially, the basic formulation of Kaldor's model was justified by graph analysis and more than a decade later Ichimura [9] presented a rigorous treatment leading to the following equations

$$\begin{cases} Y'(t) = \alpha\big(C(t) + I(t) - Y(t)\big), \\ K'(t) = I\big(Y(t), K(t)\big), \end{cases} \tag{1}$$

where C denotes the national consumption. By definition, savings S is the portion of income that it is not spent, $S = Y - C$, so the first equation of (1) can be written as $Y'(t) = \alpha(I(t) - S(t))$, that is, the economic growth is proportional to how much investment exceeds saving; and the production capacity growth, $K'(t)$, is simply the investment. Thanks to the assumptions (see [11]) over the shape of the curves I and S, such a model leads to economic oscillations either using the Poincaré-Bendixson theorem as in [3] or using the Hopf bifurcation theorem as in [6]. Ichimura [9] came to the same conclusions thanks to Liénard equation techniques.

Several different delay formulations of (1) have been considered in the last years. For instance, substituting its second equation by

$$K'(t) = I\big(Y(t-\tau_0), K(t-\tau_1)\big) - \delta K(t),$$

one obtains a formulation that considers the gestation lag of investment and the depreciation effect thanks to the positive parameter δ. The case $\tau_0 > 0 = \tau_1$ was firstly studied in [13] – where the model is thereafter called the Kaldor-Kalecki model, referring to [11], [12] – and more recently in [21], where the authors proved that the dynamic behavior is affected quantitatively by the investment delay but not qualitatively; the case $\tau_0 = \tau_1 > 0$ was considered by [10], and also by [22], adding a noise perturbation. In 2009, Zhou and Li [25] analyzed a combination of IS-LM and Kaldor's model with two time delays in the capital accumulation processes.

Following [24], Takeuchi and Yamamura [23] added the government and a delay time on its fiscal policy to the model, which were important elements missing, as pointed out in [2], [17]. Such a formulation in $\mathbb{R}^4$ consists on an adaptation on the equations in [11], a government budget constraint and a monetary market equation. To make this precise, we introduce some economic quantities:

(i) the aggregate value of bonds varies with time, so we write $t \mapsto B(t)$. Every bond is assumed to be a consol, that is, a bond with a fixed income security and no maturity date;

(ii) money supply $M(t)$ together with money demand $t \mapsto L\big(Y(t), M(t)\big)$, which is entirely controlled by the government, are the forces of the money market, in the sense that

$$M'(t) = L\big(Y(t), M(t)\big) - M(t); \tag{2}$$

(iii) the price level $t \mapsto p\big(Y(t)\big)$ is an index that corrects the real value of bonds and the money power throughout the time;

(iv) the tax revenue is

$$t \mapsto T(t) = T\big(Y(t), B(t)\big) = \theta\left(Y(t) + \frac{B(t)}{p\big(Y(t)\big)}\right) - T_0,$$

where $0 < \theta < 1$ is the tax rate over the income and the profits on the bonds, and $T_0 > 0$;

(v) government expenditure is

$$t \mapsto G(t) = G_0 + \beta\big(Y^* - Y(t-\tau)\big),$$

where G_0 is the fixed spending and $\beta > 0$ measures how the expenditure responds to the excess (or lack) of national income, assuming that the government always know the equilibrium national income Y^*. The constant delay $\tau \geqslant 0$ represents the policy lag, since it naturally takes time to recognize opportunities to implement a stabilization policy and to actually put it in practice;

(vi) the interest rate of the bonds is $t \mapsto r\big(Y(t), M(t)\big)$and it is basically the money price.

Hence the government budget constraint reads as follows:

$$\frac{M'(t)}{p(Y)} + \frac{B'(t)}{r(Y, M)p(Y)} = G(t) + \frac{B(t)}{p(Y)} - T(Y, B), \tag{3}$$

which equates the changes in the stocks of bonds and money to the government deficit, since it is assumed that selling bonds and printing banknotes finance the government deficit. Besides,

(vii) the national consumption is

$$C(t) = C_0 + c_1\left(Y(t) + \frac{B(t)}{p(Y)} - T(t)\right) + c_2\left(\frac{B(t)}{r(Y, M)p(Y)} + \frac{M(t)}{p(Y)}\right),$$

where $0 < c_1, c_2 < 1$ are the marginal propensity to consume the available income and the available wealth respectively and $C_0 > 0$ is the minimal (basically vital) consumption; and

(viii) the (nonlinear) investment function $t \mapsto I\big(Y(t), K(t), M(t)\big)$ represents the amount of money spent on buying goods for future use, which should provide more money.

By considering the money depreciation over time, the variables B and M have to be corrected by the price level, whence the values $Y, C, I, T, G, K, B/p$ and M/p are measured in real terms (let us say, euro or dollar).

An extended version of Kaldor's model in $\mathbb{R}^4$ arises by adding (2) and (3) to the original formulation (1) together with the adapted consumption and investment functions and the government expenditure; it reads as follows:

$$\begin{cases} Y'(t) = \alpha\big(C(t) + I(t) + G(t) - Y(t)\big), \\ K'(t) = I\big(Y(t), K(t), M(t)\big), \\ \dfrac{M'(t)}{p(Y)} + \dfrac{B'(t)}{r(Y, M)p(Y)} = G(t) + \dfrac{B(t)}{p(Y)} - T(Y, B), \\ M'(t) = L(Y, M) - M(t). \end{cases} \tag{4}$$

On the one hand, fiscal policy refers to the mechanism of increasing or decreasing the expenditure G, which directly affects the economic activity, stimulating it or discouraging it, respectively. The government pursues such a policy by adjusting the parameter β, which is assumed to be positive, since the Kaldor model is essentially a Keynesian one. One could consider that fiscal policy includes the alteration of taxation levels as well, which is achieved by adjusting the parameter $0 < \theta < 1$. But we do not consider this way because the tax rate θ is predetermined and nearly unchangeable by political reasons. The immediate consequence of such an assumption is that we do not analyze the stability of the equilibrium point with respect to this parameter.

On the other hand, monetary policy refers to the fact that is the government who effectively prints every banknote in circulation and consequently determines the available money quantity, which affects the price level and the interest rate and consequently investment and national production. These two policies together allow the government to promote economic stability or, unfortunately, instability.

As in [2], Takeuchi and Yamamura considered two extreme scenarios (both leading to an $\mathbb{R}^3$ formulation): money finance case by setting $B' \equiv 0$ in (4); and bond finance case by setting $M' \equiv 0$. In the former, the government controls the money supply but bonds offer keeps constant $B = \overline{B}$; and in the latter, the government controls the bonds supply in order to finance its deficit but it cannot adjust its money supply ($M = \overline{M}$). And then the model stability is analyzed under these two settings with or without delay time τ. However such scenarios separately do not fit the practical government activity, therefore we take a step forward by analyzing the model (4) in $\mathbb{R}^4$ with its full budget constraint and with or without delay time.

