BIHARMONIC GREEN’S FUNCTIONS AND
BIHARMONIC DEGENERACY

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It is well known that the harmonic Green’s function plays an important role in the harmonic classification theory (e.g. Sario–Nakai [7]). Explicitly, if we denote by $O^N_{H^X}$, $X = G, B, D, C$, the classes of Riemannian $N$-manifolds which do not carry harmonic Green’s functions, or bounded, Dirichlet finite, or bounded Dirichlet finite nonconstant harmonic functions, respectively, then we have the strict inclusion relations

$$O^N_{H^G} < O^N_{H^B} < O^N_{H^D} = O^N_{H^C}$$

for every dimension $N \geq 2$. Recently L. Sario [4] introduced the biharmonic Green’s function $\gamma$ which, roughly speaking, satisfies $\gamma = \Delta \gamma = 0$ on the ideal boundary of a Riemannian manifold. In the present paper, we shall discuss the role played by $\gamma$ in the biharmonic classification theory. It turns out that, in striking contrast with the harmonic case, the class $O^N_\Gamma$ of Riemannian $N$-manifolds which do not carry $\gamma$ neither is contained in nor contains any of the classes $O^N_{H^\Omega^X}$, $X = B, D, C$, of Riemannian $N$-manifolds which carry no bounded, Dirichlet finite, or bounded Dirichlet finite nonharmonic biharmonic functions, respectively.

1. On a Riemannian $N$-manifold $R$, take a regular subregion $\Omega$ of $R$. Let $\gamma_\Omega(x, y)$ be the biharmonic Green’s function on $\Omega$ with the biharmonic fundamental singularity at $y \in \Omega$, and with boundary data $\gamma_\partial = \Delta \gamma_\partial = 0$ on $\partial \Omega$, where $\Delta = d \delta + \delta d$ is the Laplace–Beltrami operator. Clearly

$$\gamma_\Omega(x, y) = \int_\Omega g_\Omega(x, z) g_\Omega(z, y) \, dz ,$$

where $g_\Omega(x, z)$ is the harmonic Green’s function on $\Omega$ with pole $z$, and $dz$ the volume element at $z$. The biharmonic Green’s function $\gamma$, if it exists, on $R$ is

$$\gamma(x, y) = \lim_{\Omega \to R} \gamma_\Omega(x, y) = \int_R g_R(x, z) g_R(z, y) \, dz .$$

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Take a regular subregion $R_0$ of $R$ and denote by $\omega = \omega(x, R_0)$ the harmonic measure on $R - R_0$, with $\omega|\partial R_0 = 1$. It is known that every parabolic $R$ belongs to $O_T^N$, whereas a hyperbolic $R$ belongs to $O_T^N$ if and only if $\omega \notin L^2(R - R_0)$ (Sario [4], [6]). This criterion is our main tool in testing the existence of $\gamma$.

2. We shall establish the following complete result:

**Theorem.** The classes

$$\bar{O}_T^N \cap \bar{O}_{H^2X}^N, \quad O_T^N \cap \bar{O}_{H^2X}^N, \quad \bar{O}_T^N \cap O_{H^2X}^N, \quad O_T^N \cap O_{H^2X}^N$$

are all nonvoid for $X = B, D, C$, and $N \geq 2$.

Trivially, the Euclidean unit $N$-ball is in $\bar{O}_T^N \cap \bar{O}_{H^2X}^N$, $X = B, D, C$. It remains to show that $O_T^N \cap \bar{O}_{H^2C}^N, \bar{O}_T^N \cap O_{H^2D}^N, \bar{O}_T^N \cap O_{H^2B}^N, O_T^N \cap O_{H^2B}^N$, and $O_T^N \cap O_{H^2D}^N$ are not empty. The proof will be given in Sections 3–11.

