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A THEOREM ON RESTRICTED GROUP REPRESENTATIONS

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Abstract.

We discuss the case where all irreducible representations of some finite group G contain those of a subgroup H with multiplicities 0 or 1 only. A sufficient condition is given which applies, in particular, to the symmetric group S_n with S_{n-1} as a subgroup.

1. Statement of the theorem.

We will consider the complex irreducible linear representations of the finite group G and their restrictions to the subgroup H of G. We denote by $A(G)$ the group algebra of G (see [1]), whose elements are the formal linear combinations

$$a = \sum_{g \in G} a_g g \tag{1}$$

with complex coefficients a_g. Thus $A(G)$ is a complex vector space with the group elements of G as a basis, defining, by the group product, also an algebraic structure on $A(G)$. The group algebra $A(H)$ of H is a subalgebra of $A(G)$.

Another subalgebra of $A(G)$ is the commutant $A(H)'$ of $A(H)$, defined as

$$A(H)' = \{a \in A(G) \mid ab = ba \text{ for every } b \in A(H)\}\,. \tag{2}$$

Note that $A(H)'$ is equivalently defined as the commutant of H in $A(G)$.

We want to prove the following.

THEOREM. A) *A necessary and sufficient condition that any irreducible representation of G, when restricted to the subgroup H, will contain any irreducible representation of H with multiplicity either* 0 *or* 1, *is that the commutant $A(H)'$ is commutative.*

B) *A sufficient condition is that any element $g \in G$ is conjugate to its inverse g^{-1} by some element $h \in H$, that is $g^{-1} = hgh^{-1}$.*

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2. Application.

A trivial instance of part A) is that of a commutative group G, in which all irreducible representations, of G as well as of H, are one-dimensional. This same example shows that B) is not a necessary condition, since not always $g^{-1}=g$ in commutative groups.

A less trivial instance of part B) is that of the symmetric group S_n of (all permutations of) n objects, say the numbers $1,2,\ldots,n$, with S_{n-1} as the subgroup in question (see [2]).

Any permutation $p \in S_n$ is the product of disjoint (and commuting) cycles. The usual notation for a cycle of length k is

$$s = (x_1,x_2,\ldots,x_k) \tag{3}$$

telling that $s(x_1)=x_2$, $s(x_2)=x_3,\ldots$, $s(x_k)=x_1$. Obviously

$$s^{-1} = (x_k,\ldots,x_2,x_1)\,. \tag{4}$$

Now when $r \in S_n$,

$$rsr^{-1} = \big(r(x_1),r(x_2),\ldots,r(x_k)\big)\,. \tag{5}$$

From this one deduces that two permutations $p,q \in S_n$ are conjugate,

$$q = rpr^{-1} \quad \text{for some } r \in S_n\,, \tag{6}$$

if and only if they possess the same cycle structure (cycle lengths). Furthermore, $r \in S_{n-1}$ means that $r(n)=n$, and so $p,q \in S_n$ are 'S_{n-1}-conjugate', i.e.

$$q = rpr^{-1} \text{ for some } r \in S_{n-1}\,, \tag{7}$$

if and only if their cycle structures are identical and at the same time the number n belongs to a cycle of the same length in p as in q.

Consequently, p and p^{-1} are always S_{n-1}-conjugate in S_n, and by our theorem the multiplicity of an irreducible representation of S_{n-1} in (the restriction of) an irreducible representation of S_n, is at most one. This fact is of course well-known, but is usually proved by means of the characters (as in [2, chapter 7–5]).

3. Proof of the theorem.

Let $D^{(1)},\ldots,D^{(r)}$ be a complete set of inequivalent irreducible matrix representations of G, and hence of $A(G)$, of dimensions $n_1,\ldots,n_r$. As is well-known [1], $A(G)$ is the direct sum of two-sided ideals $A^{(1)},\ldots,A^{(r)}$, in such a way that $D^{(\mu)}$ is an isomorphism between $A^{(\mu)}$ and the full algebra of $n_\mu \times n_\mu$ matrices. Therefore $A(H)'$ is commutative if and only if its representation by any $D^{(\mu)}$ is commutative.

$C = D^{(\mu)}(A(H)')$, the representation of $A(H)'$, consists of those $n_\mu \times n_\mu$ matrices that commute with every $D^{(\mu)}(h)$, $h \in H$. We can assume that $D^{(\mu)}$ represents every element $h \in H$ by a fully reduced (block-diagonal) matrix

$$D^{(\mu)}(h) = \begin{pmatrix} d^{(1)}(h) & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & d^{(s)}(h) \end{pmatrix} \tag{8}$$

where $d^{(\alpha)}$ is an irreducible matrix representation of H. (If necessary, make a similarity transformation). We can further assume that if $d^{(\alpha)}$ and $d^{(\beta)}$ are equivalent, they are equal. It is then a simple application of Schur's lemma [1] to show that C is isomorphic to the algebra consisting of all complex $s \times s$ matrices

$$\begin{pmatrix} m_{11} & \dots & m_{1s} \\ \vdots & & \vdots \\ m_{s1} & \dots & m_{ss} \end{pmatrix} \tag{9}$$

in which $m_{\alpha\beta} = 0$ whenever $d^{(\alpha)}$ and $d^{(\beta)}$ are inequivalent representations of H.

C is commutative if and only if the matrices (9) are all diagonal, and so the proof of part A) is complete.

Part B) follows from A) because the condition stated implies that the commutant $A(H)'$ is commutative. This is seen by introducing an explicit basis for $A(H)'$.

Assume that $a \in A(G)$ commutes with all $h \in H$ and write a as in equation (1). Then, with $|H|$ being the order of H, we have

$$a = |H|^{-1}\textstyle\sum_{h\in H} hah^{-1} = \sum_{g\in G} a_g |H|^{-1} \sum_{h \in H} hgh^{-1} \tag{10}$$

so that a is a linear combination of elements

$$K(g) = \textstyle\sum_{h\in H} hgh^{-1} \tag{11}$$

in $A(H)'$. The different $K(g)$'s are linearly independent and form a basis for $A(H)'$. The product of two basis elements is

$$\begin{aligned} K(g_1)K(g_2) &= \textstyle\sum_{h_1, h_2 \in H} h_1 g_1 h_1^{-1} h_2 g_2 h_2^{-1} = \sum_{h\in H} K(g_1 h g_2 h^{-1}) \\ &= \textstyle\sum_{h \in H} K(h g_1 h^{-1} g_2) \end{aligned} \tag{12}$$

(define either $h = h_1^{-1}h_2$ or $h = h_2^{-1}h_1$) .

The condition stated in part B) of the theorem is simply that $K(g^{-1}) = K(g)$ for every $g \in G$. Under this condition $A(H)'$ is commutative, because

$$
\begin{aligned}
(13) \qquad K(g_2)K(g_1) &= K(g_2^{-1})K(g_1^{-1}) = \textstyle\sum_{h\in H} K(g_2^{-1}hg_1^{-1}h^{-1}) \\
&= \textstyle\sum_{h\in H} K\big((g_2^{-1}hg_1^{-1}h^{-1})^{-1}\big) = \textstyle\sum_{h\in H} K(hg_1h^{-1}g_2) \\
&= K(g_1)K(g_2)\,.
\end{aligned}
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