By setting $u = (u_1, u_2, u_3, u_4) \equiv (Y, K, B, M)$ in (4), we obtain

$$\begin{cases} u_1'(t) = \alpha\Big(-\big(1-(1-\theta)c_1\big)u_1(t) - \beta u_1(t-\tau) + I(u) \\ \qquad\qquad + \dfrac{\big((1-\theta)c_1 + c_2/r(u)\big)u_3(t) + c_2 u_4(t)}{p(u)} \\ \qquad\qquad + C_0 + c_1 T_0 + G_0 + \beta Y^* \Big), \\ u_2'(t) = I(u), \\ u_3'(t) = r(u)p(u)\Big(-\theta u_1(t) - \beta u_1(t-\tau) \\ \qquad\qquad + \dfrac{(1-\theta)u_3(t) + u_4(t) - L(u)}{p(u)} + G_0 + \beta Y^* + T_0 \Big), \\ u_4'(t) = L(u) - u_4(t). \end{cases} \tag{5}$$

All the functions are assumed to be as smooth as necessary. Additionally, consider the following assumptions for every $u \in \mathbb{R}_+^4$:

(A1) $L(u)\big|_{u_4=0} > 0$, $\lim_{u_4\to\infty} L(u) < 0$ and $\dfrac{\partial L}{\partial u_4}(u) \leqslant 0 < \dfrac{\partial L}{\partial u_1}(u)$;

(A2) $I(u)\big|_{u_2=0} > 0$, $\lim_{u_2\to\infty} I(u) < 0$ and $\dfrac{\partial I}{\partial u_2}(u) < 0 < \dfrac{\partial I}{\partial u_1}(u), \dfrac{\partial I}{\partial u_4}(u)$;

(A3) $p(u) > 0$ and $\dfrac{dp}{du_1}(u) > 0$; and

(A4) $0 < r(u) < 1$ and $\dfrac{\partial r}{\partial u_4}(u) < 0 < \dfrac{\partial r}{\partial u_1}(u)$.

Under these assumptions, we can prove the existence and the uniqueness of a positive equilibrium point.

Assume the government establishes some equilibrium income $Y^* > 0$ as target and pursues it. By (A1), the right-hand side of the last equation in (5) applied for $u_1 = Y^*$ is a function of u_4, namely $u_4 \mapsto L(Y^*, u_4) - u_4$, such that it is positive for $u_4 = 0$ and it becomes negative as u_4 increases, since $(\partial L/\partial u_4)(u) \leqslant 0$ and $\lim_{u_4\to\infty} L(u) < 0$. Thus we obtain a unique value $u_4 = M^* > 0$ for which that expression is null.

Setting $u_1 = Y^*$ and $u_4 = M^*$ in the second equation of (4), thanks to (A2), we may argue as before to obtain a unique value $u_2 = K^* > 0$ such that $I(Y^*, K^*, M^*) = 0$. Now we set $u_1 = Y^*$, $u_2 = K^*$ and $u_4 = M^*$ in the first and third equations of (4). Their right-hand sides vanish if and only if

$$\begin{cases} 0 = C(Y^*, B, M^*) + G_0 - Y^*, \\ 0 = G_0 + \dfrac{B}{p(Y^*, M^*)} - T(Y^*, B), \end{cases} \tag{6}$$

which is a linear system on the variables B and G_0. So there exists a unique positive equilibrium point $u^* = (Y^*, K^*, B^*, M^*)$ if and only if the government can fix a compatible value $G_0 > 0$ so that the system above admits a unique positive solution $B = B^*$. It is noteworthy that u^* does not depend on β. Also, about the conditions (A3) and (A4), we just have used the fact that the functions p and r are positive.

Therefore, we have proved the following result.

LEMMA 2.1. *Suppose that the conditions* (A1)–(A4) *hold. Given $Y^* > 0$, if* (6) *admits a unique positive solution* (B^*, G_0) *then* (5) *admits a unique positive equilibrium point u^* associated to Y^*, which does not depend on β.*

In [23], under suitable additional technical assumptions one has to deal with expressions where either it is possible to extract a unique positive Y^* from one of the equations and then $M^* > 0$ from other so that $I(Y^*, K, M^*) = 0$ provides a unique $K^* > 0$; or by imposing a lower bound to Y^* it is possible to obtain $B^* > 0$ as function of Y^* and the remaining argumentation follows analogously.

However we do not have such a scenario, that is, it is not possible to determine a unique $Y^* > 0$ since each of the four equations of (4) depends nontrivially on at least two variables. Thus we assume that the government is able to establish a national income $Y^* > 0$ as target and a compatible expenditure value $G_0 > 0$. Doing so, one can obtain a unique associated equilibrium point $u^* = (Y^*, K^*, B^*, M^*)$ in $\mathbb{R}^4_+$ as we did. In our opinion, such a setting is realistic because governments pursue annual growth rates – consequently they reconsider future values of Y^* as the economy grows – and they adjust their expenditures and policies accordingly. The reader should recognize now why Lemma 2.1 requires (B^*, G_0) to be positive.

REMARK 2.2 (About the assumptions). It is natural to expect that the richer a nation the more money it demands; and clearly the money demand $L(u)$ decreases as more money $u_4 = M$ is provided, whence the derivative assumptions of (A1) are reasonable from the economic point of view.

As the infrastructure of a nation improves together with its capacity of production – K_2 increasing – the best opportunities of investment disappear; such a phenomenon is expressed by $\partial I/\partial u_2 < 0$. Besides, investment is stimulated by economic activity and it essentially requires money, whence we unsurprisingly required $\partial I/\partial u_1$ and $\partial I/\partial u_4$ to be positive.

In capitalist economies, prices rising is an intrinsic reaction to the economic growth, justifying dp/du_1 is assumed to be positive. The only point that the government should be concerned about is to keep the associated inflation under control. Clearly $p(u)$ must be positive since it is associated with a weighted mean of all prices practiced in the markets.

The interest rate $r(u)$ is a percentage that defines the remuneration over the money loaned by investors to the government. The reader knows this financial operation by bonds. The more money is available, the smaller is the necessity of the government to be financed by third parties and hence it offers lower remunerations to investors, that is, $\partial r/\partial u_4 < 0$.

Moreover, $\partial r/\partial u_1 > 0$ follows from liquidity preference theory, as in the IS-LM model, which basically states that one dollar today is worth more than one dollar tomorrow. The logic is the following: greater income implies greater money demand which increases the price of money, that is, the interest rate r.

The government cannot print banknotes as it pleases, because it would promote a scenario of hyperinflation very hard to handle and which would immediately cause loss of a prime function of money: store of value. In such an extreme situation, no one wants an additional one dollar bill: money demand is negative! This is expressed by $\lim_{u_4\to\infty} L(u) < 0$. Finally, investment refers to the gain of production capacity while depreciation refers to its loss due whether to wear and tear or to obsolete technology. If the production capacity is too high, there is no new investment projects for some time until the point where there is inevitably depreciation; and $\lim_{u_2\to\infty} I(u) < 0$ expresses it.

3. Local stability of Kaldor's model

Now we analyze the local stability of Kaldor's model (first without delay and later with it) by considering its linearization, as in [8], [14], for the nontrivial equilibrium point u^* obtained in Lemma 2.1.

3.1. Model without delay time

We shall evaluate the Jacobian matrix for the differential system (5) on u^*, omitting the arguments of functions and their derivatives or even the symbol *. For instance, we simply write r_1 to denote $(\partial r/\partial u_1)(u^*)$. This minor abuse of notation rarely causes problems and it will be very handy for the expressions to come, which will require several renamings.