3. To show that $O_T^N \cap \bar{O}_{H^2C}^N \neq \emptyset$, consider the $N$-cylinder

$$T : \{(x, y_1, \ldots, y_{N-1}) \mid |x| < \infty, |y_i| \leq 1\}$$

with each pair of opposite faces $y_i = 1, y_i = -1, i = 1, \ldots, N - 1$, identified by a parallel translation orthogonal to the $x$-axis. Endow $T$ with the metric

$$ds^2 = e^{-x^2}dx^2 + e^{-x^2/(N-1)} \sum_{i=1}^{N-1} dy_i^2.$$  

In view of $\Delta h(x) = -e^{x^2}h''$, a harmonic function of $x$ must be of the form $h(x) = ax + b$ for some constants $a, b$. Let $R_0 = \{|x| < 1\} \cap T$ and, for $x_0 > 1$, $\Omega = \{|x| < x_0\} \cap T$. The harmonic measure $\omega_\Omega(x, R_0)$ on $\bar{\Omega} - R_0$ is $(1-x_0)^{-1}(|x| - x_0)$. Letting $\Omega$ exhaust $R$, we obtain the harmonic measure on $R - R_0$, $\omega(x, R_0) = \lim_{\Omega \to R} \omega_\Omega(x, R_0) \equiv 1$. Therefore $T$ is parabolic and consequently $T \in O_T^N$.

To show that $T \in \bar{O}_{H^2C}^N$, we first note that

$$u(x) = \int_0^\infty e^{-t} dt$$

is a nonharmonic biharmonic function on $T$; in fact, $\Delta u(x) = 2x$. Clearly $u$ is bounded and its Dirichlet integral

$$D(u) = c \int_{-\infty}^{\infty} (u')^2 dx = c \int_{-\infty}^{\infty} e^{-2x^2} dx$$

is finite. Therefore $T \in \bar{O}_{H^2C}^N$. 
4. Next we shall show that \( \partial_N^N \cap O_{H^2D} \neq \emptyset \). Consider the \( N \)-ball

\[
S = \{ x \mid |x| = r < 1, \ x = (x^1, \ldots, x^N) \}
\]

with the Poincaré metric \( ds = (1 - r^2)^{-1} |dx| \), where \( |dx| \) is the Euclidean metric. If \( h(x) \) is a harmonic function, then

\[
\Delta h(x) = -(1 - r^2)^N r^{-N+1} [1 - r^{-2} - (N-2) r^{-1} h'] = 0,
\]

and consequently \( h(x) \sim a(1 - r)^{N-1} + b \) as \( r \to 1 \). Hence the harmonic measure \( \omega(x, R_0) \) of \( R_0 = \{ x \mid r < \frac{1}{2} \} \) is \( \sim (1 - r)^{N-1} \) and

\[
\|\omega\|^2 \{ \frac{1}{2} < r < 1 \} \approx c \int_{\frac{1}{2}}^1 (1 - r)^{N-2} dr < \infty.
\]

By Sario’s test for the existence of the biharmonic Green’s function \( \gamma \), we conclude that \( S \in \partial_N^N \).

5. Suppose we have a \( u \in H^2D \) on \( S \). Then

\[
|\langle \Delta u, \varphi \rangle| = |\langle du, d\varphi \rangle| \leq \sqrt{D(u)} \sqrt{D(\varphi)} = K \sqrt{D(\varphi)}
\]

for all \( \varphi \in C^\infty_0 \). We shall show that \( S \in O_{H^2D}^N \) by constructing a family of \( C^\infty_0 \)-functions \( \varphi_t, \ 0 < t \leq 1 \), on \( S \) such that \( |\langle \Delta u, \varphi_t \rangle| / \sqrt{D(\varphi_t)} \) is not bounded.

Let \( (r, \theta) = (r, \theta^1, \ldots, \theta^{N-1}) \) be the Euclidean polar coordinates on \( S \), and \( S_n(\theta) = \sum \alpha_n S_n(\theta) \) spherical harmonics of degree \( n \), that is, \( r^n S_n(\theta) \) is harmonic with respect to the Euclidean metric. Take a \( u \in H^2 \). Since \( \Delta u \in H, \Delta u \) has a representation

\[
\Delta u(r, \theta) = \sum_{n=0}^\infty f_n(r) S_n(\theta),
\]

which converges absolutely and uniformly on compacts of \( S \), with \( f_n(r) S_n(\theta) \in H(S) \) for \( n = 0, 1, 2, \ldots \). Suppose \( f_n \not\equiv 0 \) for some \( n \geq 0 \). Choose for our testing functions \( \varphi_t, \ 0 < t \leq 1 \),

