MATH. SCAND. 126 (2020), 256–258

EXTENSION OF POSITIVE MAPS

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Abstract

We prove an extension theorem for positive maps from operator systems into matrix algebras

1. Introduction

In the theory of positive maps of operator algebras Arveson's Extension Theorem for completely positive maps [1] plays a major role. In the paper [5] the author extended this result to maps with more general positivity properties, the main theorem being included in Theorem 5.2.3 in the book [6]. However, it was pointed out by D. Chruscinski to the author that V. Paulsen had a counterexample for general positive maps to this theorem in his book [3, Example 2.2], see also [2].

In the present note we show a corrected version of the above Theorem 5.2.3. The theorem is very close to the original one except that we restrict the operator system to consist only of self-adjoint matrices. This is due to the fact that Krein's Extension Theorem [6, Theorem A.3.1] is formulated for real spaces, while in Theorem 5.2.3 we wrongly applied it to complex spaces.

Our basic reference for these notes is the book [6]. We recall some concepts which we shall use. An operator system is a complex self-adjoint linear subspace A of operators in $B(H)$ such that $1 \in A$. A mapping cone $\mathscr{C}$ on H is a closed convex subcone of the cone of positive maps of $B(H)$ into itself such that $\phi \in \mathscr{C}$ implies $\alpha \circ \phi \circ \beta \in \mathscr{C}$ for all completely positive maps α and β of $B(H)$ into itself. For simplicity we assume H and K are finite-dimensional Hilbert spaces, and we let Tr denote usual trace on $B(K)$ or on $A \otimes B(K)$ if there is no confusion. Let ϕ be a linear map from A into $B(K)$. Then the dual functional $\tilde{\phi}$ on $B(H) \otimes B(K)$ is defined by the formula

$$\tilde{\phi}(a \otimes b) = \mathrm{Tr}(\phi(a)b^t),$$

where b^t is the transpose of b. This is equal to $\mathrm{Tr}(C_\phi^t(a \otimes b))$ when $A = B(H)$, and C_ϕ is the Choi matrix for ϕ, see [6, Definition 4.1.1]. If $\mathscr{C}$ is a mapping

Received 6 June 2018, in final form 16 March 2020. Accepted 19 March 2020.
DOI: https://doi.org/10.7146/math.scand.a-115974

cone on K as above, we denote by $P(A, \mathscr{C})$ the cone

$$P(A, \mathscr{C}) = \{ x \in (A \otimes B(K))_{\mathrm{sa}} : \iota \otimes \alpha(x) \geq 0, \ \forall \alpha \in \mathscr{C} \},$$

where ι is the identity map on $B(H)$. We say ϕ is $\mathscr{C}$-positive if $\tilde{\phi}$ is positive on $P(A, \mathscr{C})$.

2. Main results

As mentioned above our extension theorem is for the real version of operator systems. If A is a real linear subspace of the self-adjoint operators in $B(H)$ containing the identity operator 1, we say A is a real operator system. The definition of $\mathscr{C}$-positive maps still make sense for real operator systems. In the following H and K are still finite dimensional Hilbert spaces.

THEOREM 1. *Let A be a real operator system contained in $B(H)$. Let ϕ be a $\mathscr{C}$ positive map of A into $B(K)$ for a mapping cone $\mathscr{C}$. Then there exists a $\mathscr{C}$ positive map ψ of $B(H)$ into $B(K)$ such that $\psi(a) = \phi(a)$ for $a \in A$.*

PROOF. Let $P = P(B(H), \mathscr{C})$ be defined as above. Then $P(A, \mathscr{C}) = P \cap (A \otimes B(K))_{\mathrm{sa}}$. Since ϕ is $\mathscr{C}$-positive its dual functional $\tilde{\phi}$ is positive on $P \cap (A \otimes B(K))_{\mathrm{sa}}$. By [6, Lemma 5.2.1], $1 \otimes 1$ is an interior point of P. Thus by Krein's Extension Theorem [6, Theorem A.3.1], $\tilde{\phi}$ has an extension to a real linear functional $\tilde{\psi}_o$ on $(B(H) \otimes B(K))_{\mathrm{sa}}$, which is positive on P. Define a complex linear functional $\tilde{\psi}$ on $B(H) \otimes B(K)$ by

$$\tilde{\psi}(a + ib) = \tilde{\psi}_o(a) + i\tilde{\psi}_o(b), \quad \forall a, b \in (B(H) \otimes B(K))_{\mathrm{sa}}.$$

A straightforward computation shows that

$$\tilde{\psi}(\lambda x) = \lambda \tilde{\psi}(x), \quad \forall x \in B(H) \otimes B(K), \text{ and } \forall \lambda.$$

Thus $\tilde{\psi}$ is a complex linear functional on $B(H) \otimes B(K)$ which is positive on P. But then there exists an operator $C \in B(H) \otimes B(K)$ such that

$$\tilde{\psi}(x) = \mathrm{Tr}(Cx), \quad \forall x \in B(H) \otimes B(K).$$

By [6, Lemmas 4.2.2 and 4.2.3], there exists a linear map ψ of $B(H)$ into $B(K)$ such that $C = C_\psi$ is the Choi matrix for ψ. Then

$$\mathrm{Tr}(C_\psi x) \geq 0, \quad \forall x \in P.$$

Thus ψ is $\mathscr{C}$-positive, and if $a \in A, b \in B(K)$ then

$$\mathrm{Tr}(\psi(a) b^t) = \mathrm{Tr}(C_\psi a \otimes b) = \tilde{\psi}(a \otimes b) = \tilde{\phi}(a \otimes b) = \mathrm{Tr}(\phi(a) b^t).$$

Since this holds for all $b \in B(K)$, $\psi(a) = \phi(a)$ for all $a \in A$, completing the proof.

We need H and K finite dimensional in order to choose the operator C in the proof. The theorem can be extended to the case when K is infinite dimensional by the same proof as that of [6, Theorem 5.2.3].

COROLLARY 2. *Let B be a C^*-subalgebra of $B(H)$ and ϕ a $\mathscr{C}$-positive map of B into $B(K)$. Then ϕ has a $\mathscr{C}$-positive extension ψ of $B(H)$ into $B(K)$.*

PROOF. The corollary follows by applying Theorem 1 to $A = B_{\mathrm{sa}}$.

ACKNOWLEDGEMENTS. The author wants to thank D. Chruscinski for useful correspondence on the subject and E. Alfsen for very useful discussions on the proofs of the theorem in the paper. I am also indebted to Ken Davidson and Vern Paulsen for pointing out a mistake in an early version of this paper, see [3].

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