MATH. SCAND. 37 (1975), 271–283

THE STANDARD FORM OF VON NEUMANN ALGEBRAS

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Introduction.

To any left Hilbert algebra $\mathscr{A}$ we associate a selfdual cone P, which generalizes the cones $P^{\natural}_{\xi_0}$ in [3] and $V^{\frac{1}{4}}_{\xi_0}$ in [1]. P is defined as the closure of the set

$$\{\xi(J\xi) \mid \xi \in \mathscr{A}\}$$

in the completion H of $\mathscr{A}$. Using this cone we prove that any von Neumann algebra is isomorphic to a von Neumann algebra M on a Hilbert space H, such that there exists a conjugate linear, isometric involution J of H and a selfdual cone P in H with the properties:

1) $JMJ = M'$,
2) $JcJ = c^* \quad \forall c \in Z(M)$ (center of M),
3) $J\xi = \xi \quad \forall \xi \in P$,
4) $aa^t(P) \subseteq P \quad \forall a \in M,\ a^t = JaJ$.

A quadruple (M,H,J,P) satisfying the conditions 1)–4) is called a standard form of the von Neumann algebra M. We prove that the standard form is unique in the sense, that if (M,H,J,P) and $(\tilde{M},\tilde{H},\tilde{J},\tilde{P})$ are two standard forms, and $\Phi\colon M \to \tilde{M}$ is a *isomorphism then there is a *unique* unitary $u\colon H \to \tilde{H}$ such that

a) $\Phi(x) = uxu^* \quad \forall x \in M$,
b) $\tilde{J} = uJu^*$,
c) $\tilde{P} = u(P)$.

An easy application of this uniqueness theorem gives that the group of all *automorphisms of a von Neumann algebra on standard form has a canonical unitary implementation.

If the von Neumann algebra M admits a cyclic and separating vector, our results are more or less trivial consequences of the results of H. Araki and A. Connes in the papers [1] and [3]. Therefore the proofs are concentrated mainly on the special difficulties in the non σ-finite case.

This paper is a shortened version of my thesis [5] for the cand. scient. degree in Copenhagen 1973.

Received September 13, 1973; in revised form May 1, 1975.

1. Positive elements associated with an achieved left Hilbert algebra.

Let P be a cone in a Hilbert space H. The dual cone P° is defined by $P^\circ = \{\xi \in H \mid (\xi|\eta) \geqq 0 \ \forall \eta \in P\}$. If $P = P^\circ$, P is called selfdual.

Let $\mathscr{A}$ be an achieved left Hilbert algebra, and $\mathscr{A}'$ the corresponding right Hilbert algebra. Since $\xi \in \mathscr{A}$ implies $\xi^* = J\xi \in \mathscr{A}'$ it makes sense to put

$$P = \{\xi \cdot \xi^* \mid \xi \in \mathscr{A}\}^- ,$$

where the closure is in the completion H of $\mathscr{A}$.

The von Neumann algebra $\mathscr{L}(\mathscr{A})$ will be denoted by M.

THEOREM 1.1. *P is a cone in H with the properties*

(1) $J\xi = \xi \quad \forall \xi \in P$.
(2) $\Delta^{it}(P) = P \quad \forall t \in \mathbf{R}$.
(3) *P is selfdual.*
(4) $\forall a \in M : aa^t(P) \subseteqq P$, *where* $a^t = JaJ$.

REMARK 1.2. Let M be a von Neumann algebra with a cyclic and separating vector ξ_0. The set $M\xi_0$ is a left Hilbert algebra with product

$$(a\xi_0)(b\xi_0) = (ab)\xi_0$$

and involution

$$(a\xi_0)^\# = a^*\xi_0 .$$

An easy computation gives $P = \{aa^t\xi_0 \mid a \in M)^-$ where $a^t = JaJ$. Hence in this case P coincides with P_{ξ_0} in [3] and $V^{\frac{1}{4}}_{\xi_0}$ in [1].

For the proof of Theorem 1.1 we shall use a result from [8]. It is proved that the cones

$$P^\# = \{\xi\xi^\# \mid \xi \in \mathscr{A}\}^-, \quad P^\flat = \{\eta\eta^\flat \mid \eta \in \mathscr{A}'\}^-$$

are dual cones, i.e.

$$\xi \in P^\# \Leftrightarrow (\xi|\eta) \geqq 0 \ \forall \eta \in P^\flat \quad \text{and} \quad \eta \in P^\flat \Leftrightarrow (\xi|\eta) \geqq 0 \ \forall \xi \in P^\# .$$

LEMMA 1.3. *Let $\mathscr{A}_0$ be the maximal Tomita algebra equivalent to $\mathscr{A}$* (cf. [2, lemma 2.7]). *For $\xi \in \mathscr{A}$ there exists a sequence $\{\xi_n\} \subseteqq \mathscr{A}_0$ such that*