By (6), if $G^* = (\theta Y^* - G_0 - T_0)/(1-\theta)$ then $B^* = p^* G^*$ and the Jacobian matrix evaluated at u^* is given by

$$J = \begin{pmatrix} F_{11}(\beta) & \alpha I_2 & F_{13} & F_{14} \\ I_1 & I_2 & 0 & I_4 \\ F_{31}(\beta) & 0 & F_{33} & r^* F_{44} \\ L_1 & 0 & 0 & -F_{44} \end{pmatrix}, \tag{7}$$

where

$$k_1 = \alpha\big(1 - (1-\theta)c_1\big), \qquad k_2 = \alpha c_2,$$

$$b_{11} = k_1 + k_2\left[\frac{r_1 G^*}{(r^*)^2} + \frac{\dot{p}^* L}{(p^*)^2}\right] + \dot{p}^* F_{13} G^*,$$

$$b_{31} = r^*\big(L_1 + \theta p^* + (1-\theta)\dot{p}^* G^*\big), \qquad F_{11}(\beta) := -\alpha\beta + \alpha I_1 - b_{11},$$

$$F_{31}(\beta) := -p^* r^* \beta - b_{31}, \qquad F_{13} = \frac{k_2 + (\alpha - k_1) r^*}{p^* r^*},$$

$$F_{14} = \alpha I_4 + k_2\left[\frac{1}{p^*} - \frac{r_4 G^*}{(r^*)^2}\right], \qquad 0 < F_{33} = (1-\theta) r^* < 1,$$

$$F_{44} = 1 - L_4.$$

Furthermore, we write

$$\mu = p^* r^* F_{13} - \alpha F_{33}, \qquad \nu = F_{44} - I_2,$$

$$\sigma = \nu - F_{33}, \qquad \Gamma = k_2\left[\frac{1}{p^*} - \frac{r_4 G^*}{(r^*)^2}\right],$$

so that $k_1, k_2, b_{11}, F_{13}, b_{31}, F_{14}, \Gamma, F_{44}, \nu > 0$. The characteristic equation is

$$\lambda^4 + a_1(\beta)\lambda^3 + a_2(\beta)\lambda^2 + a_3(\beta)\lambda + a_4(\beta) = 0,$$

where $a_j \equiv a_j(\beta) := a_{j0} + a_{j1}\beta$, for $j = 1, 2, 3, 4$, are given by

$$a_{10} = b_{11} - \alpha I_1 + \sigma,$$

$$a_{11} = \alpha,$$

$$a_{20} = -I_2(\alpha I_1 + I_2) + \nu\left(-I_2 - F_{33} - \frac{F_{14} L_1}{\nu}\right) + \sigma(b_{11} - \alpha I_1) + F_{13} b_{31},$$

$$a_{21} = \alpha\nu + \mu,$$

$$
\begin{aligned}
a_{30} &= F_{13}F_{44}\big(b_{31} - r^*L_1\big) - b_{11}F_{33}I_2F_{44}\left[\frac{1}{F_{33}}\left(1 - \frac{\Gamma L_1}{b_{11}F_{44}}\right) + \frac{1}{I_2} - \frac{1}{F_{44}}\right] \\
&\quad + F_{33}\big(F_{14}L_1 + F_{44}(\alpha I_1 + I_2)\big) - b_{31}I_2F_{13}, \\
a_{31} &= -\alpha I_2F_{44} + \mu\nu, \\
a_{40} &= -I_2F_{13}F_{44}\big(b_{31} - r^*L_1\big) + b_{11}F_{33}I_2F_{44}\left[1 - \frac{\Gamma L_1}{b_{11}F_{44}}\right] \qquad \text{and} \\
a_{41} &= -\mu I_2F_{44}.
\end{aligned}
$$

We shall analyze whether or not the stability of u^* is sensitive with respect to how strong the fiscal policy is, that is, with respect to the parameter $\beta > 0$. By Routh-Hurwitz criteria, u^* is asymptotically stable if and only if $a_1, a_3, a_4 > 0$ and $a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0$.

LEMMA 3.1. *Suppose that*

(H1) $c_2 > (1-\theta)(1-c_1)r^*$ *and* $\sigma > 0$*;*

(H2) $F_{33} + F_{14}L_1/\nu < -I_2 < \alpha I_1 < b_{11}$*; and*

(H3) $\dfrac{1}{F_{33}}\left(1 - \dfrac{\Gamma L_1}{b_{11}F_{44}}\right) + \dfrac{1}{I_2} - \dfrac{1}{F_{44}} > 0.$

Then

(a) *if* $a_{40} \geqslant 0$ *then* $a_j > 0,\ j = 1, 2, 3, 4,$ *for every* $\beta > 0$*;*

(b) *if* $a_{40} < 0$ *then* $a_j > 0,\ j = 1, 2, 3, 4,$ *for every* $\beta > -a_{40}/a_{41}$.

Moreover, $a_{41} > 0$ *if and only if* $c_2 > (1-\theta)(1-c_1)r^*$.

PROOF. It is not hard to see that $\mu = \alpha\big(c_2 - (1-\theta)(1-c_1)r^*\big)$, which is positive by (H1) and then $a_{21} > 0$. Also, by (H2)

$$
a_{10} = \underbrace{b_{11} - \alpha I_1}_{>0} \underbrace{- I_2 - F_{33}}_{>0} + \underbrace{F_{44}}_{>0} > 0
$$

and

$$
a_{20} = -I_2\underbrace{(\alpha I_1 + I_2)}_{>0} + \nu\underbrace{\left(-I_2 - F_{33} - \frac{F_{14}L_1}{\nu}\right)}_{>0} + \sigma\underbrace{(b_{11} - \alpha I_1)}_{>0} + F_{13}b_{31} > 0,
$$

whence $a_1(\beta), a_2(\beta) > 0$ for every $\beta > 0$. Clearly a_{30} and a_{31} are sums of positive terms since $I_2 < 0$; whence $a_3(\beta) > 0$ for every $\beta > 0$. Although $a_{41} > 0$ by (H1), a_{40} is a sum of a positive term and a negative one, whence instead of controlling its sign we consider the sign of $a_4(\beta)$ for both cases as stated; and the proof is complete.

REMARK 3.2. Since ν is a sum of two (possibly large) positive numbers and F_{33} is a product of two numbers which lie in (0, 1), it is not restrictive to assume that $\sigma = \nu - F_{33} > 0$ in (H1).

As the reader may promptly realize, even for $a_j(\beta)$, which depends linearly on β, the main challenge is renaming, rearranging and noticing conveniently expressions and hypotheses in order to guarantee the positive sign of large sums and then to fulfill the Routh-Hurwitz conditions. The main theorem below deals with the sign of $p_{RH}(\beta) = a_1a_2a_3 - a_3^2 - a_1^2a_4$, which is a cubic function of β and which has over 500 terms if it is fully expanded. Although computing systems, such as Wolfram Mathematica, are very handy for symbolic expressions, they are not able to assimilate the concept of 'convenient rearrangements' and hence we must deal with some hard parts by ourselves.