\[
\varphi_t(r, \theta) = \varphi_t(r) S_n(\theta), \quad \varphi_t(r) = g((1 - r)/t),
\]

where \( g \) is a fixed nonnegative \( C^\infty_0 \)-function with \( \text{supp} g \subset (\beta, \gamma), \ 0 < \beta < \gamma < 1 \). Clearly

\[
|\langle \Delta u, \varphi_t \rangle| = \text{const} |\int_{\frac{1}{2} - \gamma t}^{\frac{1}{2} - \beta t} f_n(r) \varphi_t(r) r^{-N-1} (1 - r^2)^{-N} dr|
\]

\[
> \text{const} (1 - \gamma)^{N-1} (\gamma t)^{-N} |\int_{\frac{1}{2} - \gamma t}^{\frac{1}{2} - \beta t} f_n(r) \varphi_t(r) dr| .
\]

Since \( f_n(r) S_n(\theta) \) is harmonic and \( \not\equiv 0 \), we have \( f_n(r) \not\equiv 0 \) for all \( r \), and \( \lim_{r \to 0} f_n(r) \not\equiv 0 \), where the limit exists in view of the monotonicity of \( f_n \), entailed by the maximum principle. For \( t \) sufficiently small, we obtain
\[ |(\Delta u, \varphi_t)| > \text{const} t^{-N} \int_{1-r^2}^{1-r^t} \xi_t(r) \, dr \]
\[ = \text{const} t^{-N} \int_{\beta}^{1-r^t} \xi_g(s) \, ds \]
\[ = \text{const} t^{-N+1}. \]

On the other hand, the Dirichlet integral of \( \varphi_t \) is

\[
D(\varphi_t) = \int_S |\nabla \varphi_t|^2 \, dV
\]
\[ = \int_{1-r^t}^{1-r^2} (1-r^2)^2 (c_1 \xi_t(r)^2 + c_2 r^{-2} \xi_t(r^2))^r \left(1-r^2\right)^{-N} \, dr
\]
\[ < t^{-N+2} \left( d_1 \int_{1-r^t}^{1-r^2} \xi_t(r)^2 \, dr + d_2 \int_{1-r^t}^{1-r^2} \xi_t(r^2)^2 \, dr \right)
\]
\[ = t^{-N+2} \left( d_1 t^{-1} \int_{\beta}^{1-r^t} \xi_g(s)^2 \, ds + d_2 t^{-1} \int_{\beta}^{1-r^t} \xi_g(s)^2 \, ds \right)
\]
\[ = e_1 t^{-N+1} + e_2 t^{-N+3} < e t^{-N+1}. \]

Hence for \( t \) sufficiently small, the ratio

\[
\frac{|(\Delta u, \varphi_t)|}{\sqrt{D(\varphi_t)}} > \text{const} \frac{t^{-N+1}}{t^{-N+1}/2}
\]

is not bounded, and we have \( f_n \equiv 0 \) for every \( n \). A fortiori \( \Delta u = 0 \), and \( S \in O_{H^2}^N \).

6. To show that \( \bar{U}_R \cap O_{H^2}^N \neq \emptyset \), consider the \( N \)-ball \( B_\varepsilon = \{ r < 1 \} \) with the metric

\[
ds = (1-r^2)^{-1-\varepsilon} |dx|,
\]

where \( \varepsilon > 0 \). In the same manner as in Section 4, we see that the harmonic measure \( \omega \) of \( \{ x \mid r < \frac{1}{2} \} \) is \( (1-r)^{(N-2)(1+\varepsilon)+1} \) as \( r \to 1 \), and

\[
\| \omega \|_2^2 |\{ \frac{1}{2} < r < 1 \} \approx \int_{\frac{1}{2}}^1 (1-r)^{2(N-2)(1+\varepsilon)+2} (1-r)^{-N(1+\varepsilon)} \, dr < \infty.
\]

Thus \( B_\varepsilon \in \bar{U}_R \).