(i) $\xi_n \to \xi$, $\xi_n^\# \to \xi^\#$,
(ii) $\|\pi(\xi_n)\| \leqq \|\pi(\xi)\| \ \forall n \in \mathbf{N}$,
(iii) $\pi(\xi_n) \to \pi(\xi)$, $\pi(\xi_n^\#) \to \pi(\xi^\#)$ *strongly.*

PROOF. $\mathscr{A}_0$ consists of the elements $\xi \in H$ for which

(a) $\xi \in D(\Delta^\alpha) \quad \forall \alpha \in \mathbf{C}$
(b) $\Delta^\alpha \xi \in \mathscr{A} \quad \forall \alpha \in \mathbf{C}$.

Put

$$f_n(x) = \exp(-x^2/2n^2) \quad \text{and} \quad \xi_n = f_n(\log \Delta)\xi .$$

Obviously $\xi_n \in D(\Delta^\alpha)$ for any $\alpha \in \mathbf{C}$. Note that

$$\Delta^\alpha(f_n(\log \Delta)) = \varphi_{\alpha,n}(\log \Delta)$$

where $\varphi_{\alpha,n}(x) = \exp(\alpha x - x^2/2n^2)$. Since $\varphi_{\alpha,n}$ is a linear combination of positive definite functions, $\varphi_{\alpha,n}(\log \Delta)$ maps $\mathscr{A}$ into $\mathscr{A}$ (see [9, lemma 10.1]). Hence $\Delta^\alpha \xi_n \in \mathscr{A}$ $\forall \alpha \in \mathbf{C}$.

(i) Since $f_n(\log \Delta)$ converges strongly to 1 we get

$$\xi_n = f_n(\log \Delta)\xi \to \xi$$
$$\xi_n{}^\# = f_n(\log \Delta)\xi^\# \to \xi^\#$$

(cf. [9, lemma 10.1])

(ii) Since

$$f_n(x) = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-n^2 t^2/2) e^{ixt} dt$$

$f_n(x)$ are positive definite functions. By the proof of [9, lemma 10.1] we find that

$$\|\pi(\xi_n)\| \leqq f_n(0)\|\pi(\xi)\| = \|\pi(\xi)\| .$$

(iii) For each $\eta \in \mathscr{A}'$:

$$\pi(\xi_n)\eta = \pi'(\eta)\xi_n \to \pi'(\eta)\xi = \pi(\xi)\eta .$$

Since $\mathscr{A}'$ is dense in H and $\sup \|\pi(\xi_n)\| < \infty$ we conclude that $\pi(\xi_n) \to \pi(\xi)$ strongly. The same argument gives $\pi(\xi_n{}^\#) \to \pi(\xi^\#)$ strongly.

LEMMA 1.4. *Put*

$$P_0 = \{\xi\xi^* \mid \xi \in \mathscr{A}_0\}, \quad P_0{}^\# = \{\xi\xi^\# \mid \xi \in \mathscr{A}_0\}, \quad P_0{}^\flat = \{\xi\xi^\flat \mid \xi \in \mathscr{A}_0\} .$$

Then P (respectively $P^\#, P^\flat$) is the closure of P_0 (respectively $P_0{}^\#, P_0{}^\flat$).

PROOF. (i) It is enough to show that the closure of P_0 contains $\{\xi\xi^* \mid \xi \in \mathscr{A}\}$. Let $\xi \in \mathscr{A}$, and let $\{\xi_n\} \subseteq \mathscr{A}_0$ be a sequence satisfying the conditions of lemma 1.3. Then

$$\xi_n \xi_n{}^* = \pi(\xi_n)\xi_n{}^* \to \pi(\xi)\xi^* = \xi \cdot \xi^* .$$

(ii) By the same arguments we get $P^\sharp = (P_0{}^\sharp)^-$.

(iii) $P^\flat = (P_0{}^\flat)^-$ follows from (ii) because $P^\flat = J(P^\sharp)$ and $P_0{}^\flat = J(P_0{}^\sharp)$.

LEMMA 1.5. *P is the closure of $\Delta^{\frac{1}{4}}(P^\sharp)$ (respectively $\Delta^{-\frac{1}{4}}(P^\flat)$). In particular P is a closed convex cone.*

PROOF. Since

$$P^\sharp \subseteq D(S) = D(\Delta^{\frac{1}{2}}) \quad \text{and} \quad P^\flat \subseteq D(F) = D(\Delta^{-\frac{1}{2}})$$

the two sets are well defined. Since

$$\Delta^{\frac{1}{2}}(P^\sharp) = JS(P^\sharp) = J(P^\sharp) = P^\flat$$

we get $\Delta^{\frac{1}{4}}(P^\sharp) = \Delta^{-\frac{1}{4}}(P^\flat)$. Therefore it is enough to prove $P = (\Delta^{\frac{1}{4}}(P^\sharp))^-$.

