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ON POINT CLASSIFICATION IN CONVEX SETS

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Abstract.

A classification of points of convex sets in a Euclidean space E_n is given. The classification discriminates four point types and is based on a function specially defined on a convex set as well as on the concept of the point's cone with respect to the set. The fourth-type points in this classification are simply the locally polyhedral points. The properties of the first-type points are closely connected to the well-known results on sections, projections and asymptotes of convex sets [2]. These properties also lead to a theorem on the closedness of the sum of convex closed cones.

0.

Let $[x,y\rangle$ denote the ray $\{z=x+\lambda(y-x) : \lambda \geqq 0\}$ emanating from the point x towards the point y. For a point p and a set A, let $[p,A\rangle$ denote the set $\bigcup_{x\in A}[p,x\rangle$ and $\operatorname{cone}(p,A)$ the cone $[0,A-p\rangle$ (so that $[p,A\rangle = \operatorname{cone}(p,A)+p$).

1.

For a convex $A \subset E_n$ and $x \in A$, define

$$\mu_A(x) = \inf_{y\in A\setminus\{x\}}\|A \cap [x,y\rangle\| ,$$

where $\|\cdot\|$ is the Euclidean length.

If $A\setminus\{x\}=\emptyset$, that is, $A=\{x\}$, then, by definition, $\mu_{\{x\}}(x)=+\infty$. A sequence $\{y_k\}_1^\infty \subset A\setminus\{x\}$, for which $\|A\cap[x,y_k\rangle\| \to \mu_A(x)$ can be chosen in such a way that it converges and the sequence $\{l_k\}_1^\infty$ of the rays $l_k = [x,y_k\rangle$ converges too. We shall call such sequences, $\{y_k\}_1^\infty$ of points and $\{l_k\}_1^\infty$ of rays, *realizing sequences* for the point x. Any convex set A can be partitioned into two disjoint subsets

$$A_0 = \{x\in A : \mu_A(x)=0\}, \quad A_{+0} = \{x\in A : \mu_A(x) \neq 0\} .$$

Obviously the relative interior $\operatorname{ri} A \subset A_{+0}$, so that $A_{+0}\neq\emptyset$. Thus A_0 is a subset of the relative boundary of A:

$$A_0 \subset \operatorname{rb} A = A\setminus \operatorname{ri} A .$$

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2.

The following example will be used extensively in the sequel. Consider a plane E_2 in E_3, a circumference arc in E_2 and a point sequence $\{c_i\}_1^\infty$ on the arc converging monotonously to a point b. Let $a \in E_3 \setminus E_2$, $l_i = [a, c_i\rangle$, and suppose a point c_i' is chosen in such a way that $c_i' \in l_i$, $\|a - c_i'\| \to 0$ and it lies on the same side of the plane defined by the points b, c'_{i-2}, c'_{i-1}, as the point a (see fig.1). The set $A = \operatorname{conv}\{a, b, c_1', c_2' \ldots\}$ is a convex compact set and the points $a, b, c_1', c_2' \ldots$ comprise the complete list of its extreme points. It is easy to verify that $A_0 = [a, b)$.

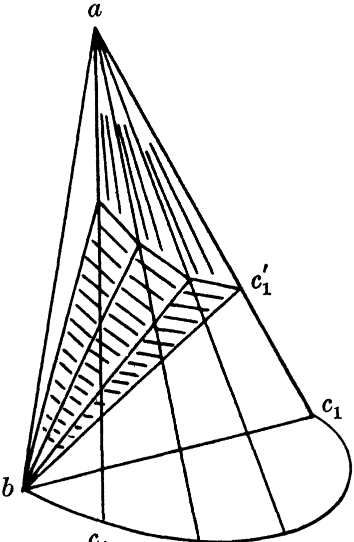

Fig. 1.

3.

Obviously if $x \in A$, Π is a hyperplane and $x \in \Pi$, then

$$\mu_{A \cap \Pi}(x) \geqq \mu_A(x) .$$

In particular, it may happen that $x \in A_0$, while $x \in (A \cap \Pi)_{+0}$.

