THE STANDARD FORM OF VON NEUMANN ALGEBRAS

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Introduction.

To any left Hilbert algebra \( \mathcal{A} \) we associate a selfdual cone \( P \), which generalizes the cones \( P^b_{\ell_0} \) in [3] and \( V^1_{\ell_0} \) in [1]. \( P \) is defined as the closure of the set

\[ \{ \xi(J\xi) \mid \xi \in \mathcal{A} \} \]

in the completion \( H \) of \( \mathcal{A} \). Using this cone we prove that any von Neumann algebra is isomorphic to a von Neumann algebra \( M \) on a Hilbert space \( H \), such that there exists a conjugate linear, isometric involution \( J \) of \( H \) and a selfdual cone \( P \) in \( H \) with the properties:

1) \( JMJ = M' \),
2) \( JcJ = c^* \quad \forall c \in Z(M) \) (center of \( M \)),
3) \( J\xi = \xi \quad \forall \xi \in P \),
4) \( a\alpha'(P) \subseteq P \quad \forall a \in M, \quad a^* = JaJ \).

A quadruple \((M, H, J, P)\) satisfying the conditions 1)–4) is called a standard form of the von Neumann algebra \( M \). We prove that the standard form is unique in the sense, that if \((M, H, J, P)\) and \((\bar{M}, \bar{H}, \bar{J}, \bar{P})\) are two standard forms, and \( \Phi : M \to \bar{M} \) is a *isomorphism then there is a unique unitary \( u : H \to \bar{H} \) such that

a) \( \Phi(x) = u x u^* \quad \forall x \in M \),
b) \( \bar{J} = uJ u^* \),
c) \( \bar{P} = u(P) \).

An easy application of this uniqueness theorem gives that the group of all *automorphisms of a von Neumann algebra on standard form has a canonical unitary implementation.

If the von Neumann algebra \( M \) admits a cyclic and separating vector, our results are more or less trivial consequences of the results of H. Araki and A. Connes in the papers [1] and [3]. Therefore the proofs are concentrated mainly on the special difficulties in the non \( \sigma \)-finite case.

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1. Positive elements associated with an achieved left Hilbert algebra.

Let $P$ be a cone in a Hilbert space $H$. The dual cone $P^\circ$ is defined by $P^\circ = \{ \xi \in H \mid (\xi | \eta) \geq 0 \ \forall \eta \in P \}$. If $P = P^\circ$, $P$ is called selfdual.

Let $\mathcal{A}$ be an achieved left Hilbert algebra, and $\mathcal{A}'$ the corresponding right Hilbert algebra. Since $\xi \in \mathcal{A}$ implies $\xi^* = J\xi \in \mathcal{A}'$ it makes sense to put

$$P = \{ \xi \cdot \xi^* \mid \xi \in \mathcal{A} \}^-,$$

where the closure is in the completion $H$ of $\mathcal{A}$.

The von Neumann algebra $\mathcal{L}(\mathcal{A})$ will be denoted by $M$.

**Theorem 1.1.** $P$ is a cone in $H$ with the properties

1. $J\xi = \xi \ \forall \xi \in P$.
2. $\Delta^t(P) = P \ \forall t \in \mathbb{R}$.
3. $P$ is selfdual.
4. $\forall a \in M : aa^t(P) \subseteq P$, where $a^t = jaJ$.

**Remark 2.** Let $M$ be a von Neumann algebra with a cyclic and separating vector $\xi_0$. The set $M\xi_0$ is a left Hilbert algebra with product

$$(a\xi_0)(b\xi_0) = (ab)\xi_0$$

and involution

$$(a\xi_0)^\# = a^* \xi_0.$$

An easy computation gives $P = \{ aa^t \xi_0 \mid a \in M \}^-$ where $a^t = jaJ$. Hence in this case $P$ coincides with $P_{\xi_0}$ in [3] and $V_{\xi_0}^1$ in [1].

For the proof of Theorem 1.1 we shall use a result from [8]. It is proved that the cones

$$P^\# = \{ \xi \xi^* \mid \xi \in \mathcal{A} \}^- , \quad P^b = \{ \eta \eta^b \mid \eta \in \mathcal{A}' \}^-$$

are dual cones, i.e.

$$\xi \in P^\# \iff (\xi | \eta) \geq 0 \ \forall \eta \in P^b \quad \text{and} \quad \eta \in P^b \iff (\xi | \eta) \geq 0 \ \forall \xi \in P^\#.$$ 

**Lemma 1.3.** Let $\mathcal{A}_0$ be the maximal Tomita algebra equivalent to $\mathcal{A}$ (cf. [2, lemma 2.7]). For $\xi \in \mathcal{A}$ there exists a sequence $\{ \xi_n \} \subseteq \mathcal{A}_0$ such that