Let $\xi \in \mathscr{A}_0$:

$$\Delta^{\frac{1}{4}}(\xi\xi^\sharp) = (\Delta^{\frac{1}{4}}\xi)(\Delta^{\frac{1}{4}}\xi^\sharp) = (\Delta^{\frac{1}{4}}\xi)(\Delta^{\frac{1}{4}}\xi)^* .$$

Since $\Delta^{\frac{1}{4}}(\mathscr{A}_0) = \mathscr{A}_0$ we conclude that $\Delta^{\frac{1}{4}}(P_0{}^\sharp) = P_0$.

Let $\xi \in P^\sharp$, and choose a sequence $\{\xi_n\} \subseteq P_0{}^\sharp$ so that $\xi_n \to \xi$. Since $S\xi_n = \xi_n$, $S\xi = \xi$ and $\Delta^{\frac{1}{2}} = JS$ we have $\Delta^{\frac{1}{2}}\xi_n \to \Delta^{\frac{1}{2}}\xi$. Hence

$$\|\Delta^{\frac{1}{4}}\xi_n - \Delta^{\frac{1}{4}}\xi\|^2 = (\Delta^{\frac{1}{2}}(\xi_n - \xi) \,|\, \xi_n - \xi) \to 0 .$$

Since $\Delta^{\frac{1}{4}}\xi_n \in P_0$ we have $\Delta^{\frac{1}{4}}(P^\sharp) \subseteq P$. On the other hand $\Delta^{\frac{1}{4}}(P^\sharp) \supseteq P_0$. Hence $(\Delta^{\frac{1}{4}}(P^\sharp))^- = P$.

PROOF OF THEOREM 1.1. (1) Let $\xi \in \mathscr{A}_0$. Then

$$J(\xi\xi^*) = (\xi\xi^*)^* = \xi\xi^* .$$

Hence by lemma 1.4 $J\xi = \xi \ \forall \xi \in P$.

(2) Let $\xi \in \mathscr{A}_0$. Then

$$\Delta^{it}(\xi\xi^*) = (\Delta^{it}\xi)(\Delta^{it}\xi^*) = (\Delta^{it}\xi)(\Delta^{it}\xi)^* .$$

Hence $\Delta^{it}(P_0) = P_0$, and $\Delta^{it}(P) = P$.

(3) Let $\xi \in \Delta^{\frac{1}{4}}(P^\sharp)$ and $\eta \in \Delta^{-\frac{1}{4}}(P^\flat)$. Then

$$(\xi \,|\, \eta) = (\Delta^{-\frac{1}{4}}\xi \,|\, \Delta^{\frac{1}{4}}\eta) \geqq 0$$

because $P^\sharp$ and $P^\flat$ are dual cones. Hence

$$(\xi \,|\, \eta) \geqq 0 \quad \forall \xi, \eta \in P$$

(by lemma 1.5).

Let now $\xi \in H$ and assume that $(\xi | \eta) \geqq 0 \ \forall \eta \in P$. We shall prove that $\xi \in P$. Put

$$\xi_n = f_n(\log \Delta)\xi$$

where $f_n(x) = \exp(-x^2/2n^2)$ as in lemma 1.3. Note that $\xi_n \in D(\Delta^{\frac{1}{4}})$ for any $n \in \mathbf{N}$. Since

$$\xi_n = \int_{-\infty}^{\infty} g_n(t) \Delta^{it} \xi \, dt$$

where $g_n(t) = n(2\pi)^{-\frac{1}{2}} \exp(-n^2t^2/2)$, we get using (2) that for any $\eta \in P$:

$$(\xi_n | \eta) = \int_{-\infty}^{\infty} g_n(t)(\xi | \Delta^{-it}\eta) dt \geqq 0 .$$

Hence for any $\zeta \in P^\flat$:

$$0 \leqq (\xi_n | \Delta^{-\frac{1}{4}}\zeta) = (\Delta^{-\frac{1}{4}}\xi_n | \zeta) .$$

Therefore $\Delta^{-\frac{1}{4}}\xi_n \in P^\sharp$ (dual cone of $P^\flat$). Hence $\xi_n \in \Delta^{\frac{1}{4}}(P^\sharp) \subseteq P$. Since $\xi_n \to \xi$ we get $\xi \in P$.