PROPOSITION. *For a convex $A \subset E_n$, let $x \in A_0$, and suppose that x is not an extreme point of A, that is, $x \in (p,q), [p,q] \subset A$. Let Π be a hyperplane such that $(p,q) \cap \Pi = \{x\}$. Then $x \in B_0$, where $B = A \cap \Pi$.*

PROOF. Assume $\mu_B(x) > 0$. Let $0 < \varepsilon \leqq \min\{\mu_B(x), \|p-x\|, \|q-x\|\}$ and let α be the angle between the interval $[p,q]$ and the hyperplane Π, $0 < \alpha \leqq \frac{1}{2}\pi$. A contradiction will be obtained by showing that

$$\mu_A(x) \geqq \varepsilon', \quad \text{where } \varepsilon' = \tfrac{1}{2} \cdot \varepsilon \cdot \cos \tfrac{1}{2}(\pi - \alpha) .$$

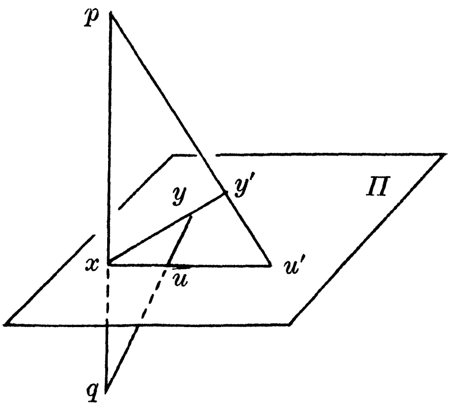

Fig. 2.

Assume $y \in A$ and $y \notin [p,q] \cup \Pi$ (otherwise $\|A \cap [x,y\rangle\| \geqq \varepsilon > \varepsilon'$). Assume, for instance, that y and q lie on the opposite sides of Π. Then

$$\Pi \cap [q,y] = u \in B, \quad u \neq x ,$$

and therefore a point $u' \in B$ exists such that $u' \in [x,u\rangle, \|x-u'\| = \frac{1}{2}\varepsilon$ (see fig.2). Consider the point

$$y' = [p,u'] \cap [x,y\rangle, \quad y' \in A .$$

Obviously

$$\|A \cap [x,y\rangle\| \geqq \|x-y'\| .$$

On the other hand, the fact that the sides xp and xu' in the triangle xpu' are not less than $\frac{1}{2}\varepsilon$ and $\angle pxu' \leqq \pi - \alpha$, implies that $\|x-y'\| \geqq \varepsilon'$. Thus

$$\|A \cap [x,y\rangle\| \geqq \varepsilon' \quad \text{for all } y \in A \setminus \{x\} .$$

4.

Every open interval in A belongs either to A_0 or to A_{+0}. This is proved in the following

PROPOSITION. *For convex $A \subset E_n$, let $x \in (a,b) \subset A, x \in A_0$. Then $(a,b) \subset A_0$.*

PROOF. Let Π be a hyperplane such that $\Pi \cap (a,b) = \{x\}$ and let $B = A \cap \Pi$. By proposition 3, $\mu_B(x) = 0$. Let $x' \in (a,b)$ and let Π' be a hyperplane such that Π' is parallel to Π and $x' \in \Pi'$. Denote $B' = A \cap \Pi'$. We shall prove that $x' \in B_0'$. Let us choose $[p,q] \subset (a,b)$, such that $x,x' \in (p,q)$. Assume, for instance, that x' lies between x and p. Let

$$\{y_k\}_1^\infty \subset B \setminus \{x\}, \quad l_k = [x,y_k\rangle, \quad \|B \cap l_k\| \to 0 .$$

Denote $y_k' = \Pi' \cap [p, y_k]$ and $l_k' = [x', y_k'\rangle$. Obviously, $\{y_k'\}_1^\infty \subset B' \setminus \{x'\}$ and l_k' is parallel to l_k. Assume $\mu_{B'}(x') > \varepsilon > 0$. Then for every k there exists $z_k' \in B' \cap l_k'$ such that $\|x' - z_k'\| \geqq \varepsilon$. For $z_k = \Pi \cap [q, z_k']$ we have

$$z_k \in B \cap l_k, \quad \|x - z_k\| = \lambda \|x' - z_k'\|,$$

where $\lambda = \|q - x\| / \|q - x'\|$. Thus

$$\lim_{k \to \infty} \|B \cap l_k\| \geqq \lim_{k \to \infty} \|x - z_k\| \geqq \lambda \varepsilon > 0 ,$$

a contradiction.

5.

As applied to the partition $\mathscr{F}$ of a convex set A into the family of the relative interiors of the faces of A (see [1], theorem 18.2), proposition 4 gives the following

COROLLARY. *Each element of the family $\mathscr{F}$ either is included in A_0 or is disjoint from A_0 (that is, included in $A_{\neq 0}$).*

6.