1. $\xi_n \to \xi$, $\xi_n^\# \to \xi^\#$,
2. $\|\pi(\xi_n)\| \leq \|\pi(\xi)\| \ \forall n \in \mathbb{N},$
3. $\pi(\xi_n) \to \pi(\xi)$, $\pi(\xi_n^\#) \to \pi(\xi^\#)$ strongly.
THE STANDARD FORM OF VON NEUMANN ALGEBRAS

\textbf{Proof.} $\mathcal{A}_0$ consists of the elements $\xi \in H$ for which

(a) $\xi \in D(\Delta^\alpha)$ \quad \forall \alpha \in \mathbb{C}$
(b) $\Delta^\alpha \xi \in \mathcal{A}$ \quad \forall \alpha \in \mathbb{C}.

Put

\[ f_n(x) = \exp(-x^2/2n^2) \quad \text{and} \quad \xi_n = f_n(\log \Delta)\xi. \]

Obviously $\xi_n \in D(\Delta^\alpha)$ for any $\alpha \in \mathbb{C}$. Note that

\[ \Delta^\alpha(f_n(\log \Delta)) = \varphi_{\alpha,n}(\log \Delta) \]

where $\varphi_{\alpha,n}(x) = \exp(\alpha x - x^2/2n^2)$. Since $\varphi_{\alpha,n}$ is a linear combination of positive definite functions, $\varphi_{\alpha,n}(\log \Delta)$ maps $\mathcal{A}$ into $\mathcal{A}$ (see [9, lemma 10.1]). Hence $\Delta^\alpha \xi_n \in \mathcal{A} \quad \forall \alpha \in \mathbb{C}$.

(i) Since $f_n(\log \Delta)$ converges strongly to 1 we get

\[ \xi_n^* = f_n(\log \Delta)^* \xi \rightarrow \xi^* \]
\[ \xi_n^{**} = f_n(\log \Delta)^{**} \xi^{**} \rightarrow \xi^{**} \]

(cf. [9, lemma 10.1])

(ii) Since

\[ f_n(x) = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^\infty \exp(-n^2 t^2/2)e^{i\xi t} dt \]

$f_n(x)$ are positive definite functions. By the proof of [9, lemma 10.1] we find that

\[ ||\pi(\xi_n)|| \leq f_n(0)||\pi(\xi)|| = ||\pi(\xi)||. \]

(iii) For each $\eta \in \mathcal{A}'$:

\[ \pi(\xi_n)\eta = \pi'(\eta)\xi_n \rightarrow \pi'(\eta)\xi = \pi(\xi)\eta. \]

Since $\mathcal{A}'$ is dense in $H$ and $\sup||\pi(\xi_n)|| < \infty$ we conclude that $\pi(\xi_n) \rightarrow \pi(\xi)$ strongly. The same argument gives $\pi(\xi_n^{**}) \rightarrow \pi(\xi^{**})$ strongly.

\textbf{Lemma 1.4.} \textbf{Put}

\[ P_0 = \{\xi \xi^* \mid \xi \in \mathcal{A}_0\}, \quad P_0^{**} = \{\xi^{**} \mid \xi \in \mathcal{A}_0\}, \quad P_0^b = \{\xi \xi^b \mid \xi \in \mathcal{A}_0\}. \]

Then $P$ (respectively $P^{**}, P^b$) is the closure of $P_0$ (respectively $P_0^{**}, P_0^b$).

\textbf{Proof.} (i) It is enough to show that the closure of $P_0$ contains $\{\xi \xi^* \mid \xi \in \mathcal{A}\}$. Let $\xi \in \mathcal{A}$, and let $\{\xi_n\} \subseteq \mathcal{A}_0$ be a sequence satisfying the conditions of lemma 1.3. Then

\[ \xi_n \xi_n^* = \pi(\xi_n)\xi_n^* \rightarrow \pi(\xi)\xi^* = \xi \cdot \xi^*. \]
(ii) By the same arguments we get \( P^\# = (P_0^\#)^- \).
(iii) \( P^b = (P_0^b)^- \) follows from (ii) because \( P^b = J(P^\#) \) and \( P_0^b = J(P_0^\#) \).

**Lemma 1.5.** \( P \) is the closure of \( \Lambda^1(P^\#) \) (respectively \( \Lambda^{-1}(P^b) \)).
In particular \( P \) is a closed convex cone.