(4) Let $\xi, \eta \in \mathscr{A}_0$ and put $\pi(\xi)^t = J\pi(\xi)J$. Then

$$\begin{aligned} \pi(\xi)\pi(\xi)^t(\eta\eta^*) &= \pi(\xi)J\pi(\xi)J(\eta\eta^*) \\ &= \pi(\xi)J(\pi(\xi)\eta\eta^*) \\ &= \xi(\xi\eta\eta^*)^* = (\xi\eta)(\xi\eta)^* . \end{aligned}$$

Hence by lemma 1.4, $\pi(\xi)\pi(\xi)^t$ maps the cone P into itself. An easy application of Kaplansky's density theorem gives now that

$$aa^t(P) \subseteq P$$

for any a in the von Neumann algebra associated with the left Hilbert algebra $\mathscr{A}_0$, i.e. for any $a \in M$.

From theorem 1.1 and the basic results of the Tomita–Takesaki theory we get:

THEOREM 1.6. *Any von Neumann algebra is isomorphic to a von Neumann algebra M on a Hilbert space H, such that there exists a conjugate linear isometric involution $J: H \to H$, and a selfdual cone P in H with the following properties:*

(1) $JMJ = M'$

(2) $JcJ = c^* \quad \forall c \in Z(M)$ (*the center of M*).

(3) $J\xi = \xi \quad \forall \xi \in P$

(4) $aa^t(P) \subseteq P \quad \forall a \in M$ *where* $a^t = JaJ$.

2. The standard form of von Neumann algebras.

DEFINITION 2.1. A quadruple (M,H,J,P) satisfying the conditions of Theorem 1.6 is called a standard form of the von Neumann algebra M.

REMARK 2.2. Usually a von Neumann algebra on a Hilbert space H is called standard if there exists a conjugate linear isometric involution J_0 of H, such that $J_0 M J_0 = M'$. Such a von Neumann algebra is spatially isomorphic to the von Neumann algebra associated with some left Hilbert algebra. Hence if M is standard on H, we can choose J and P in H, such that (M,H,J,P) is a standard form (Theorem 1.1). It can happen that $J \neq J_0$ for any possible choice of (J,P) (cf. [5, proposition 5.3]).

The main result of this section asserts that the standard form is unique in the following strict sense:

THEOREM 2.3. *Let (M,H,J,P) and $(\tilde{M},\tilde{H},\tilde{J},\tilde{P})$ be two standard forms, and let $\Phi: M \to \tilde{M}$ be a *isomorphism. There exists one and only one unitary $u: H \to \tilde{H}$ such that*

(1) $\Phi(x) = uxu^{-1} \quad \forall x \in M,$
(2) $\tilde{J} = uJu^{-1},$
(3) $\tilde{P} = u(P).$

LEMMA 2.4. *Let M be a von Neumann algebra on a Hilbert space H, and let q be a projection of the form $q = pp'$ where $p \in M$ and $p' \in M'$ are two projections. Put $qMq = \{qaq \mid a \in M\}$ regarded as a set of operators on $q(H)$. Then:*

(i) *qMq is a von Neumann algebra*
(ii) *$(qMq)' = qM'q$.*
(iii) *$Z(qMq) = qZ(M)q$, where $Z(\cdot)$ denotes the center.*
(iv) *If $c(p) \leq c(p')$ the map $pxp \to qxq$ is a *isomorphism of pMp onto qMq. ($c(\cdot)$ denotes the central support).*

PROOF. In [4] Chapter 1, § 2 the lemma is proved if $q \in M$ (reduction) or $q \in M'$ (induction). The general case $q = pp'$ can easily be reduced to these cases, because the map $x \to qxq$ is composed of a reduction $x \to pxp$ of M onto pMp followed by an induction $pxp \to qxq$ of pMp onto qMq, where q is regarded as an element of $(pMp)'$.

COROLLARY 2.5. *Let (M,H,J,P) be a standard form, and let p be a projection in M. Put $q = pp^t$ ($p^t = JpJ$). Then the induction $pxp \to qxq$ is an isomorphism of pMp onto qMq. In particular $p \neq 0$ iff $q \neq 0$.*

PROOF. Since J commutes with central projections in M we have $c(p^t) = Jc(p^t)J \geqq Jp^tJ = p$. Hence $c(p^t) \geqq c(p)$.

LEMMA 2.6. *Let (M,H,J,P) be a standard form, p a projection in M and $q = pp^t$. Then $(qMq, q(H), qJq, q(P))$ is a standard form.*

PROOF. Since $Jq = qJ$, J leaves $q(H)$ invariant. Hence qJq is an isometric involution in $q(H)$. Obviously $(\xi|\eta) \geqq 0 \ \forall \xi,\eta \in q(P)$ because $q(P) \subseteqq P$.