The partition of A into A_0 and $A_{\neq 0} = A \setminus A_0$ can be also described in terms of continuity of the μ function. Namely, it turns out that

$$A_0 = \{x \in \operatorname{rb} A : \mu_A \text{ is continuous at } x\} .$$

THEOREM. *For a convex $A \subset E_n$, the function μ_A is continuous at $x \in A$ iff $x \in \operatorname{ri} A \cup A_0$.*

PROOF. Assume, for convenience, that $\dim A = n$. Let $x \in \operatorname{int} A$, $\varepsilon > 0$,

$$U = \{z : \|z - x\| \leqq \varepsilon\} \subset A .$$

Obviously, $\mu_A(x) = \alpha \geqq \varepsilon > 0$. Let

$$\{z_s\}_1^\infty \subset A, \quad z_s \to x, \quad \mu_A(z_s) \to \beta .$$

Suppose that $\beta > \alpha$. Fix a positive integer k and any $\beta', \alpha < \beta' < \beta$. Let

$$\{y_m\}_1^\infty \subset A \setminus \{x\} \quad \text{and} \quad \|A \cap l_m\| \to \alpha ,$$

where $l_m = [x, y_m\rangle$. Choose a point z_s' such that

$$z_s' \in [z_s, y_k\rangle, \quad \|z_s' - z_s\| = \beta' .$$

For big s, $z_s' \in A$. Consider the interval

$$[x, y_s'] = \operatorname{conv}(U \cup \{z_s'\}) \cap l_k .$$

From $\|y_s' - x\| \to \beta'$ we obtain $\|A \cap l_k\| \geqq \beta'$, that is,

$$\lim_{k\to\infty}\|A \cap l_k\| \geqq \beta',$$

which is a contradiction.

Suppose now that $\beta < \alpha$. Obviously $\beta > 0$. Fix a number $\alpha', \beta < \alpha' < \alpha$. For every s choose $z_s' \in A$ such that $\|z_s' - z_s\| \to \beta$. Then, for $l_s = [z_s, z_s'\rangle$,

$$\|A \cap l_s\| \to \beta .$$

Choose a point v_s on the ray $[x, z_s'\rangle$ such that $\|v_s - x\| = \alpha'$. Obviously, $v_s \in A$ and, for big s,

$$z_s' \in (x, v_s) .$$

Consider the interval

$$[z_s, z_s''] = \operatorname{conv}(U \cup \{v_s\}) \cap l_s .$$

Obviously,

$$z_s' \in (z_s, z_s''), \quad \|v_s - z_s''\| \to 0, \quad \|z_s'' - z_s\| \to \alpha' .$$

Hence

$$\lim_{s\to\infty}\|A \cap l_s\| \geqq \lim_{s\to\infty}\|z_s'' - z_s\| \geqq \alpha' > \beta ,$$

a contradiction.

If $x \in \operatorname{bd}A$ and $\mu_A(x) \neq 0$, then x is a point of discontinuity, since for every sequence

$$\{y_s\}_1^\infty \subset \operatorname{int}A, \quad y_s \to x ,$$

the inequality

$$0 \leqq \lim_{s\to\infty}\mu_A(y_s) \leqq \lim_{s\to\infty}\|y_s - x\| = 0$$

holds.

Suppose now that $x \in A_0$, and let $\{y_k\}_1^\infty \subset A \setminus \{x\}$,

$$y_k \to x, \quad \mu_A(y_k) > \varepsilon > 0 \quad \text{for all } k .$$

Consider the face F of the set A for which $x \in \operatorname{ri}F$. For big k, $y_k \in F$ since $x \in \operatorname{ri}(A \cap [y_k, x\rangle)$ and, moreover, $y_k \in \operatorname{ri}F$ since $\|y_k - x\| \to 0$. By corollary 5, $y_k \in A_0$, which is a contradiction.

7.

Let $\operatorname{ex}A$ denote the extreme point set of a convex set A. If $x \in \operatorname{cl}(\operatorname{ex}A \setminus \{x\})$ then clearly $x \in A_0$. It turns out that the converse becomes true in a proper section of A.

THEOREM. *For a convex $A \subset E_n$ and $x \in A_0$ the following holds:*

a) *There exists a plane section D of A such that*

$$x \in D_0, \quad x \in \operatorname{ex}D, \quad 2 \leqq \dim D \leqq n .$$

b) *If $x \in \operatorname{ex}A$ and A is closed, then $x \in \operatorname{cl}(\operatorname{ex}A \setminus \{x\})$.*

PROOF. a) If $x \notin \operatorname{ex} A$, then, by proposition 3, a section $B = A \cap \Pi$ exists such that $x \in B_0$, $\dim B = \dim A - 1$. If still $x \notin \operatorname{ex} B$, we shall make further section, etc. Finally a section D will be obtained such that $x \in D_0$, $x \in \operatorname{ex} D$. At the same time $\dim D \geqq 2$ since $\dim D = 0$ or 1 implies $D_0 = \emptyset$.