**Proof.** Since

\[
P^\# \subseteq D(S) = D(\Lambda^1) \quad \text{and} \quad P^b \subseteq D(F) = D(\Lambda^{-1})
\]

the two sets are well defined. Since

\[
\Lambda^1(P^\#) = JS(P^\#) = J(P^\#) = P^b
\]

we get \( \Lambda^1(P^\#) = \Lambda^{-1}(P^b) \). Therefore it is enough to prove \( P = (\Lambda^1(P^\#))^-. \)

Let \( \xi \in \mathcal{A}_0 \):

\[
\Lambda^1(\xi^\#) = (\Lambda^1 \xi)(\Lambda^1 \xi^\#) = (\Lambda^1 \xi)(\Lambda^1 \xi)^*.
\]

Since \( \Lambda^1(\mathcal{A}_0) = \mathcal{A}_0 \) we conclude that \( \Lambda^1(P_0^\#) = P_0 \).

Let \( \xi \in P^\# \), and choose a sequence \( \{\xi_n\} \subseteq P_0^\# \) so that \( \xi_n \to \xi \). Since \( S\xi_n = \xi_n, S\xi = \xi \) and \( \Lambda^1 = JS \) we have \( \Lambda^1 \xi_n \to \Lambda^1 \xi \). Hence

\[
||\Lambda^1 \xi_n - \Lambda^1 \xi||^2 = (\Lambda^1(\xi_n - \xi) | \xi_n - \xi) \to 0.
\]

Since \( \Lambda^1 \xi_n \in P_0 \) we have \( \Lambda^1(P^\#) \subseteq P \). On the other hand \( \Lambda^1(P^\#) \supseteq P_0 \). Hence \( (\Lambda^1(P^\#))^- = P \).

**Proof of Theorem 1.1.** (1) Let \( \xi \in \mathcal{A}_0 \). Then

\[
J(\xi \xi^*) = (\xi \xi^*)^* = \xi \xi^*.
\]

Hence by lemma 1.4 \( J \xi = \xi \forall \xi \in P \).

(2) Let \( \xi \in \mathcal{A}_0 \). Then

\[
\Lambda^{\ast u}(\xi \xi^*) = (\Lambda^{\ast u} \xi)(\Lambda^{\ast u} \xi^*) = (\Lambda^{\ast u} \xi)(\Lambda^{\ast u} \xi)^*.
\]

Hence \( \Lambda^{\ast u}(P_0) = P_0 \), and \( \Lambda^{\ast u}(P) = P \).

(3) Let \( \xi \in \Lambda^1(P^\#) \) and \( \eta \in \Lambda^{-1}(P^b) \). Then

\[
(\xi | \eta) = (\Lambda^{-1} \xi | \Lambda^1 \eta) \geq 0
\]

because \( P^\# \) and \( P^b \) are dual cones. Hence

\[
(\xi | \eta) \geq 0 \quad \forall \xi, \eta \in P
\]

(by lemma 1.5).
Let now $\xi \in H$ and assume that $(\xi | \eta) \geq 0 \forall \eta \in P$. We shall prove that $\xi \in P$. Put
\[
\xi_n = f_n(\log \Lambda)\xi
\]
where $f_n(x) = \exp(-x^2/2n^2)$ as in lemma 1.3. Note that $\xi_n \in D(\Lambda^\dagger)$ for any $n \in \mathbb{N}$. Since
\[
\xi_n = \int_{-\infty}^{\infty} g_n(t)\Lambda^\mu \xi \, dt
\]
where $g_n(t) = n(2\pi)^{-1/2} \exp(-nt^2/2)$, we get using (2) that for any $\eta \in P$:
\[
(\xi_n | \eta) = \int_{-\infty}^{\infty} g_n(t)(\xi | \Lambda^{-\mu}\eta) \, dt \geq 0.
\]
Hence for any $\zeta \in P^b$:
\[
0 \leq (\xi_n | \Lambda^{-\dagger}\zeta) = (\Lambda^{-\dagger}\xi_n | \zeta).
\]
Therefore $\Lambda^{-\dagger}\xi_n \in P^\#_d$ (dual cone of $P^b$). Hence $\xi_n \in A^\dagger(P^\#_d) \subseteq P$. Since $\xi_n \to \xi$ we get $\xi \in P$.

(4) Let $\xi, \eta \in \mathcal{A}_0$ and put $\pi(\xi)' = J\pi(\xi)J$. Then
\[
\pi(\xi)\pi(\xi)'(\eta\eta^*) = \pi(\xi)J\pi(\xi)J(\eta\eta^*)
= \pi(\xi)J(\pi(\xi)\eta\eta^*)
= \xi(\xi\eta\eta^*)^* = (\xi\eta)(\xi\eta)^*.
\]
Hence by lemma 1.4, $\pi(\xi)\pi(\xi)'$ maps the cone $P$ into itself. An easy application of Kaplansky's density theorem gives now that
\[
a\Lambda'(P) \subseteq P
\]
for any $a$ in the von Neumann algebra associated with the left Hilbert algebra $\mathcal{A}_0$, i.e. for any $a \in M$.