We may write $p_{RH}(\beta) = Q_0 + Q_1\beta + Q_2\beta^2 + Q_3\beta^3$, where

$$Q_0 = \frac{a_{30}}{2}(a_{10}a_{20} - 2a_{30}) + \frac{a_{10}}{2}(a_{20}a_{30} - 2a_{10}a_{40}),$$

$$Q_1 = a_{31}\underbrace{(a_{10}a_{20} - 2a_{30})}_{\text{(E1.1)}} + \alpha\underbrace{(a_{20}a_{30} - 2a_{10}a_{40})}_{\text{(E1.2)}} + a_{10}\underbrace{(a_{21}a_{30} - a_{10}a_{41})}_{\text{(E1.3)}},$$

$$Q_2 = a_{10}\underbrace{(a_{21}a_{31} - 2\alpha a_{41})}_{\text{(E2.1)}} + a_{31}\underbrace{(\alpha a_{20} - a_{31})}_{\text{(E2.2)}} + \alpha\underbrace{(a_{21}a_{30} - \alpha a_{40})}_{\text{(E2.3)}} \quad \text{and}$$

$$Q_3 = \alpha(a_{21}a_{31} - \alpha a_{41}).$$

THEOREM 3.3. *Suppose that* (E1.1)*,* (E1.2)*,* (E1.3) *and* (E2.2) *are positive. Under the assumptions of Lemma 3.1, we have*

(a) *if* $a_{40} \geqslant 0$ *then* u^* *is asymptotically stable for every* $\beta > 0$.

(b) *if* $a_{40} < 0$ *then* u^* *is asymptotically stable for every* $\beta > -a_{40}/a_{41}$.

PROOF. Note that Q_0 is a linear combination of (E1.1) and (E1.2) with positive weights (under the assumptions of Lemma 3.1) and that if (E2.1) is positive then Q_3 is positive as well. Hence it is sufficient to control the sign of the expressions on Q_1 and Q_2 in order to obtain $p_{RH}(\beta) > 0$ possibly adding a restriction on β.

On the one hand, if (H1) holds then

$$a_{21}a_{31} - 2\alpha a_{41} = \mu^2\nu - \alpha^2\nu I_2F_{44} + \alpha\mu(I_2^2 + F_{44}^2 - I_2F_{44})$$

is a sum of positive terms and (E2.1) is positive, which immediately implies that p_{RH} is positive for $\beta > 0$ large enough. Also, if (H1) holds then (E2.3) is positive independently of the sign of a_{40}:

$$\begin{aligned} a_{21}a_{30} - \alpha a_{40} &> (\mu + \alpha\nu)F_{13}F_{44}\big(b_{31} - r^* L_1\big) + \alpha I_2 F_{13}F_{44}\big(b_{31} - r^* L_1\big) \\ &= F_{13}F_{44}\big(b_{31} - r^* L_1\big)\big(\mu + \alpha F_{44} - \alpha I_2 + \alpha I_2\big) > 0. \end{aligned}$$

On the other hand, let us deal with (E2.2). From all possible assumptions, the cleanest is $\alpha a_{20} - a_{31} > 0$, but we could pursue others conditions. For instance, it is easy to see that $\alpha a_{20} - a_{31}$ is greater than

$$\alpha\sigma\left(b_{11} - \alpha I_1 - \frac{I_2}{\sigma}(\alpha I_1 + I_2 - F_{44})\right) + \alpha\nu\left(-I_2 - F_{33} - \frac{F_{14}L_1}{\nu} - \frac{\mu}{\alpha}\right),$$

which is positive if each term is; note that these two conditions are slightly stronger than (H2), since $I_2/\sigma, \mu/\alpha \in (0, 1)$. Or yet, $\alpha a_{20} - a_{31}$ is greater than

$$\alpha\nu\left(-I_2 - F_{33} - \frac{F_{14}L_1}{\nu} + I_2\frac{F_{44}}{\nu} - \frac{\mu}{\alpha}\right)$$

and asking this expression to be positive is again a slightly stronger condition than (H2), since $F_{44}/\nu, \mu/\alpha \in (0, 1)$. In both cases, the new conditions are considerably larger though. Similarly one can obtain conditions for (E1.1), (E1.2), (E1.3) and (E2.2) to be positive, where (E1.1) is the one which demands more effort since it has a longer expression to be dealt with. However we abide by the cleanest assumptions sparing the reader the gruesome estimates and their details; and the proof is complete.

REMARK 3.4. Actually we proved that if (H1) holds then $Q_3 > 0$ which implies that a strong fiscal policy (that is, a scenario where $\beta > 0$ is large enough) always promotes a long-term stable economy, as long as the government does not delay its implementation (since we are dealing with the model without delay so far).

3.2. Model with delay time in fiscal policy

Invariably economic dynamics involves human behavior, which is a decisive factor to be taken into account. It basically refers to the capacity of making decisions after recognizing opportunities and evaluating available resources. Such an aspect can be added appropriately to an economic model by formulating it with delay; that is, instead of considering differential equations where the variables react instantly to external forces independently of the past, a delay formulation does take into account the fact that the past is important when comes to making decisions. A formulation with a nonconstant delay function $t \mapsto \tau(t)$ or considering the government expenditure as function of a weighted average of the national income, let us say

$$\beta \int_{-\tau(t)}^{0} \big(Y^* - Y(s)\big) f(s)\, ds,$$

provides a more realistic modeling. We shall analyze the constant delay case.

In (5), delay time τ models the government capacity of recognizing, formulating and implementing fiscal policies. To obtain such a fixed value τ, one may evaluate the mean policy lag of a nation considering a given period of time. The associated linearized model evaluated at u^* is given by $u'(t) = J_0 u(t) + J_\tau u(t-\tau)$, where

$$J_0 = \begin{pmatrix} \alpha I_1 - b_{11} & I_2 & F_{13} & F_{14} \\ I_1 & I_2 & 0 & I_4 \\ -b_{31} & 0 & F_{33} & r^* F_{44} \\ L_1 & 0 & 0 & -F_{44} \end{pmatrix} \quad \text{and} \quad J_\tau = \begin{pmatrix} -\alpha\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p^* r^* \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so that $J = J_0 + J_\tau$ and its characteristic equation can be written as

$$Q_0(\lambda) + e^{-\lambda\tau} Q_\tau(\lambda) = 0, \tag{8}$$

where

$$Q_0(\lambda) := \lambda^4 + a_{10}\lambda^3 + a_{20}\lambda^2 + a_{30}\lambda + a_{40} \qquad \text{and}$$
$$Q_\tau(\lambda;\beta) \equiv Q_\tau(\lambda) := \big(a_{11}\lambda^3 + a_{21}\lambda^2 + a_{31}\lambda + a_{41}\big)\beta.$$

By Theorem 3.3, the equilibrium point u^* is locally stable for every $\beta > 0$, whenever $\tau = 0$. For $\tau > 0$, we know that u^* is locally asymptotically stable if and only if every root of (8) has negative real part, see [8], [14]. Also instability is equivalent to the existence of at least one root with positive real part.

REMARK 3.5. Under the assumptions of Lemma 3.1, if the additional assumption

(H4) $a_{40} > 0$

holds then the real part of every root of Q_0 is negative, by Routh-Hurwitz criteria.

From now on, we assume that (H1)–(H4) hold. We shall study how the local stability of u^* responds to fiscal policy strength, $\beta > 0$, and time lag, τ. First we apply the following stability switch result to the delayed model (4). For a complex number z, we write its real and imaginary parts as $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively.