b) It is sufficient to consider the case of a compact A; otherwise A can be replaced by

$$A \cap \{y : \|y - x\| \leqq r\}, \quad r > 0 .$$

The conditions $x \in A_0$ and $\operatorname{cl} A = A$ imply the existence of such a sequence $\{y_k\}_1^\infty \subset A \setminus \{x\}$ that

$$\begin{aligned} &1)\ y_k \to x, \\ &2)\ A \cap [x, y_k\rangle = [x, y_k] . \end{aligned}$$

By the Caratheodory theorem, there exists for every k a representation

$$y_k = \textstyle\sum_{i=1}^{n+1} \alpha_{ki} z_{ki}, \quad \alpha_{ki} > 0, \ \sum_{i=1}^{n+1} \alpha_{ki} = 1 ,$$

where $z_{ki} \in \operatorname{ex} A$ for all i. Here for a fixed k some $z_{ki}, i = 1, 2, \ldots, n+1$ may coincide. Still always $z_{ki} \neq x$ since otherwise a contradiction to 2) follows immediately.

Choose a subsequence $\{y_k\}_{k \in K}$ such that for all $i = 1, 2, \ldots, n+1$,

$$\lim_{k \in K} z_{ki} = q_i \quad \text{and} \quad \lim_{k \in K} \alpha_{ki} = \beta_i .$$

It follows immediately that

$$q_i \in A, \quad \beta_i \geqq 0, \quad \textstyle\sum_{i=1}^{n+1} \beta_i = 1 ,$$
$$x = \textstyle\sum_{i=1}^{n+1} \beta_i q_i .$$

Since $x \in \operatorname{ex} A$, one of the points q_i, say q_1, coincides with x, and therefore

$$\{z_{1k}\}_{k \in K} \subset \operatorname{ex} A \setminus \{x\} \quad \text{and} \quad \lim_{k \in K} z_{1k} = x .$$

8.

Thus we obtain, with the additional assumption of closedness, the following characteristic property of the A_0 points.

COROLLARY. *For a convex closed set $A \subset E_n$ and $x \in A$, $x \in A_0$ iff a plane section B of the set A exists such that $2 \leqq \dim B \leqq n$,*

$$x \in \operatorname{ex} B, \quad x \in \operatorname{cl}(\operatorname{ex} B \setminus \{x\}) .$$

In particular, a convex compact set A is a polytope iff $A_0 = \emptyset$.

9.

We are now in a position to develop a more detailed classification of points of convex sets. To this end we shall use the function μ and the structure of the point's cone with respect to the set. But first several simple facts are given in the following

LEMMA. *For a convex set* $A \subset E_n$ *and* $(p,q) \subset A$, *the following holds:*

a) $\operatorname{cone}(x,A) = \operatorname{cone}(y,A)$ *for every* $x,y \in (p,q)$.

b) *If* Π *is a hyperplane,* $\Pi \cap (p,q) = \{x\}$ *and* $B = A \cap \Pi$, *then*

$$\operatorname{cone}(x,A) = \operatorname{cone}(x,B) + [0,p-x\rangle + [0,q-x\rangle .$$

In particular, the cones $\operatorname{cone}(x,A)$ *and* $\operatorname{cone}(x,B)$ *are simultaneously closed or not, polyhedral or not.*

We shall say that the point $x \in A$ is of the *first type*, and write $x \in A_1$, if $\operatorname{cone}(x,A)$ is not closed. It follows immediately from the lemma that the propositions 3 and 4, corollary 5 and part a) of theorem 7 hold when index 1 is substituted for index 0. Part b) of theorem 7 also remains true, since a convex set A being closed implies $A_1 \subset A_0$ (that is $A_1 \cap A_{+0} = \emptyset$).

The set

$$A_2 = A_0 \setminus A_1 = \{x \in A : \mu_A(x) = 0, \operatorname{cone}(x,A) \text{ is closed}\}$$

is not, generally speaking, empty. In example 2, $A_1 = (a,b)$ and $A_2 = \{a\}$. Points of A_2 will be called the *second-type* points.

As above, propositions 3 and 4, corollary 5 and theorem 7 hold when index 2 is substituted for index 0 (the only change is $3 \leqq \dim D \leqq n$ in 7b)).

10.

We shall study now the special role played in the representation $A_0 = (A_0 \cap A_1) \cup A_2$ by its first component.

LEMMA. *For a convex* $A \subset E_n$ *and* $x \in A_0$, *let* $\{l_k\}_1^\infty$ *be a realizing ray sequence for* x *with a limit ray* l. *Then* $\operatorname{ri}(A \cap l) \subset A_0$.