From theorem 1.1 and the basic results of the Tomita–Takesaki theory we get:

**Theorem 1.6.** Any von Neumann algebra is isomorphic to a von Neumann algebra $M$ on a Hilbert space $H$, such that there exists a conjugate linear isometric involution $J: H \to H$, and a selfdual cone $P$ in $H$ with the following properties:

1. $JMJ = M'$
2. $JcJ = c^*$ $\forall c \in Z(M)$ (the center of $M$).
3. $J\xi = \xi$ $\forall \xi \in P$
4. $a\Lambda'(P) \subseteq P$ $\forall a \in M$ where $a' = JaJ$. 

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3. $J\xi = \xi$ $\forall \xi \in P$
4. $a\Lambda'(P) \subseteq P$ $\forall a \in M$ where $a' = JaJ$. 

2. The standard form of von Neumann algebras.

**Definition 2.1.** A quadruple \((M,H,J,P)\) satisfying the conditions of Theorem 1.6 is called a standard form of the von Neumann algebra \(M\).

**Remark 2.2.** Usually a von Neumann algebra on a Hilbert space \(H\) is called standard if there exists a conjugate linear isometric involution \(J_0\) of \(H\), such that \(J_0MJ_0 = M'\). Such a von Neumann algebra is spatially isomorphic to the von Neumann algebra associated with some left Hilbert algebra. Hence if \(M\) is standard on \(H\), we can choose \(J\) and \(P\) in \(H\), such that \((M,H,J,P)\) is a standard form (Theorem 1.1). It can happen that \(J = J_0\) for any possible choice of \((J,P)\) (cf. [5, proposition 5.3]).

The main result of this section asserts that the standard form is unique in the following strict sense:

**Theorem 2.3.** Let \((M,H,J,P)\) and \((\tilde{M}, \tilde{H}, \tilde{J}, \tilde{P})\) be two standard forms, and let \(\Phi: M \to \tilde{M}\) be a *isomorphism. There exists one and only one unitary \(u: H \to \tilde{H}\) such that

1. \(\Phi(x) = u xu^{-1}\) \(\forall x \in M\),
2. \(\tilde{J} = uJu^{-1}\),
3. \(\tilde{P} = u(P)\).

**Lemma 2.4.** Let \(M\) be a von Neumann algebra on a Hilbert space \(H\), and let \(q\) be a projection of the form \(q = pp'\) where \(p \in M\) and \(p' \in M'\) are two projections. Put \(qMq = \{gaq \mid a \in M\}\) regarded as a set of operators on \(q(H)\). Then:

1. \(qMq\) is a von Neumann algebra
2. \((qMq)' = qM'q\).
3. \(Z(qMq) = qZ(M)q\), where \(Z(\cdot)\) denotes the center.
4. If \(c(p) \leq c(p')\) the map \(pxp \to qxq\) is a *isomorphism of \(pMp\) onto \(qMq\). (\(c(\cdot)\) denotes the central support).

**Proof.** In [4] Chapter 1, § 2 the lemma is proved if \(q \in M\) (reduction) or \(q \in M'\) (induction). The general case \(q = pp'\) can easily be reduced to these cases, because the map \(x \to qxq\) is composed of a reduction \(x \to pxp\) of \(M\) onto \(pMp\) followed by an induction \(pxp \to qxq\) of \(pMp\) onto \(qMq\), where \(q\) is regarded as an element of \((pMp)'\).

**Corollary 2.5.** Let \((M,H,J,P)\) be a standard form, and let \(p\) be a projection in \(M\). Put \(q = pp'\) \((p' = JpJ)\). Then the induction \(pxp \to qxq\) is an isomorphism of \(pMp\) onto \(qMq\). In particular \(p \perp 0\) iff \(q \perp 0\).
THE STANDARD FORM OF VON NEUMANN ALGEBRAS

Proof. Since $J$ commutes with central projections in $M$ we have $c(p') = Jc(p')J \geq Jp'J = p$. Hence $c(p') \geq c(p)$.

Lemma 2.6. Let $(M,H,J,P)$ be a standard form, $p$ a projection in $M$ and $q = pp'$. Then $(qMq,q(H),qJq,q(P))$ is a standard form.

Proof. Since $Jq = qJ$, $J$ leaves $q(H)$ invariant. Hence $qJq$ is an isometric involution in $q(H)$. Obviously $(\xi|\eta) \geq 0 \quad \forall \xi, \eta \in q(P)$ because $q(P) \subseteq P$.

Assume that $\xi \in q(H)$ and $(\xi|\eta) \geq 0 \quad \forall \eta \in q(P)$. Then $\forall \xi \in P$:

$$0 \leq (\xi|q\xi) = (q\xi|\xi) = (\xi|\xi).$$

Hence $\xi \in P$ and $\xi = q\xi \in q(P)$. Therefore $q(P)$ is a selfdual cone in $q(H)$. We now verify the conditions 1)–4) in Theorem 1.6.