Assume that $\xi \in q(H)$ and $(\xi|\eta) \geqq 0 \ \forall \eta \in q(P)$. Then $\forall \zeta \in P$:

$$0 \leqq (\xi|q\zeta) = (q\xi|\zeta) = (\xi|\zeta) .$$

Hence $\xi \in P$ and $\xi = q\xi \in q(P)$. Therefore $q(P)$ is a selfdual cone in $q(H)$. We now verify the conditions 1)–4) in Theorem 1.6.

(1) $(qJq)(qMq)(qJq) = q(JMJ)q = qM'q = (qMq)'$.

(2) If $c \in Z(qMq) = qZ(M)q$ then $c = qxq$ for some $x \in Z(M)$. Hence

$$(qJq)(qxq)(qJq) = q(JxJ)q = qx^*q = (qxq)^*$$

(3) and (4) are trivial because $q(P) \subseteqq P$.

REMARK 2.7. Any selfdual cone P in a Hilbert space H is total. For if $(\xi|\eta) = 0 \ \forall \eta \in P$, then both ξ and $-\xi$ belong to $P^\circ = P$. Hence $(\xi|-\xi) \geqq 0$.

Let M be a von Neumann algebra on a Hilbert space H, and let ξ be a vector in H. Then

a) $e(\xi)$ (respectively $e'(\xi)$) denotes the projection on the closure of $M'\xi$ (respectively $M\xi$).

b) ω_ξ (respectively ω'_ξ) denotes the restriction of the vector functional $x \to (x\xi|\xi)$ to M (respectively M').

Note that $e(\xi) = s(\omega_\xi)$ and $e'(\xi) = s'(\omega_\xi)$ where $s(\cdot)$ is the support of the functional.

LEMMA 2.8. *Let (M,H,J,P) be a standard form, and M σ-finite, then there exists a cyclic and separating vector $\xi \in P$.*

PROOF. Take a maxinal family $(\xi_i)_{i\in I}$ of vectors in $P \setminus \{0\}$ such that $(e(\xi_i))_{i\in I}$ are mutually orthogonal. Assume that

$$p = 1 - \textstyle\sum_{i\in I} e(\xi_i) \neq 0 .$$

By corollary 2.5, $q=p\cdot p^t \neq 0$ and since $q(P)$ is a selfdual cone in $q(H)$, there exists $\xi \in q(P)\setminus\{0\}$. However, $e(\xi)\leqq p$, which contradicts the maximality of $(\xi_i)_{i\in I}$. Hence $\sum_{i\in I} e(\xi_i)=1$.

Since M is σ-finite the index set I is at most countable. Thus we may assume that

$$\sum_{i\in I} \|\xi_i\|^2 < \infty .$$

Now put $\xi=\sum_{i\in I}\xi_i \in P$.

Using that $M'\xi_i \perp M'\xi_j$ if $i\neq j$ and that $M'=JMJ$ we get

$$M\xi_i \perp M\xi_j \quad \text{if } i\neq j .$$

Hence $\omega_\xi=\sum_{i\in I}\omega_{\xi_i}$ and

$$e(\xi) = s(\omega_\xi) = \sum_{i\in I} s(\omega_{\xi_i}) = \sum_{i\in I} e(\xi_i) = 1 .$$

Therefore ξ is separating. Using that $e'(\xi)=e'(J\xi)=Je(\xi)J$ we find that ξ is also cyclic.

LEMMA 2.9. *Let (M,H,J,P) be a standard form and ξ a cyclic and separating vector in P. Then $J_\xi=J$ and $P_\xi=P$, where J_ξ and P_ξ is the involution and the selfdual cone associated with the left Hilbert algebra $M\xi$* (cf. Remark 1.2).

PROOF. That $J_\xi=J$ follows from [10, lemma 4.2] (see also [1, theorem 1]).

Since $aa^t=a(JaJ)$ maps P into P for any $a\in M$ we get

$$P_\xi = \{a(J_\xi aJ_\xi)\xi \mid a\in M\}^{-} \subseteq P .$$

Hence $P_\xi=P$, because both P_ξ and P are selfdual.

LEMMA 2.10. *Let (M,H,J,P) be a standard form.*

(1) *Any $\varphi\in M_*^+$ has the form $\varphi=\omega_\xi$ for a unique vector $\xi\in P$.*
(2) *For $\xi,\eta\in P$:*

$$\|\xi-\eta\|^2 \leqq \|\omega_\xi-\omega_\eta\| \leqq \|\xi-\eta\|\,\|\xi+\eta\| .$$

In particular $\xi\to\omega_\xi$ is a homeomorphism of P onto M_*^+.