PROOF. The case $A \cap l = \{x\}$ is trivial. Now let $p \in (x,y) \subset A \cap l$. For a given k consider in the plane of the rays l, l_k the right-angled triangle $xpr_k, \angle p = \frac{1}{2}\pi$. For a big k, $r_k \notin A$ (see fig.3). On the other hand, there exists $y_k \in (x,r_k), y_k \in A$. Hence there is a point $q_k \in (p,r_k)$ such that $q_k \in A$, that is, $l_k' = [p,r_k\rangle \subset [p,A\rangle$. Then

$$\|A \cap l_k'\| \leqq \|r_k - p\| = \|x-p\| \operatorname{tg} \alpha_k \quad \text{where } \alpha_k = (\widehat{l l_k}) .$$

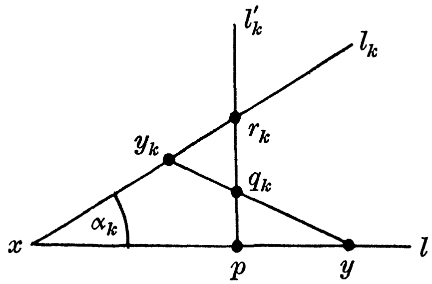

Fig. 3.

Since $\alpha_k \to 0$ we have $\|A \cap l_k'\| \to 0$, hence $\mu_A(p)=0$.

THEOREM. *For a convex $A \subset E_n$, $A_0 \neq \emptyset$ implies $A_0 \cap A_1 \neq \emptyset$.*

PROOF. It may be assumed that $\dim A \geqq 2$, for otherwise $A_0 = \emptyset$. In the case $\dim A = 2$ it is easy to verify that $A_2 = \emptyset$, that is, $A_0 \subset A_1$. The rest of the proof is carried out by the induction on $\dim A$. For $x \in A_0$, let $\{l_k\}_1^\infty$ be its realizing ray sequence, $l_k \to l$. If $A \cap l = \{x\}$, then

$$l \not\subset [x,A\rangle, \quad l \subset \mathrm{cl}[x,A\rangle\,,$$

hence $x \in A_1$. Therefore let $A \cap l \neq \{x\}$ and $p \in \mathrm{ri}(A \cap l)$. By Lemma, $p \in A_0$. Let $B = A \cap \Pi$, where Π is a hyperplane such that $\Pi \cap l = \{p\}$. Clearly, $\dim B = \dim A - 1$. By proposition 3, $p \in B_0$, and therefore $B_0 \neq \emptyset$. By the inductive hypothesis, there exists a point $q \in B_0 \cap B_1$. The obvious inclusions $B_0 \subset A_0$ and $B_1 \subset A_1$ imply $q \in A_0 \cap A_1$.

11.

Better insight into the special role of the $A_0 \cap A_1$ points (in the case of a closed set A, $A_0 \cap A_1 = A_1$) will be achieved through the further classification of the points of A_{+0}. The set A_{+0}, as distinct from A_0, is the set of points at which A is locally a cone. The points of A_{+0} will be discriminated through their cones' structure. The *fourth-type* points are defined as follows:

$$A_4 = \{x \in A : \ \mathrm{cone}(x,A) \text{ is polyhedral}\}\,.$$

It is easy to see that $A_4 \subset A_{+0}$. Indeed, if $[x,A\rangle$ is the convex hull of k rays

$$l_i = [x,y_i\rangle, \quad y_i \in A \setminus \{x\}, \quad i=1,2,\ldots,k\,,$$

then for the polytope P which is the convex hull of the points $x,y_1,\ldots,y_k$, we have $[x,A\rangle = [x,P\rangle, P \subset A$, and hence $\mu_A(x) \geqq \mu_P(x) > 0$. Thus $A_4 \subset A_{+0} \setminus A_1$.

The set of the *third-type* points is defined as follows: $A_3 = A_{\neq 0} \setminus (A_1 \cup A_4)$, that is,

$$A_3 = \{x \in A : \mu_A(x) > 0, \text{cone}(x,A) \text{ is closed and not polyhedral}\}.$$

(In the case of a closed set A, $A_{\neq 0} = A_3 \cup A_4$). In example 2, $A_3 = \{b\}$.

12.

Lemma *For convex sets A and B the following holds*:

a) *If $x \in (A \cap B)_0$ and $x \notin B_0$, then $x \in A_0$.*
b) *If $x \in (A \cap B)_1$ and $x \notin B_1$, then $x \in A_1$.*
c) *If $x \notin (A \cap B)_4$ and $x \in B_4$, then $x \notin A_4$.*

Proof. a) follows from the easily verifiable inequality

$$\mu_{A \cap B}(x) \geqq \min\{\mu_A(x), \mu_B(x)\}.$$

The relation

$$\text{cone}(x, A \cap B) = \text{cone}(x,A) \cap \text{cone}(x,B)$$

implies b) and c).