1. $(qJq)(qMq)(qJq) = q(JMJ)q = qM'q = (qMq)'$.
2. If $c \in Z(qMq) = qZ(M)q$ then $c = qxq$ for some $x \in Z(M)$. Hence

$$(qJq)(qxq)(qJq) = q(JxJ)q = qx^*q = (qxq)^*$$

(3) and (4) are trivial because $q(P) \subseteq P$.

Remark 2.7. Any selfdual cone $P$ in a Hilbert space $H$ is total. For if $(\xi|\eta) = 0 \quad \forall \eta \in P$, then both $\xi$ and $-\xi$ belong to $P^\circ = P$. Hence $(\xi|-\xi) \geq 0$.

Let $M$ be a von Neumann algebra on a Hilbert space $H$, and let $\xi$ be a vector in $H$. Then

a) $e(\xi)$ (respectively $e'(\xi)$) denotes the projection on the closure of $M'\xi$ (respectively $M\xi$).

b) $\omega_\xi$ (respectively $\omega_\xi'$) denotes the restriction of the vector functional $x \rightarrow (x\xi|\xi)$ to $M$ (respectively $M'$).

Note that $e(\xi) = s(\omega_\xi)$ and $e'(\xi) = s'(\omega_\xi)$ where $s(\cdot)$ is the support of the functional.

Lemma 2.8. Let $(M,H,J,P)$ be a standard form, and $M$ $\sigma$-finite, then there exists a cyclic and separating vector $\xi \in P$.

Proof. Take a maximal family $(\xi_i)_{i \in I}$ of vectors in $P \setminus \{0\}$ such that $(e(\xi_i))_{i \in I}$ are mutually orthogonal. Assume that

$$p = 1 - \sum_{i \in I} e(\xi_i) \neq 0.$$
By corollary 2.5, $q = p \cdot p' \neq 0$ and since $q(P)$ is a selfdual cone in $q(H)$, there exists $\xi \in q(P) \setminus \{0\}$. However, $e(\xi) \leq p$, which contradicts the maximality of $(\xi_i)_{i \in I}$. Hence $\sum_{i \in I} e(\xi_i) = 1$.

Since $M$ is $\sigma$-finite the index set $I$ is at most countable. Thus we may assume that

$$\sum_{i \in I} ||\xi_i||^2 < \infty.$$ 

Now put $\xi = \sum_{i \in I} \xi_i \in P$.

Using that $M'_j \xi_i \perp M'_j \xi_j$ if $i \neq j$ and that $M' = JMJ$ we get

$$M \xi_i \perp M \xi_j$$ 

if $i \neq j$.

Hence $\omega_\xi = \sum_{i \in I} \omega_{\xi_i}$ and

$$e(\xi) = s(\omega_\xi) = \sum_{i \in I} s(\omega_{\xi_i}) = \sum_{i \in I} e(\xi_i) = 1.$$ 

Therefore $\xi$ is separating. Using that $e'(\xi) = e'(J \xi) = Je(\xi) J$ we find that $\xi$ is also cyclic.

**Lemma 2.9.** Let $(M, H, J, P)$ be a standard form and $\xi$ a cyclic and separating vector in $P$. Then $J_\xi = J$ and $P_\xi = P$, where $J_\xi$ and $P_\xi$ is the involution and the selfdual cone associated with the left Hilbert algebra $M_\xi$ (cf. Remark 1.2).

**Proof.** That $J_\xi = J$ follows from [10, lemma 4.2] (see also [1, theorem 1]).

Since $aa' = a(JaJ)$ maps $P$ into $P$ for any $a \in M$ we get

$$P_\xi = \{a(J_\xi aJ_\xi) \xi \mid a \in M\}^- \subseteq P.$$ 

Hence $P_\xi = P$, because both $P_\xi$ and $P$ are selfdual.

**Lemma 2.10.** Let $(M, H, J, P)$ be a standard form.

1. Any $\varphi \in M_+^*$ has the form $\varphi = \omega_\xi$ for a unique vector $\xi \in P$.
2. For $\xi, \eta \in P$:

$$||\xi - \eta||^2 \leq ||\omega_\xi - \omega_\eta|| \leq ||\xi - \eta|| ||\xi + \eta||.$$ 

In particular $\xi \mapsto \omega_\xi$ is a homeomorphism of $P$ onto $M_+^*$.