PROOF. Note that the inequality $\|\omega_\xi-\omega_\eta\|\leqq\|\xi-\eta\|\,\|\xi+\eta\|$ is trivial because

$$\omega_\xi-\omega_\eta = \tfrac{1}{2}(\omega_{\xi-\eta,\,\xi+\eta}+\omega_{\xi+\eta,\,\xi-\eta}) .$$

If M is σ-finite the lemma follows from [1, Theorem 4 and Theorem 6], because $P=P_\xi$ and $J=J_\xi$ for some cyclic and separating vector $\xi\in P$ (see also [3, Theorem 2.7]).

Let now M be arbitrary:

(1): Take $\varphi \in M_*^+$, let p be the support of φ and $q = pp'$, where $p' = JpJ$. Since the induction $pMp \to qMq$ is an isomorphism, there exists a functional $\psi \in (qMq)_*$ such that

$$\varphi(x) = \psi(qxq) \quad \forall x \in M .$$

Since qMq is σ-finite and (qMq, qH, qJq, qP) is a standard form, there exists $\xi \in q(P) \subseteqq P$ so that $\psi(y) = (y\xi \mid \xi)$ $\forall y \in qMq$. Hence

$$\varphi(x) = (x\xi \mid \xi), \quad x \in M .$$

The uniqueness of ξ follows when the inequality (2) is proved.

(2): The inequality follows from the σ-finite case by regarding the reduced standard form (qMq, qH, qJq, qP) corresponding to $q = pp'$ where $p = e(\xi) \vee e(\eta)$.

Proof of Theorem 2.3. Assume that u_1 and u_2 satisfy the conditions 1)–3).

Let $\xi \in P$. By 3), $u_1\xi \in \tilde{P}$ and $u_2\xi \in \tilde{P}$. Moreover:

$$(\Phi(a)u_1\xi \mid u_1\xi) = (a\xi \mid \xi) = (\Phi(a)u_2\xi \mid u_2\xi) .$$

Since the map $\eta \to \omega_\eta$ is a bijection of $\tilde{P}$ on $\tilde{M}_*$ we get $u_1\xi = u_2\xi$. Consequently $u_1 = u_2$, because a selfdual cone is total (by Remark 2.7). To prove the existence we assume first that M is σ-finite. Then M has a cyclic and separating vector $\xi \in P$. By Lemma 2.9 there exists $\eta \in \tilde{P}$ so that

$$\omega_\eta(\Phi(x)) = \omega_\xi(x) \quad \forall x \in M .$$

η is separating for $\tilde{M}$ and therefore $J\eta = \eta$ is cyclic for $\tilde{M}$. The equation

$$\|\Phi(a)\eta\|^2 = \omega_\eta(\Phi(a^*a)) = \omega_\xi(a^*a) = \|a\xi\|^2 \quad \forall a \in M .$$

shows that the map $a\xi \to \Phi(a)\eta$, $a \in M$ can be extended to a unitary $u: H \to \tilde{H}$. We claim that u satisfies the conditions 1)–3):

(1) Let $\zeta \in \tilde{M}\eta$, $\zeta = \Phi(a)\eta$ for some $a \in M$. Then

$$\Phi(b)\zeta = \Phi(ba)\eta = u(ba\xi) = ubu^{-1}(\Phi(a)\eta) = ubu^{-1}\zeta \quad \forall b \in M .$$

Hence $\Phi(b) = ubu^{-1}$ because $\tilde{M}\eta$ is dense in $\tilde{H}$.

(2) Let S_ξ (respectively S_η) be the closure of the operator $a\xi \to a^*\xi$, $a \in M$ (respectively $b\eta \to b^*\eta$, $b \in \tilde{M}$). Then it is easy to check that

$$S_\eta = uS_\xi u^{-1} .$$

By polar decomposition

$$S_\xi = J_\xi \Delta_\xi^{\frac{1}{2}}, \quad S_\eta = J_\eta \Delta_\eta^{\frac{1}{2}} .$$

Thus $J_\eta = uJ_\xi u^{-1}$. But $J = J_\xi$ and $\tilde{J} = J_\eta$ (by Lemma 2.9). Hence $\tilde{J} = uJu^{-1}$.

(3) Clearly

$$\begin{aligned} P = P_\eta &= \{a\tilde{J}a\eta \mid a \in \tilde{M}\}^- \\ &= \{(uau^{-1})(uJu^{-1})(uau^{-1})\eta \mid a \in M\}^- \\ &= u\{aJa\xi \mid a \in M\}^- = u(P) . \end{aligned}$$

In the general case let p be a σ-finite projection in M. Put $q = pp^t$ and $r = \Phi(p)\Phi(p)^t$. Since the inductions

$$pMp \to qMq \quad \text{and} \quad \Phi(p)\tilde{M}\Phi(p) \to r\tilde{M}r$$

are isomorphisms, there is a unique isomorphism $\Phi_q\colon qMq \to r\tilde{M}r$ so that

$$\Phi_q(qxq) = r\Phi(x)r, \quad x \in M .$$

Using the first part of the proof on the reduced standard forms we find that there is a unique isometry u_q of $q(H)$ on $r(H)$ satisfying

(a) $r\Phi(x)r = u_q(qxq)u_q \quad \forall x \in M$,
(b) $r\tilde{J}r = u_q(qJq)u_q$,
(c) $r(\tilde{P}) = u_q q(\tilde{P})$.