A polytope $P(x,\varepsilon)$ satisfying

$$\{y : \|y-x\| \leqq \tfrac{1}{2}\varepsilon\} \subset P(x,\varepsilon) \subset \{y : \|y-x\| \leqq \varepsilon\}$$

will be called a *polyhedral ε-neighbourhood* of x. The lemma shows the convenience of polyhedral neighbourhoods on the study of point type distribution in the vicinity of a given point. Indeed, for any polytope $P, P = P_4$. Hence, in view of the lemma,

$$(P \cap A)_0 \subseteqq P \cap A_0, \quad (P \cap A)_1 \subseteqq P \cap A_1,$$
$$x \notin (P \cap A)_4 \text{ implies } x \notin P \cap A_4.$$

13.

Theorem. *For a convex set $A \subset E_n$,*

$$A_2 \cup A_3 \subset \text{cl}(A_0 \cap A_1), \quad A_4 \cap \text{cl}(A_0 \cap A_1) = \emptyset.$$

(For a closed A, simply $\text{cl} A_1 = A_1 \cup A_2 \cup A_3 = A \setminus A_4$).

Proof. Let $\{\varepsilon_k > 0\}_1^\infty$ converge to 0 and let $B^{(k)} = A \cap P(x, \varepsilon_k)$. Suppose $x \in A_2$. Then $x \in B_2^{(k)}$, that is $B_0^{(k)} \neq \emptyset$.

Suppose $x \in A_3$. Since $\mu_A(x) > 0$, for big k (such that $\varepsilon_k < \mu_A(x)$) we have

$$B^{(k)} = [x, A\rangle \cap P(x, \varepsilon_k).$$

Moreover, $B^{(k)}$ is a compact set which is not polyhedral, for otherwise $\operatorname{cone}(x,A) = \operatorname{cone}(x,B^{(k)})$ would imply $x \in A_4$. Therefore the set $\operatorname{ex} B^{(k)}$ has a limit point, that is, $B_0^{(k)} \neq \emptyset$. Thus, $x \in A_2 \cup A_3$ implies $B_0^{(k)} \neq \emptyset$ for big k. By theorem 10,

$$B_0^{(k)} \cap B_1^{(k)} \neq \emptyset .$$

In view of a) and b) of lemma 12,

$$(A_0 \cap A_1) \cap P(x,\varepsilon_k) \neq \emptyset ,$$

that is, $x \in \operatorname{cl}(A_0 \cap A_1)$.

Let now $x \in A_4$ and hence $\mu_A(x) > 0$. Take a positive $\varepsilon < \mu_A(x)$. Then for $B = A \cap P(x,\varepsilon)$, we have

$$B = [x,A\rangle \cap P(x,\varepsilon) ,$$

and so B is polyhedral. If $y \in \operatorname{int} P(x,\varepsilon)$, then

$$\operatorname{cone}(y,A) = \operatorname{cone}(y,B) ,$$

that is $y \in A_4$, and so $x \notin \operatorname{cl}(A_0 \cap A_1)$.

14.

In the following theorem it is proved that for a point $z \in A_2 \cup A_3$ the relation $z \in \operatorname{cl}(A_0 \cap A_1)$ is realized by an interval (z,u) which lies in $A_0 \cap A_1$.

THEOREM. *For a convex set $A \subset E_n$, let $z \in A, z \notin A_0 \cap A_1$, and $z \in \operatorname{cl}(A_0 \cap A_1)$ (in the case of a closed set A this is equivalent to $z \in A_2 \cup A_3$). There exists a point $u \in A$ such that $(z,u) \subset A_0 \cap A_1$.*

PROOF. Let $\mu_A(z) = \alpha > 0$ and $\{y_k\}_1^\infty \subset A_0 \cap A_1$ $y_k \to z$. Since for $l_k = [z,y_k\rangle$ we have $\|A \cap l_k\| \geqq \alpha$, there exists, for a big s a point $u \in A \cap l_s$ such that $y_s \in (z,u)$. By a) of lemma 9 and proposition 4, $(z,u) \subset A_0 \cap A_1$.