THEOREM 3.6 ([14, Theorem 3.4.1]). *Consider the equations* (8) *on* λ *and*

$$F(y) := |Q_0(iy)|^2 - |Q_\tau(iy)|^2 = 0, \qquad \text{for} \quad y \in \mathbb{R}. \tag{9}$$

Suppose that $\lambda \mapsto Q_0(\lambda)$, $Q_\tau(\lambda)$ *are analytic functions for* $\operatorname{Re}\lambda > 0$ *and that*

(i) *there is no common pure imaginary roots of* Q_0 *and* Q_τ*;*
(ii) $\overline{Q_0(-iy)} = Q_0(iy)$ *and* $\overline{Q_\tau(-iy)} = Q_\tau(iy)$*, for every* $y \in \mathbb{R}$*;*
(iii) $\lambda = 0$ *is not a root for* (8)*;*
(iv) $\limsup_{|\lambda|\to\infty, \operatorname{Re}\lambda \geqslant 0} \left| \frac{Q_\tau(\lambda)}{Q_0(\lambda)} \right| < 1$*; and*
(v) *the equation* (9) *admits only finitely many real roots.*

Then

(a) *if* $F(y) = 0$ *has no positive roots then no stability switch occurs;*
(b) *if* $F(y) = 0$ *has at least one positive root and each of them is simple then, as* τ *increases, a finite number of stability switches occurs and eventually* u^* *becomes unstable.*

The assumption (i) holds by Remark 3.5 and (iii) holds because $a_4(\beta) > 0$ for every positive β. Since Q_0 and Q_λ are polynomials with real coefficients and $\deg Q_0 > \deg Q_\tau$, assumptions (ii), (iv) and (v) hold. Although F clearly depends on β, we shall omit such a dependence from time to time. Thus we shall analyze the stability of u^* by studying the positive roots of $F(y) = 0$. Setting $z = y^2$, the function F can be written as

$$F(z;\beta) \equiv F(z) = z^4 + b_1 z^3 + b_2 z^2 + b_3 z + b_4,$$

where $b_j \equiv b_j(\beta)$, $j = 1, 2, 3, 4$, are given by

$$b_1(\beta) = a_{10}^2 - 2a_{20} - a_{11}^2\beta^2,$$

$$b_2(\beta) = a_{20}^2 - 2a_{10}a_{30} + 2a_{40} + (2a_{11}a_{31} - a_{21}^2)\beta^2,$$

$$b_3(\beta) = a_{30}^2 - 2a_{20}a_{40} + (2a_{21}a_{41} - a_{31}^2)\beta^2 \qquad \text{and}$$

$$b_4(\beta) = a_{40}^2 - a_{41}^2\beta^2.$$

Clearly, $F(y) = 0$ has no positive roots whenever $F(z) = 0$ has no positive roots. Actually, they have the same number of positive simple roots.

It is noteworthy that $b_4(\beta) < 0$ if and only if $\beta > a_{40}/a_{41}$, and in this case the number of positive roots of F can be 1, 2 or 3 only. As we shall see with simulations in §4, under a weak fiscal policy scenario – more precisely, for $0 < \beta < a_{40}/a_{41}$ – the government efficiency on implementing it does not harm the economic stability because $z \mapsto F(z)$ has no positive zeros. On the other hand, a more careful analysis is required if $\beta > a_{40}/a_{41}$.

Now we discuss the relationship between the parameters β and τ. First note that a pure complex number $\lambda = iy$, where $y > 0$, is a root of (8) if and only if y is a positive root of (9).

We shall show only the sufficiency in order to fix some notations. Note that $|Q_0(iy)/Q_\tau(iy)| = 1$ so that there exists a unique value $\phi(y) \in [0, 2\pi)$ such that $-e^{-i\phi(y)} = Q_0(iy)/Q_\tau(iy)$ and hence $\lambda = iy$ is a root of (8) whenever τ is of the form $(\phi(y) + 2n\pi)/y$, with $n \in \mathbb{Z}_+$.

The reader should promptly see that, by hypothesis (i) of Theorem 3.6, $Q_\tau(iy)$ cannot be zero, whenever iy is a root of (8) or y is a root of (9). If we write $Q_\ell(iy) = Q_{\ell,\mathrm{Re}}(y) + i\,Q_{\ell,\mathrm{Im}}(y)$, $\ell = 0, \tau$, then after some computations we see that $\phi(y) \in [0, 2\pi)$ is the angle that satisfies the equations

$$\begin{cases} \cos\big(\phi(y)\big) = -\dfrac{Q_{0,\mathrm{Re}}(y)Q_{\tau,\mathrm{Re}}(y) + Q_{0,\mathrm{Im}}(y)Q_{\tau,\mathrm{Im}}(y)}{|Q_\tau(y)|^2} =: -\dfrac{A(y)}{|Q_\tau(y)|^2}, \\[2ex] \sin\big(\phi(y)\big) = \dfrac{-Q_{0,\mathrm{Re}}(y)Q_{\tau,\mathrm{Im}}(y) + Q_{0,\mathrm{Im}}(y)Q_{\tau,\mathrm{Re}}(y)}{|Q_\tau(y)|^2} =: \dfrac{B(y)}{|Q_\tau(y)|^2}, \end{cases}$$

so that $(0, \infty) \ni y \mapsto \phi(y) \in (0, 2\pi)$ is defined by

$$\phi(y) = \begin{cases} \arctan\left(\dfrac{-B(y)}{A(y)}\right), & \text{if } \cos(\phi), \sin(\phi) > 0, \\ \pi/2, & \text{if } \cos(\phi) = 0 \text{ and } \sin(\phi) = 1, \\ \pi + \arctan\left(\dfrac{-B(y)}{A(y)}\right), & \text{if } \cos(\phi) < 0, \\ 3\pi/2, & \text{if } \cos(\phi) = 0 \text{ and } \sin(\phi) = -1, \\ 2\pi + \arctan\left(\dfrac{-B(y)}{A(y)}\right), & \text{if } \cos(\phi) > 0 \text{ and } \sin(\phi) < 0. \end{cases}$$

We regard the root of (8) as a function of τ by writing $\tau \mapsto \lambda(\tau) = x(\tau) + iy(\tau)$ and then we study the sign of the derivative of $\mathrm{Re}\,\lambda(\tau)$ at the points where $\lambda(\tau)$ is purely imaginary, which are precisely where a stability switch may occur, since $\lambda = 0$ is not a root of (8). Arguing as in the proof of Theorem 3.6, we see that $\lambda(\tau)$ is differentiable at $\tau = \tau^*$ whenever $\lambda(\tau^*)$ is a simple root. And if, in addition, $\lambda(\tau^*) = iy(\tau^*)$ then we explicitly obtain

$$\left(\frac{d\lambda(\tau)}{d\tau}\bigg|_{\tau=\tau^*} \right)^{-1} = \left[-\frac{Q_0'(\lambda)}{\lambda Q_0(\lambda)} + \frac{Q_\tau'(\lambda)}{\lambda Q_\tau(\lambda)} - \frac{\tau}{\lambda} \right]\bigg|_{\tau=\tau^*},$$

so we can determine the direction of motion of $x(\tau)$ as τ passes through τ^* according to

$$S := \operatorname{sign} \left. \frac{d \operatorname{Re} \lambda(\tau)}{d\tau} \right|_{\tau=\tau^*} = \operatorname{sign} \left. \frac{dF(y)}{dy} \right|_{y=y(\tau^*)}. \tag{10}$$

LEMMA 3.7 ([14, Theorem 3.4.1]). *If y^* is a simple positive root of* (9) *then there exists a pair of simple conjugate pure imaginary roots $\lambda(\tau^*) = \pm i y(\tau^*)$ of* (8) *at $\tau^* = \phi(y^*)/y^*$ which crosses the imaginary axis according to* (10). *More precisely,*

(a) *if $S > 0$ then $\lambda(\tau)$ crosses the imaginary axis at $\tau = \tau^*$ from left to right, that is, u^* becomes unstable; and*

(b) *if $S < 0$ then $\lambda(\tau)$ crosses the imaginary axis at $\tau = \tau^*$ from right to left, that is, u^* becomes stable.*

If $y_1^* > \cdots > y_m^* > 0$ are the simple positive roots of (9) then, by making explicit the dependence on β, we write

$$S_{i,n}(\tau) := \tau - \frac{\phi\big(y_i^*(\beta)\big) + 2n\pi}{y_i^*(\beta)},$$

for $i = 1, \dots, m$ and $n \in \mathbb{Z}_+$. It is an auxiliary function whose zero is the value τ at which $\lambda(\tau)$ crosses the imaginary axis. These are the tools needed to study numerically the stability switch.