This construction can be carried out for any σ-finite projection. If $p_1 \leqq p_2$ then $q_1 \leqq q_2$ and $r_1 \leqq r_2$. The uniqueness of u_q, shows that in this case $u_{q_1} \subseteqq u_{q_2}$. Choose a net $(p_i)_{i\in I}$ of σ-finite projections in M so that $p_i \nearrow 1$. Then

$$q_i = p_i p_i^t \nearrow 1 \quad \text{and} \quad r_i \nearrow 1 .$$

Since $u_{q_i} \subseteqq u_{q_j}$ when $p_i \leqq p_j$, there exists an isometry u of H onto $\tilde{H}$ which extend every u_{q_i}, $i \in I$. Using

$$\begin{aligned} H &= \left(\textstyle\bigcup_{i\in I} q_i(H)\right)^- \qquad & \tilde{H} &= \left(\textstyle\bigcup_{i\in I} r_i(H)\right)^- , \\ P &= \left(\textstyle\bigcup_{i\in I} q_i(P)\right)^- \qquad & \tilde{P} &= \left(\textstyle\bigcup_{i\in I} r_i(\tilde{P})\right)^- , \end{aligned}$$

we find that u has the required properties.

REMARK 2.11. Condition 2) in Theorem 2.3 is not essential. Since P and $\tilde{P}$ are total, we have 3) $\Rightarrow$ 2).

3. Unitary implementation of automorphism groups.

DEFINITION 3.1. Let M be a von Neumann algebra on a Hilbert space H, and G a group of *automorphisms of M. A unitary implementation of G is a unitary representation $g \to u_g$ of G on H, such that

$$g(a) = u_g a u_g^* \quad \forall g \in G,\ \forall a \in M .$$

As an easy application of Theorem 2.3 we get

THEOREM 3.2 (cf. [1, theorem 11] and [3]). *Let (M,H,J,P) be a standard form. The group* $\mathrm{aut}(M)$ *of *automorphisms of M has a unique unitary implementation $g \to u_g$, such that*

$$(*) \qquad J = u_g J u_g^{-1} \quad \text{and} \quad u_g(P) = P \quad \text{for any } g \in \mathrm{aut}(M)\,.$$

PROOF. By Theorem 2.3 we get that for each $g \in \mathrm{aut}(G)$, there is a unique unitary on H which satisfies $(*)$. It follows from the uniqueness that

$$u_{g\cdot h} = u_g \cdot u_h \quad \forall g,h \in G\,.$$

DEFINITION 3.3. The map $g \to u_g$ in Theorem 3.2 will be called the canonical implementation of $\mathrm{aut}(M)$.

DEFINITION 3.4. Let M be a von Neumann algebra. On the set of bounded, σ-weak continuous operators on M we define the *p-topology* by the semi-norms

$$T \to \langle Tx, \varphi \rangle \quad x \in M,\ \varphi \in M_*$$

and the *u-topology* by the semi-norms

$$T \to \|T_* \varphi\|, \quad \varphi \in M_*$$

where $T_* : \varphi \to \varphi \circ T$ is the transposed action on the predual.

PROPOSITION 3.5. *Let (M,H,J,P) be a standard form. The canonical implementation $g \to u_g$ of* $\mathrm{aut}(M)$ *is a homeomorphism of* $\mathrm{aut}(M)$ *onto a closed subgroup of the unitary group on H, when the first is equipped with u-topology and the latter with strong (=weak) operator topology.*

PROOF. Since the map $\xi \to \omega_\xi$ is a homeomorphism of P onto M_*^+ we find by repeating the arguments of [1, Remark following Theorem 11] that the map $g \to u_g$ is a homeomorphism on its range. Since $\{u_g \mid g \in \mathrm{aut}(M)\}$ is equal to the set of unitaries for which

$$uMu^* = M, \quad uJu^* = J, \quad u(P) = P\,,$$

the set is strongly closed relative to the unitary group.