Let $\mu_A(z) = 0$, that is, $z \in A_2$. Then if $\{l_k\}_1^\infty$, $l_k \to l$, is a realizing ray sequence for z, then $l \cap A \supset (z,u)$, $u \neq z$. By lemma 10, $(z,u) \subset A_0$. Suppose $z' \in (z,u)$ and Π is a hyperplane such that $\Pi \cap (z,u) = z'$ and let $B = A \cap \Pi$. Clearly, $\dim B = \dim A - 1$, $z' \in B_0$. The rest of the proof will be carried out by the induction on $m = \dim A$. If $m < 3$, then $A_2 = \emptyset$. Suppose $m = 3$. Then $\dim B = 2, B_2 = \emptyset, z' \in B_0 \cap B_1, z' \in A_0 \cap A_1$ and $(z,u) \subset A_0 \cap A_1$. In the general case either $z' \in B_0 \cap B_1$ and then $(z,u) \subset A_0 \cap A_1$, or $z' \in B_2$ and then, by the inductive hypothesis, an interval $[z',u'] \in B$ exists such that $(z',u') \subset B_0 \cap B_1$. But then an interval $[z,u''] \subset A$ exists such that

$$(z,u'') \cap (z',u') = \{a\} .$$

Now $a \in B_0 \cap B_1$ implies $a \in A_0 \cap A_1$, wherefrom $(z,u'') \subset A_0 \cap A_1$.

15.

Note that lemma 9 strengthens the assertions of corollary 5 and proposition 3 in the sense that they hold for the more detailed point classification.

COROLLARY. *For a convex set $A \subset E_n$, the following holds:*

a) *if F is a face of A, then* $\mathrm{ri}\,F$ *lies in one of the sets A_i $(i = 1,2,3,4)$.*
b) *if $x \in (p,q) \subset A_i$ and Π is a hyperplane such that $(p,q) \cap \Pi = \{x\}$, then for $B = A \cap \Pi$ there holds $x \in B_i$.*

We shall say that F is a *face of the i-th type* if $\mathrm{ri}\,F \subset A_i$. Concerning the first type faces a more precise statement can in fact be made: if $\mathrm{ri}\,F \subset A_1$, then either $\mathrm{ri}\,F \subset A_1 \cap A_0$, or $\mathrm{ri}\,F \subset A_1 \cap A_{\neq 0}$. Observe that for the more interesting case of a closed convex set we have $A_1 \cap A_{\neq 0} = \varnothing$. Theorem 14 implies a definite relation between first-type faces on one hand, and second- and third-type faces, on the other. We will establish it for the case of a closed set.

THEOREM. *If F^α is a second- or third type face of a convex closed set A, then F^α is a face of a certain first-type face F^β, that is,*

$$\mathrm{ri}\,F^\beta \subset A_1, \quad F^\alpha \subset \mathrm{rb}\,F^\beta$$

PROOF. Let F^α be a face of A, $\mathrm{ri}\,F^\alpha \subset A_2 \cup A_3$, and $z \in \mathrm{ri}\,F^\alpha$. By theorem 14, there exists an interval $[z,u] \subset A$ such that $(z,u) \subset A_1$. Let $z' \in (z,u)$ and

$$H = \mathrm{conv}(F^\alpha \cup \{u\}) .$$

Clearly, $F^\alpha \subset H \subset A$, $\dim H = \dim F^\alpha + 1$ and $z' \in \mathrm{ri}\,H \subset A_1$. There exists a face F^β of the set A such that $z' \in \mathrm{ri}\,F^\beta$ and therefore $\mathrm{ri}\,F^\beta \subset A_1$. Since $z' \in \mathrm{ri}\,H$ and $z' \in F^\beta$ imply $H \subset F^\beta$, we have $F^\alpha \subset F^\beta$, hence F^α is a face of F^β.

16.

COROLLARY. *In the partition $A = \bigcup_\gamma \mathrm{ri}\,F^\gamma$ of a convex closed set $A \subset E_n$ an element of the maximal dimension outside A_4 is in A_1. In other words, if $d_i (i = 1,2,3,4)$ are the maximal face dimensions of all the four types, then*

$$d_4 = \dim A, \quad d_1 \leqq \dim A - 2, \quad d_2, d_3 < d_1 .$$

The question of what dimension distributions of the faces over the four types are possible, remains open.

17.

Everything we said so far is applicable to convex cones. Observe only that every ray

$$(0,x\rangle = \{\lambda x : \lambda > 0\}$$

of a convex cone C, lies entirely, by corollary 15, in one of the sets C_i $(i=1,2,3,4)$. Accordingly we shall call such a ray an *i-th type ray.* Note also that if a cone C is convex and closed but not polyhedral, then it contains a ray of the first type. Indeed, if C is pointed (that is, $C \cap (-C) = \{0\}$), then there exists a hyperplane $\Pi, 0 \notin \Pi$, such that $A = C \cap \Pi$ is a compact non-polyhedral set. By 8 and theorem 10, $A_1 \neq \emptyset$. Then $(0,x\rangle \subset C_1$, if $x \in A_1$. If C is not pointed, then $C = C' + \mathscr{L}$, where $\mathscr{L}$ is a linear subspace,

$$C' = C \cap \mathscr{L}', \quad \mathscr{L} + \mathscr{L}' = E_n,$$

and C' is pointed and non-polyhedral. The observation that $C_1 = C_1' + \mathscr{L}$ completes the proof.