4. Numerical simulations

First, we emphasize that u^* does not depend on β but its stability may do. By running simulations in Wolfram Mathematica 11.3, we shall analyze several aspects: how the number of positive simple roots of $F(z;\beta) = 0$ changes as β varies; how the convergence of the solution responds to greater values of τ, with eventual instability; the sensitiveness of hypotheses with respect to economic parameters; the stability region in the $\beta\tau$-plane; and so on. For $u = (u_1, u_2, u_3, u_4) \in \mathbb{R}_+^4$, set

$$\begin{aligned} &\alpha = 0.40, &&C_0 = 10, &&p(u) = 0.4u_1 + 10,\\ &c_1 = 0.40, &&T_0 = 20, &&r(u) = \frac{1+u_1}{1+u_1+5u_4},\\ &c_2 = 0.15, &&\theta = 0.35, &&L(u) = 5u_1 - u_4 + 50. \end{aligned}$$

As for the investment function, we consider two different formulations, namely,

$$I(u) = -0.25u_2 + 5r(u)u_4 + 100 \tag{11}$$

and

$$\Im(u) = \eta\tilde{I}(u) + (1-\eta)I(u), \tag{12}$$

where $\eta \in (0, 1)$ is fixed and

$$\tilde{I}(u_1, u_2) := 25\exp\left(\frac{-\log 2}{\left(\frac{15}{1000}u_1 + 10^{-5}\right)^2}\right) + \frac{u_1}{100} + 5\frac{320^3}{(u_2+1)^3}.$$

The first one satisfies the original assumptions of Kaldor's paper [11] concerning the nonlinearity of I with respect to $u_1 = Y$ but not with respect to $u_2 = K$. For numerical simulations and to verify the sufficient assumptions, the linear dependence on u_2 is convenient though. The second one is an adaption of the investment function which appears in [21] and it completely satisfies Kaldor's assumptions over the shape of I curve with respect to $u_1 = Y$ and $u_2 = K$.

We will cover each choice of investment function in a section of its own.

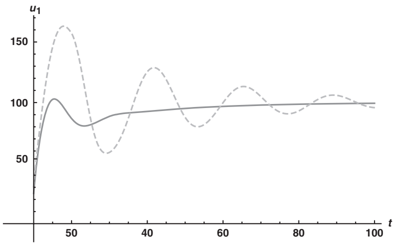
REMARK 4.1. The delay equations demand a function $\psi\colon[-\tau, 0] \to \mathbb{R}^4$ as initial data. In the simulations, we considered exponential functions with a slow increasing rate, for instance $t \mapsto \exp(0.02t)25$ for $u_1(t)$. Not that an economy increases indefinitely exponentially in time, but in short-time it is reasonable that a nation has an economic growth of 2 % and that is precisely the point.

4.1. Investment function given by (11)

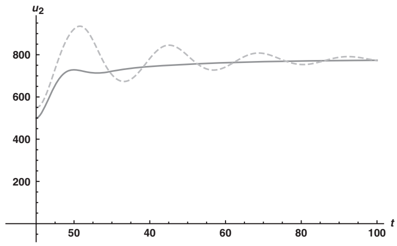
The economic assumptions (A1)–(A4) and the technical hypotheses (H1)–(H3) are satisfied. If the government pursues the national income $Y^* = 100$ then it must fix $G_0 = G(u^*) = 0.51$ to obtain $u^* = (100.00, 776.36, 1114.68, 275.00)$ as the unique positive equilibrium point of (4), which is asymptotically stable for every $\beta > 0$, by Lemma 3.1 and Theorem 3.3. In other words, in such a scenario, assuming that the government instantly applies its fiscal policies then no matter how strong they are, the economy is always stable.

First, we set $\beta = 0.40$, then the eigenvalues of the associated Jacobian matrix at u^* are

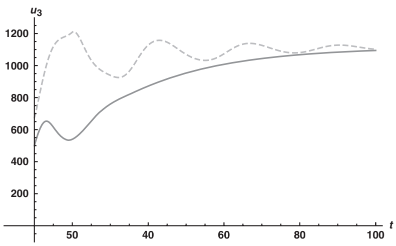
$$-2.2112, \quad -0.0415 \quad \text{and} \quad -0.2097 \pm 0.3409i$$



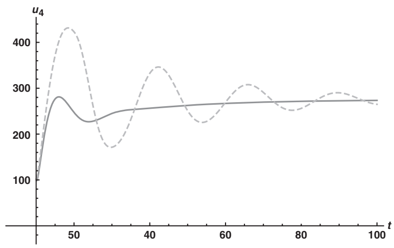
(a) The national income $u_1 = Y$.



(b) The capital stock $u_2 = K$.



(c) The governments bonds $u_3 = B$.



(d) The money supply $u_4 = M$.

FIGURE 1. The evolution in time with and without delay.

and we obtain the graphs in Figure 1 comparing the numerical solutions for the model with or without delay of $\tau = 10$, represented by the dashed and continuous lines, respectively.

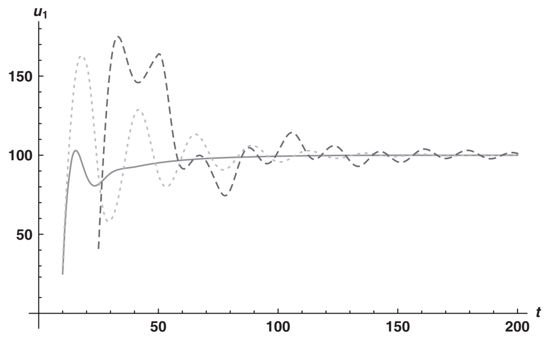
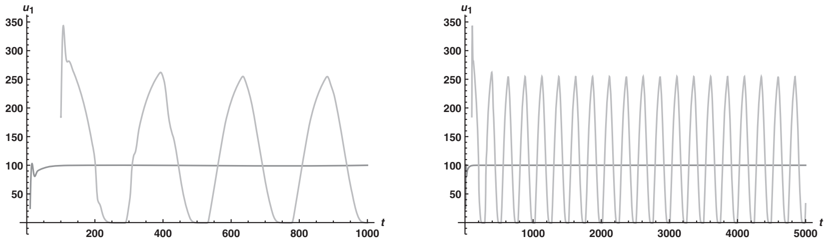


FIGURE 2. The evolution of $u_1(t)$ under different values of τ.