COROLLARY 3.6. *Let (M,H,J,P) be a standard form, G a locally compact group and $\alpha : G \to$* $\mathrm{aut}(M)$ *a σ-weakly continuous representation of G on M. Then the canonical unitary implementation $g \to u_{\alpha(g)}$ of G is strongly continuous.*

PROOF. Since $g \to \langle \alpha(g)x, \varphi \rangle$ is continuous for $x \in M$, $\varphi \in M_*$ the action of G on the predual $g \to \alpha(g)_*$ is $\sigma(M_*, M)$-continuous. Hence by [6, p. 23] the action is also strongly continuous, i.e.

$$\|\alpha(g_i)_* \varphi - \alpha(g)_* \varphi\| \to 0 \quad \text{for any } \varphi \in M_* .$$

REMARK. Theorem 3.2 and Corollary 3.6 are generalizations of theorem 6.10 and proposition 6.11 of [7].

PROPOSITION 3.7. *Let φ be a normal, faithful, semifinite weight on a von Neumann algebra M. The p-topology and the u-topology coincide on the group* $\mathrm{aut}_\varphi(M)$ *of *automorphisms on M, which leaves φ invariant.*

PROOF. Let (π_φ, H_φ) be the representation of M induced by φ. H_φ is obtained as completion of the pre-Hilbert-space

$$n_\varphi = \{x \in M \mid \varphi(x^*x) < \infty\} .$$

We let Λ_φ denote the injection of n_φ in H_φ. The set $\mathscr{A} = \Lambda_\varphi(n_\varphi \cap n_\varphi^*)$ is an achieved left Hilbert algebra (cf. [2]). Let (M, H, J, P) be the standard form associated with this left Hilbert algebra as in section 1. (We identify M and $\pi_\varphi(M)$.) For any $g \in \mathrm{aut}_\varphi(M)$ the map $\Lambda_\varphi(x) \to \Lambda_\varphi(g(x))$ can be extended to a unitary u_g on H. It is easily seen that u_g implements the automorphism u_g. We will prove that $g \to u_g$ is the canonical implementation. For $x \in n_\varphi \cap n_\varphi^*$ we have

$$u_g S \Lambda_\varphi(x) = u_g \Lambda_\varphi(x^*) = \Lambda_\varphi(g(x^*)) = S\Lambda_\varphi(g(x)) = S u_g \Lambda_\varphi(x)$$

Since S is the closure of the map $\Lambda_\varphi(x) \to \Lambda_\varphi(x^*)$, $x \in n_\varphi \cap n_\varphi^*$ we get $S = u_g S u_g^*$, and by polar decomposition

$$J = u_g J u_g^* \quad \text{and} \quad \Delta^{\frac{1}{2}} = u_g \Delta^{\frac{1}{2}} u_g^* .$$

Since $P^\sharp$ in this setting is the closure of

$$\{\Lambda_\varphi(x^*x) \mid x \in n_\varphi \cap n_\varphi^*\}$$

it is easily seen that $u_g(P^\sharp) = P^\sharp$. Using $P = (\Delta^{\frac{1}{4}}(P^\sharp))^-$ we get $u_g(P) = P$. Hence $g \to u_g$ is the canonical implementation of g. Obviously the p-topology is weaker than the u-topology on $\mathrm{aut}_\varphi(M)$. To prove the converse let $\xi \in \mathscr{A}$ and $\eta \in (\mathscr{A}')^2$. Now η has the form

$$\eta = \sum_{i=1}^n \eta_i \zeta_i^\flat \quad \eta_i, \zeta_i \in \mathscr{A}' .$$

Thus $\forall g \in \mathrm{aut}_\varphi(M)$:

$$\begin{aligned}(u_g\xi\,|\,\eta) &= \textstyle\sum_{i=1}^n (u_g\xi\,|\,\eta_i\zeta_i^\flat) = \sum_{i=1}^n (\pi'(\zeta_i)u_g\xi\,|\,\eta_i)\\ &= \textstyle\sum_{i=1}^n (\pi(u_g\xi)\zeta_i\,|\,\eta_i) = \sum_{i=1}^n \big(g(\pi(\xi))\zeta_i\,|\,\eta_i\big)\end{aligned}$$

Since $\mathscr{A}$ and $(\mathscr{A}')^2$ are dense in H we conclude that if $g_i \to g$ in the p-topology on $\mathrm{aut}_\varphi(M)$ then $u_{g_i} \to u_g$ weakly. Hence by Proposition 3.5, $g_i \to g$ in the u-topology.

COROLLARY 3.8. *If M is a factor of type* I *or of type* II_1, *the u-topology and the p-topology coincide on* $\mathrm{aut}(M)$.

PROOF. Every automorphism of these factors leaves the trace invariant.

REMARK 3.9. In general the p-topology on $\mathrm{aut}(M)$ is strictly weaker than the u-topology. An example is given in [5, corollary 3.15].

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