18.

Consider the sum $C+L$, where C is a convex closed cone and L is a one-dimensional subspace. It is easy to verify that if $L \cap C = L$ or $L \cap C = \{0\}$, then the sum $C+L$ is closed. Therefore, $C+L$ is not closed only if $C \cap L = l$, that is,

$$l \subset C, \quad -l \not\subset C, \quad l \cup (-l) = L.$$

Note that $x \in l \subset C, x \neq 0$, implies

$$C+L = C+(-l) = \operatorname{cone}(x,C),$$

that is $C+L$ is not closed iff the intersection $C \cap L$ is a first-type ray in C.

If $E_n = L + E_{n-1}$ and π is the linear projection onto E_{n-1} parallel to L, then $C+L = \pi C + L$, and hence πC is closed iff $C+L$ is closed. In other words, πC is not closed iff L is a first-type direction in C. In particular, the existence of a non-closed $(n-1)$-projection of a convex closed cone C is equivalent to the condition $C_1 \neq \emptyset$, that is to C not being polyhedral. The latter is central to the theorem of Mirkil [3] and its stronger version given in [2].

19.

As above let C be a convex closed cone. Let, in addition, K be a polyhedral cone. The question of whether the sum $C+K$ is not closed appears to be more complicated (when K is a supspace it is equivalent to the problem of characterizing the non-closed projections of the cone C). The question arises, for instance, in connection with Tucker's key theorem [4]. The theorem can be formulated as follows. If K is a non-negative orthant with respect to some orthogonal basis and C is a polyhedral cone, then

$$\operatorname{int} K \cap [C \cap K + C^* \cap K] \neq \emptyset ,$$

where C^* is the dual of C.

It can be shown that the key theorem holds whether the cone polyhedral or not if $C+K^*$ is closed.

20.

The following theorem gives only a necessary condition for the sum not beeing closed; it holds, however, for any two closed convex cones.

THEOREM. *Let C' and C'' be convex closed cones in $E_n (n \geqq 3)$ and suppose the sum $C'+C''$ is not closed. Then there exists a straight line*

$$L = l+(-l), \quad l = (0,x\rangle$$

such that $l \subset C', -l \subset C''$ and either $l \subset C_1'$ or $-l \subset C_1''$ or both.

PROOF. Since $C'+C''$ is not closed, $C' \cap (-C'') \neq \{0\}$ (see [1, Theorem 9.1.2]). Let

$$l = (0,x\rangle \subset C', \quad -l \subset C'', \quad L = l+(-l) .$$

If $l \subset C'_1$ or $-l \subset C_1''$, then the proof is completed. Otherwise, by 18, $\pi C'$ and $\pi C''$ are convex closed cones, where π is the linear projection onto E_{n-1} parallel to L,

$$E_n = L \dotplus E_{n-1} .$$

Since $C'+C''=L+\pi C'+\pi C''$ the sum $\pi C'+\pi C''$ is not closed. The rest of the proof will be carried out by the induction on n. For $n=3$ the closedness of $\pi C'$ and $\pi C''$ implies the closedness of their sum. Hence, either $l \subset C_1'$ or $-l \subset C_1''$ or both. As for the general case, the inductive hypothesis as applied to $\pi C', \pi C'' \subset E_{n-1}$ implies the existence of a straight line $L'=l'+(-l')$ such that

$$l' \subset \pi C', \quad -l' \subset \pi C''$$

and, for instance, $l' \subset (\pi C')_1$.

Let Π be a hyperplane parallel to $E_{n-1}, x \in \Pi$ and $B=C' \cap \Pi$. From

$$C'+L = \pi C'+L = [x,B\rangle+L$$

follows

$$(C'+L)_1 = (\pi C')_1+L = [x,B\rangle_1+L .$$

This and $[x,B\rangle = \pi C' + x$ imply $[x,B\rangle_1 = (\pi C')_1 + x$. Therefore, if $y' \in l'$ then

$$y = y' + x \in [x,B\rangle_1 .$$

Hence there exists $z \in B$ such that $y \in (x,z\rangle \subset [x,B\rangle_1$. Choose an arbitrary $u \in (x,z) \subset B$. It easy to verify that

$$\operatorname{cone}(u,B) = \operatorname{cone}(u,[x,B\rangle)$$

and hence $u \in B_1$. Thus $u \in C'_1$. Then for the ray $m = (0,u\rangle$ we have $m \subset C_1'$. Since $\pi m = l'$ and $-l' \subset \pi C''$, we obtian $-m \subset C''$ for an appropriate choice of the point $u \in (x,z)$.

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