We have $T(u^*) = 22.80$, which says that in equilibrium the government revenue represents about 20 % of the national richness, a compatible idea with the capitalist philosophy about a moderate size for the state accounts. Also, it is reasonable that an interest rate of $r^* = 6.84$ % in equilibrium promotes high



(a) Until $t = 1000$.

(b) Until $t = 5000$.

FIGURE 3. The evolution of u_1 with $\beta = 0.40$ and $\tau = 100$.

values in the bonds market, that is, $B^* = 1114.68$.

In Figure 2, we compare the numerical solutions of u_1 associated to $\tau = 0$ (the continuous line), $\tau = 10$ (the dotted line) and $\tau = 25$ (the dashed line). As τ increases the solution associated to its delay equation becomes more erratic but $u_1^* = 100$ still is asymptotically stable for $\tau = 25$; actually even for $\tau = 50$.

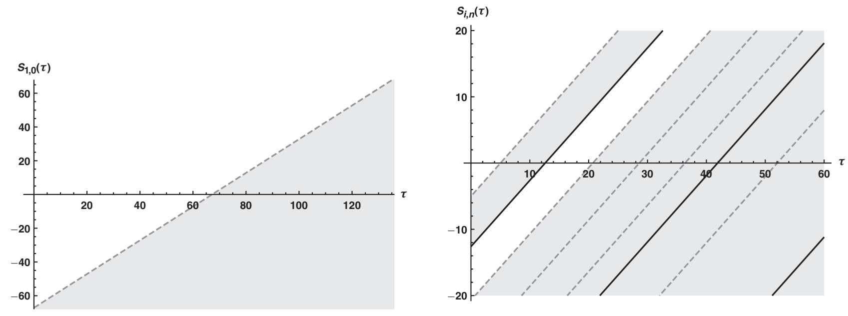
However, since $a_{40} = 0.0048$, Theorem 3.6 holds and then we shall study how τ affects the stability of (4). If $N(\beta)$ denotes the number of positive simple roots of $F(y; \beta) = 0$ then

$$N(\beta) = \begin{cases} 0, & \text{if } 0 < \beta < 0.1964, \\ 1, & \text{if } 0.1964 < \beta < 0.6071 \text{ or } \beta > 0.7790, \\ 3, & \text{if } 0.6071 < \beta < 0.7790. \end{cases}$$

The unique positive simple root of $F(y; 0.40) = 0$ is $y^* = 0.035472$ and the derivative $(dF/dy)(y; 0.40)$ is always positive for $y > 0$, whence the crossing the imaginary axis is always to the right half-plane, that is, stability switch occurs only toward instability. The evolution of u_1 for $\tau = 100$ is showed in Figure 3. Actually, Figure 4(a) points out that the stability switch already occurs toward instability at $\tau = 67.28$.

On the other hand, the switch stability analysis for $\beta = 0.70$ is far more involving since now we have three positive simple roots of $F(y; 0.70) = 0$, namely, $y_1^* = 0.4010$, $y_2^* = 0.214$ and $y_3^* = 0.0878$. Besides, $\frac{d}{dy}F(y; 0.70)$ is negative if $0.1599 < y < 0.3432$ and it is positive otherwise; whence it follows that $\frac{d}{dy}F(y_1^*; 0.70)$ and $\frac{d}{dy}F(y_3^*; 0.70)$ are positive but $\frac{d}{dy}F(y_2^*; 0.70) < 0$. Thus the crossing at iy_1^* and iy_3^* must be to the right half-plane; and the crossing at iy_2^* must be to the left half-plane.

Recall that (8) admits infinitely many complex roots $\lambda(\tau)$. Let $\tau_{i,n}$ be the zero of $S_{i,n}$. In Figure 4(b), the interceptions of the lines with the τ-axis are



(a) For $\beta = 0.40$, u^* becomes unstable at $\tau = 67.28$.

(b) For $\beta = 0.70$, u^* becomes unstable as τ enters in some shaded area and it becomes stable as τ leaves some shaded area.

FIGURE 4. The auxiliary function $S_{i,n}$ and the stability switch of u^*.

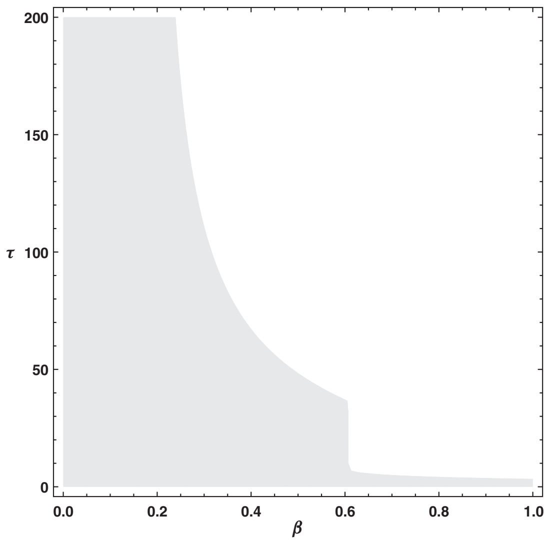


FIGURE 5. Stability of u^* with respect to β and τ.

at $\tau_{1,0} < \tau_{2,0} < \tau_{1,1} < \tau_{3,0} < \tau_{1,2} < \tau_{2,1} < \tau_{1,3} < \cdots$ and we see how the stability of u^* changes as τ increases. At $\tau = \tau_{1,0} = 5.01$, one of the roots of (8) crosses to the right half-plane and then the stability switch occurs toward instability; at the second value $\tau = \tau_{2,0} = 12.60$, such a root crosses back to the left half-plane and the switch occurs toward stability. As one may note, as τ increases, passing by $\tau_{1,1}$, $\tau_{3,0}$ and $\tau_{1,2}$, three roots of (8) cross to the right half-plane but only one of them crosses back to the left half-plane at

$\tau_{2,1} = 41.88$ and the instability persists thereafter because the number of roots crossing to the right half-plane exceeds the number of those crossing back.

Finally, by setting $\beta = 0.15$, we have that $b_4(0.15) > 0$ and all roots of $F(y) = 0$ are complex, consequently u^* is (locally) asymptotically stable for every time delay $\tau > 0$. All such conclusions are summarized in Figure 5, which shows the relationship between the stability of the equilibrium point u^* and the parameters β and τ.

The boundary of the region consists of the pairs (β, τ) such that $\tau = \tau(\beta)$ is the smallest delay time at which a stability switch occurs toward instability. In the shaded region, u^* is asymptotically stable and out of it, there are three possibilities. If $\beta < a_{40}/a_{41} = 0.1964$ then $N(\beta) = 0$, which means that no stability switch occurs. If $0.6071 < \beta < 0.7790$ then finitely many stability switches occur as τ increases, as we discussed above. If β is greater than 0.1964 but is not in (0.6071, 0.7790), we have instability for every τ such that (β, τ) lays outside the shaded region.

4.2. Investment function given by (12)

Let $\eta = 0.4$, $\alpha = 0.2$ and consider the investment function in (12).