**Proof.** Note that the inequality $||\omega_\xi - \omega_\eta|| \leq ||\xi - \eta|| ||\xi + \eta||$ is trivial because

$$\omega_\xi - \omega_\eta = \frac{1}{2}(\omega_{\xi - \eta} + \omega_{\xi + \eta}).$$ 

If $M$ is $\sigma$-finite the lemma follows from [1, Theorem 4 and Theorem 6], because $P = P_\xi$ and $J = J_\xi$ for some cyclic and separating vector $\xi \in P$ (see also [3, Theorem 2.7]).
Let now $M$ be arbitrary:

(1): Take $\varphi \in M^*_+$, let $p$ be the support of $\varphi$ and $q = pp^t$, where $p^t = JpJ$.

Since the induction $pMp \to qMq$ is an isomorphism, there exists a functional $\psi \in (qMq)_*$ such that

$$\varphi(x) = \psi(qxq) \quad \forall x \in M.$$  

Since $qMq$ is $\sigma$-finite and $(qMq,qH,qJq,qP)$ is a standard form, there exists $\xi \in q(P) \subseteq P$ so that $\psi(y) = (y\xi | \xi) \quad \forall y \in qMq$. Hence

$$\varphi(x) = (x\xi | \xi), \quad x \in M.$$  

The uniqueness of $\xi$ follows when the inequality (2) is proved.

(2): The inequality follows from the $\sigma$-finite case by regarding the reduced standard form $(qMq,qH,qJq,qP)$ corresponding to $q = pp^t$ where $p = e(\xi)v e(\eta)$.

PROOF OF THEOREM 2.3. Assume that $u_1$ and $u_2$ satisfy the conditions 1)–3).

Let $\xi \in P$. By 3), $u_1\xi \in \bar{P}$ and $u_2\xi \in \bar{P}$. Moreover:

$$(\Phi(a)u_1\xi | u_1\xi) = (a\xi | \xi) = (\Phi(a)u_2\xi | u_2\xi).$$  

Since the map $\eta \to \omega_\eta$ is a bijection of $\bar{P}$ on $\bar{M}_*$ we get $u_1\xi = u_2\xi$. Consequently $u_1 = u_2$, because a selfdual cone is total (by Remark 2.7). To prove the existence we assume first that $M$ is $\sigma$-finite. Then $M$ has a cyclic and separating vector $\xi \in P$. By Lemma 2.9 there exists $\eta \in \bar{P}$ so that

$$\omega_\eta(\Phi(x)) = \omega_\xi(x) \quad \forall x \in M.$$  

$\eta$ is separating for $\bar{M}$ and therefore $J_\eta = \eta$ is cyclic for $\bar{M}$. The equation

$$||\Phi(a)\eta||^2 = \omega_\eta(\Phi(a^*a)) = \omega_\xi(a^*a) = ||a\xi||^2 \quad \forall a \in M.$$  

shows that the map $a\xi \to \Phi(a)\eta$, $a \in M$ can be extended to a unitary $u: H \to \bar{H}$. We claim that $u$ satisfies the conditions 1)–3):

(1) Let $\zeta \in \bar{M}\eta$, $\zeta = \Phi(a)\eta$ for some $a \in M$. Then

$$\Phi(b)\zeta = \Phi(ba)\eta = u(ba\xi) = ubu^{-1}(\Phi(a)\eta) = ubu^{-1}\zeta \quad \forall b \in M.$$  

Hence $\Phi(b) = ubu^{-1}$ because $\bar{M}\eta$ is dense in $\bar{H}$.

(2) Let $S_\xi$ (respectively $S_\eta$) be the closure of the operator $a\xi \to a^*\xi$, $a \in M$ (respectively $b\eta \to b^*\eta$, $b \in \bar{M}$). Then it is easy to check that

$$S_\eta = uS_\xi u^{-1}.$$  

By polar decomposition

$$S_\xi = J_\xi A_\xi^+, \quad S_\eta = J_\eta A_\eta^+.$$
Thus \( J_\eta = uJ_\xi u^{-1} \). But \( J = J_\xi \) and \( J = J_\eta \) (by Lemma 2.9). Hence \( J = uJu^{-1} \).

(3) Clearly
\[
P = P_\eta = \{ aJ_\eta \mid a \in \overline{M} \}
= \{ (uau^{-1})(uJau^{-1})(uau^{-1})a \mid a \in M \}
= u\{ aJa_a \mid a \in M \} = u(P).
\]

In the general case let \( p \) be a \( \sigma \)-finite projection in \( M \). Put \( q = pp^t \) and \( r = \Phi(p)\Phi(p)^t \). Since the inductions
\[
pMp \to qMq \quad \text{and} \quad \Phi(p)\overline{M}\Phi(p) \to r\overline{M}r
\]
are isomorphisms, there is a unique isomorphism \( \Phi_q : qMq \to r\overline{M}r \) so that
\[
\Phi_q(qxq) = r\Phi(x)r, \quad x \in M.
\]

Using the first part of the proof on the reduced standard forms we find that there is a unique isometry \( u_q \) of \( q(H) \) on \( r(H) \) satisfying
\[
(a) \quad r\Phi(x)r = u_q(qxq)u_q \quad \forall x \in M,
(b) \quad r\overline{I}r = u_q(qJ^2q)u_q,
(c) \quad r(\overline{P}) = u_qq(\overline{P}).
\]

This construction can be carried out for any \( \sigma \)-finite projection. If \( p_1 \leq p_2 \) then \( q_1 \leq q_2 \) and \( r_1 \leq r_2 \). The uniqueness of \( u_q \), shows that in this case \( u_{q_1} \leq u_{q_2} \). Choose a net \( (p_i)_{i \in I} \) of \( \sigma \)-finite projections in \( M \) so that \( p_i \to 1 \). Then \( q_i = p_i p_i^t \to 1 \) and \( r_i \to 1 \).

Since \( u_{q_i} \leq u_{q_j} \) when \( p_i \leq p_j \), there exists an isometry \( u \) of \( H \) onto \( \overline{H} \) which extend every \( u_{q_i}, \ i \in I \). Using
\[
H = (\bigcup_{i \in I} q_i(H))^- \quad \overline{H} = (\bigcup_{i \in I} r_i(H))^-,
P = (\bigcup_{i \in I} q_i(P))^- \quad \overline{P} = (\bigcup_{i \in I} r_i(\overline{P}))^-,
\]
we find that \( u \) has the required properties.

**Remark 2.11.** Condition 2) in Theorem 2.3 is not essential. Since \( P \) and \( \overline{P} \) are total, we have 3) \( \Rightarrow \) 2).

3. Unitary implementation of automorphism groups.

**Definition 3.1.** Let \( M \) be a von Neumann algebra on a Hilbert space \( H \), and \( G \) a group of \(*\)automorphisms of \( M \). A unitary implementation of \( G \) is a unitary representation \( g \to u_g \) of \( G \) on \( H \), such that
\[
g(a) = u_g aw_g^* \quad \forall g \in G, \forall a \in M.
\]
As an easy application of Theorem 2.3 we get

**Theorem 3.2** (cf. [1, theorem 11] and [3]). Let \((M, H, J, P)\) be a standard form. The group \(\text{aut}(M)\) of \(*\)-automorphisms of \(M\) has a unique unitary implementation \(g \rightarrow u_g\), such that

\[(*) \quad J = u_g J u_g^{-1} \quad \text{and} \quad u_g(P) = P \quad \text{for any} \; g \in \text{aut}(M).\]

**Proof.** By Theorem 2.3 we get that for each \(g \in \text{aut}(G)\), there is a unique unitary on \(H\) which satisfies (\(*\)). It follows from the uniqueness that

\[u_{g \cdot h} = u_g \cdot u_h \quad \forall g, h \in G.\]

**Definition 3.3.** The map \(g \rightarrow u_g\) in Theorem 3.2 will be called the canonical implementation of \(\text{aut}(M)\).

**Definition 3.4.** Let \(M\) be a von Neumann algebra. On the set of bounded, \(\sigma\)-weak continuous operators on \(M\) we define the \(p\)-topology by the semi-norms

\[T \rightarrow \langle Tx, \varphi \rangle \quad x \in M, \; \varphi \in M_*\]

and the \(u\)-topology by the semi-norms

\[T \rightarrow \| T_* \varphi \|, \quad \varphi \in M_*\]

where \(T_* : \varphi \rightarrow \varphi \circ T\) is the transposed action on the predual.

**Proposition 3.5.** Let \((M, H, J, P)\) be a standard form. The canonical implementation \(g \rightarrow u_g\) of \(\text{aut}(M)\) is a homeomorphism of \(\text{aut}(M)\) onto a closed subgroup of the unitary group on \(H\), when the first is equipped with \(u\)-topology and the latter with strong (=weak) operator topology.

**Proof.** Since the map \(\xi \rightarrow \omega_\xi\) is a homeomorphism of \(P\) onto \(M_*\) we find by repeating the arguments of [1, Remark following Theorem 11] that the map \(g \rightarrow u_g\) is a homeomorphism on its range. Since \(\{u_g \mid g \in \text{aut}(M)\}\) is equal to the set of unitaries for which

\[uMu^* = M, \quad uJu^* = J, \quad u(P) = P,\]

the set is strongly closed relative to the unitary group.