In this setting, the economic assumptions (A1)–(A4) and the technical hypotheses (H1)–(H4) are satisfied. Although (E1.1), (E1.2) and (E1.3) are positive, unfortunately (E2.2) is negative (even after several attempts with different values of parameters) and hence we cannot apply Theorem 3.3. However we can verify numerically that the maximum value of the real part of the eigenvalues of the Jacobian matrix at the equilibrium point is negative for every $0 < \beta < 1$. Thus Kaldor's model (4), with $\tau = 0$, is locally asymptotically stable even without fulfilling the conditions of our results, making explicit the fact that they are not necessary. Furthermore, we can verify that Theorem 3.6 holds because, for every $0 < \beta < 1$, there is no common pure imaginary roots of Q_0 and Q_τ and $\lambda = 0$ is not a root for (8).

REMARK 4.2. We have not been able to establish general necessary conditions for existence of a positive equilibrium point. However, we know that a_{41} is positive if and only if $c_2 > (1-\theta)(1-c_1)r^*$, see Lemma 3.1; whence it is a necessary condition to fulfill Routh-Hurwitz criteria. This very same condition was already required by [23, Theorem 3.2] to obtain the stability of the equilibrium point.

Proceeding as before, we have that

$$N(\beta) = \begin{cases} 0, & \text{if } 0 < \beta < 0.1964, \\ 1, & \text{if } 0.1964 < \beta < 0.4523 \text{ or } \beta > 0.5980, \\ 3, & \text{if } 0.4523 < \beta < 0.5980, \end{cases}$$

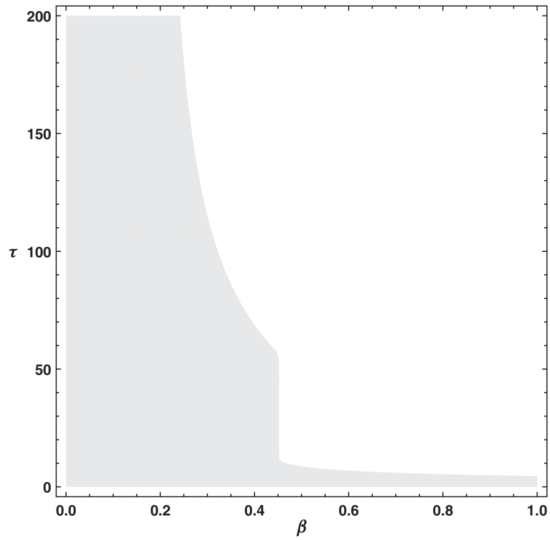
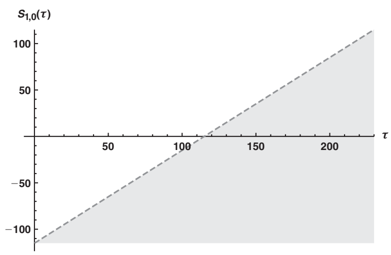
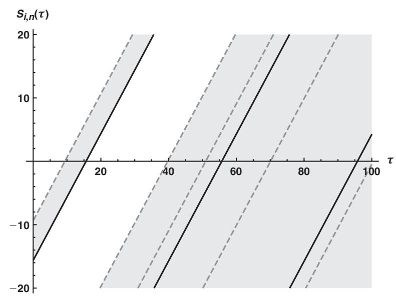


FIGURE 6. Stability of u^* with respect to β and τ.



(a) For $\beta = 0.30$, u^* becomes unstable at $\tau = 115.19$.



(b) For $\beta = 0.48$: u^* becomes unstable as τ passes by some dashed line into a shaded area and it becomes stable as τ passes by some solid line and into shaded areas.

FIGURE 7. The stability switch of u^*.

and the region of stability of u^* in the $\beta\tau$-plane is given by Figure 6. Under a moderate fiscal policy, $\beta = 0.30$, the government inefficiency does not harm the economic stability until $\tau_{1,0} = 115.19$, see Figure 7(a). On the other hand, if the fiscal policy is slightly stronger, let us say $\beta = 0.48$, several switch stabilities occur as τ increases. More precisely, at $\tau_{1,0} = 9.36$, the stability switch occurs toward instability; at the second value $\tau_{2,0} = 15.63$, the switch occurs toward stability and so on, depending on whether $\tau = \tau_{2,n}$ or $\tau = \tau_{1,n}, \tau_{3,n}$. See Figure 7(b), where $\tau_{1,0} < \tau_{2,0} < \tau_{1,1} < \tau_{3,0} < \tau_{2,1} < \tau_{1,2} < \tau_{2,2} < \tau_{1,3} < \cdots$.

5. Conclusions

We consider an extended version of the classical Kaldor's economic growth model adding the government role to the economic dynamics: monetary and fiscal policies and the government budget constraint are taken into account, leading to a differential system in $\mathbb{R}^4$, with or without a delay time on the fiscal policy. An analysis of the model stated in (5) is itself an improvement over [23] who turned it into two simpler versions in $\mathbb{R}^3$ by imposing either $B' = 0$ or $M' = 0$, that is, extreme scenarios where either the government is incapable to manage its bonds supply or it is incapable to establish its money supply.

TABLE 1. The effects of the fiscal policy on the economy.

β	Strength of the fiscal policy	First value τ under which u^* is unstable	Conclusion
0.15	weak	∞	The economy is always stable
0.40	moderate	67.28	An inefficient government can lead the economy to instability
0.70	strong	5.01	The economic stability is very sensitive to the government efficiency

Firstly we have proved the existence and uniqueness of a positive equilibrium point under reasonable economic assumptions (which represent an improvement over those technical ones required by [23]). Secondly we have established sufficient conditions under which (5), with $\tau = 0$, is locally asymptotically stable with a possible restriction over the fiscal policy strength. Under a simple additional assumption, namely (H4), we have applied a classical stability switch result to study how the fiscal policy delay time may lead to an unstable economic scenario.

In §4 we have run simulations with two different investment functions, splitting it into two case. On the one hand, all assumptions needed for the results we have presented are satisfied by the investment function (11) and by the other functions and parameters. Table 1 summarizes the results of §4.1.

Here government efficiency refers to the time efficiency on recognizing opportunities to implement a fiscal policy, formulating it and then implementing it. Curiously, if the fiscal policy is very strong, let us say $\beta = 0.9$, the conclusion are quite the same as those for $\beta = 0.4$.

On the other hand, in §4.2, the investment function is a convex combination of the previous one with the investment function suggested by [21]. Although we cannot apply Theorem 3.3 for $\alpha = 0.2$, we were able to verify numerically that the equilibrium point is always locally asymptotically stable for $0 < \beta < 1$ and $\tau = 0$; and that the switch stability theorem holds as well.

The less simplifications are imposed and the more relevant aspects are considered, the more realistic a model is. For instance, one should expect that the government capacity of recognizing, formulating and implementing fiscal policies varies with time, that is, it is more reasonable to assume a delay function $t \mapsto \tau(t)$ instead of a fixed delay time. Also the economy intrinsically carries a volatility which comes from the human behavior factor and which can be appropriately added to the model by considering certain economic parameters random. For instance, $0 < c_1, c_2 < 1$ dictate how big is the portion of the income that will be spent, which are associated with the (microeconomic) perception whether or not the economy prospers and it will continue to do so. And as we have discussed in § 2, one could aggregate a delayed investment formulation of Kaldor-Kalecki's model suitably adapted; as in [21]. Besides, a question of structural stability arises. Comparing (1) and (4), one may wonder if the limit cycle structure of (1) is present in the extended model. More precisely, is it possible to obtain the original $\mathbb{R}^2$ dynamics from (4) by deforming it appropriately?

Our future aims concerns these subjects and other related ones.

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