**Corollary 3.6.** Let \((M, H, J, P)\) be a standard form, \(G\) a locally compact group and \(\alpha : G \rightarrow \text{aut}(M)\) a \(\sigma\)-weakly continuous representation of \(G\) on \(M\). Then the canonical unitary implementation \(g \rightarrow u_{\alpha(g)}\) of \(G\) is strongly continuous.
Proof. Since $g \to \langle \alpha(g)x, \varphi \rangle$ is continuous for $x \in M$, $\varphi \in M_*$ the action of $G$ on the predual $g \to \alpha(g)_*$ is $\sigma(M_*, M)$-continuous. Hence by [6, p. 23] the action is also strongly continuous, i.e.

$$
\|\alpha(g)_* \varphi - \alpha(g)_* \varphi\| \to 0 \quad \text{for any } \varphi \in M_* .
$$

Remark. Theorem 3.2 and Corollary 3.6 are generalizations of theorem 6.10 and proposition 6.11 of [7].

Proposition 3.7. Let $\varphi$ be a normal, faithful, semifinite weight on a von Neumann algebra $M$. The $p$-topology and the $u$-topology coincide on the group $\text{aut}_\varphi(M)$ of *automorphisms on $M$, which leaves $\varphi$ invariant.

Proof. Let $(\pi_\varphi, H_\varphi)$ be the representation of $M$ induced by $\varphi$. $H_\varphi$ is obtained as completion of the pre-Hilbert-space

$$
n_\varphi = \{x \in M \mid \varphi(x^*x) < \infty\} .
$$

We let $\Lambda_\varphi$ denote the injection of $n_\varphi$ in $H_\varphi$. The set $\mathcal{A} = \Lambda_\varphi(n_\varphi \cap n_\varphi^*)$ is an achieved left Hilbert algebra (cf. [2]). Let $(M, H, J, P)$ be the standard form associated with this left Hilbert algebra as in section 1. (We identify $M$ and $\pi_\varphi(M)$.) For any $g \in \text{aut}_\varphi(M)$ the map $\Lambda_\varphi(x) \to \Lambda_\varphi(g(x))$ can be extended to a unitary $u_g$ on $H$. It is easily seen that $u_g$ implements the automorphism $u_g$. We will prove that $g \to u_g$ is the canonical implementation. For $x \in n_\varphi \cap n_\varphi^*$ we have

$$
u_g S \Lambda_\varphi(x) = u_g \Lambda_\varphi(x^*) = \Lambda_\varphi(g(x^*)) = S \Lambda_\varphi(g(x)) = S u_g \Lambda_\varphi(x)
$$

Since $S$ is the closure of the map $\Lambda_\varphi(x) \to \Lambda_\varphi(x^*)$, $x \in n_\varphi \cap n_\varphi^*$ we get $S = u_g S u_g^*$, and by polar decomposition

$$
J = u_g J u_g^* \quad \text{and} \quad \Lambda^1 = u_g \Lambda^1 u_g^* .
$$

Since $P^\#$ in this setting is the closure of

$$\{\Lambda_\varphi(x^*x) \mid x \in n_\varphi \cap n_\varphi^*\}
$$

it is easily seen that $u_g(P^\#) = P^\#$. Using $P = (\Lambda^1(P^\#))^{-1}$ we get $u_g(P) = P$. Hence $g \to u_g$ is the canonical implementation of $g$. Obviously the $p$-topology is weaker than the $u$-topology on $\text{aut}_\varphi(M)$. To prove the converse let $\xi \in \mathcal{A}$ and $\eta \in (\mathcal{A}')^2$. Now $\eta$ has the form

$$
\eta = \sum_{i=1}^n \eta_i \zeta_i
$$

$\eta_i, \zeta_i \in \mathcal{A}'$.
Thus $\forall g \in \text{aut}_\phi(M)$:

$$
(u_0 \xi | \eta) = \sum_{i=1}^n (u_0 \xi | \eta_i \xi_i^b) = \sum_{i=1}^n (\tau'(\xi_i)u_0 \xi | \eta_i) = \sum_{i=1}^n (\tau(u_0 \xi)\xi_i | \eta_i) = \sum_{i=1}^n (g(\tau(\xi))\xi_i | \eta_i)
$$

Since $\mathcal{A}$ and $(\mathcal{A}')^2$ are dense in $H$ we conclude that if $g_i \to g$ in the $p$-topology on $\text{aut}_\phi(M)$ then $u_{g_i} \to u_g$ weakly. Hence by Proposition 3.5, $g_i \to g$ in the $u$-topology.

**Corollary 3.8.** If $M$ is a factor of type I or of type $\Pi_1$, the $u$-topology and the $p$-topology coincide on $\text{aut}(M)$.

**Proof.** Every automorphism of these factors leaves the trace invariant.

**Remark 3.9.** In general the $p$-topology on $\text{aut}(M)$ is strictly weaker than the $u$-topology. An example is given in [5, corollary 3.15].

**References**


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