Suppose there exists a \( u \in H^2 \). Then \( |(\Delta u, \varphi)| = |(u, \Delta \varphi)| \leq K(1, |\Delta \varphi|) \), with \( K = \sup_{B_\varepsilon} |u| \), for every \( C_0^\infty \)-function \( \varphi \). Again we have \( \Delta u = \sum_{n=0}^\infty f_n(r) S_n(\theta) \). Suppose \( f_n \equiv 0 \) for some \( n \geq 0 \). Choose testing functions \( \varphi_t = \xi_t(r) S_n(\theta), \ 0 < t \leq 1 \), as in Section 5. For \( t \) sufficiently small, we have

\[
|(\Delta u, \varphi_t)| > \text{const} t^{-N(1+\varepsilon)} \int_{1-r^t}^{1-r^2} \xi_t(r) \, dr > \text{const} t^{-N(1+\varepsilon)+1}.
\]

On the other hand,

\[
\Delta \varphi_t = -(1-r^2)^{2(1+\varepsilon)} \left[ \xi_t'' + \left( \frac{N-1}{r} + \frac{2(N-2)(1+\varepsilon)r}{1-r^2} \right) \xi_t'
\]
\[ - n(n+N-2)r^{-2} \xi_t \right] S_n.
\]
It follows that, for $t$ sufficiently small,
\[
(1, |\Delta \varphi|) < t^{-(N-2)(1+\delta)} \int_1^{t^{-\bar{\gamma}}} (c_1 \partial_t'' + c_2 t^{-1} \partial_t' + c_3 \partial_t) \, dr < \text{const} t^{-(N-2)(1+\delta) - 1}.
\]

Therefore, $|(\Delta u, \varphi)|/(1, |\Delta \varphi|) > \text{const} t^{-2\delta}$, which is unbounded. This contradiction shows that every $f_n \equiv 0$, hence $\Delta u = 0$, and $B_\varepsilon \in O_{H^2B}^N$.

7. To show that $O_{r}^N \cap O_{H^2B}^N \cap O_{H^2D}^N = \emptyset$, consider the $N$-space

\[ E = \{0 < r < \infty, r = |x|, x = (x^1, \ldots, x^N)\}
\]

with the metric

\[ ds = r^{-1} |dx|, \]

where $|dx|$ is the Euclidean metric. A harmonic function of $r$, is of the form $a \log r + b$ for some constants $a, b$. In particular, the harmonic measure on the region bounded by $r = 1$ and $r = r_0 > 1$, is

\[ \omega_{r_0} = 1 - (\log r_0)^{-1} \log r. \]

As $r_0 \to \infty$ or $r_0 \to 0$, $\omega_{r_0} \to 1$, and therefore $E \in O_r^N$.

8. For each harmonic function $h$ on $E$, we have the expansion of $h$

\[ h(r, \theta) = \sum_{n=0}^\infty f_n(r) S_n(\theta), \]

where $(r, \theta) = (r, \theta^1, \ldots, \theta^{N-1})$, and $f_n S_n \in H$ for every $n$. By a straightforward computation of $\Delta (f_n S_n) = 0$, we find that $f_0(r) = a \log r + b$ and, for $n > 0$,

\[ f_n(r) = a_n r^{p_n} + b_n r^{q_n}, \quad p_n, q_n = \pm \sqrt{n(n+N-2)}, \]

with $a, b, a_n, b_n$ constants. Thus $h$ has the expansion

\[ h(r, \theta) = \sum_{n=1}^\infty (a_n r^{p_n} + b_n r^{q_n}) S_n(\theta) + a \log r + b. \]

9. Next we expand a biharmonic function $u$ on $E$. First we observe that $s(r) = -\frac{1}{2} (\log r)^2$ and $\tau(r) = -\frac{4}{3} (\log r)^3$ are solutions of $\Delta s(r) = 1$ and $\Delta \tau(r) = \log r$. We also note that

\[ u_n = -(2p_n)^{-1} r^{p_n} \log r \cdot S_n, \quad v_n = (2p_n)^{-1} r^{q_n} \log r \cdot S_n\]

satisfy the equations $\Delta u_n = r^{p_n} S_n$ and $\Delta v_n = r^{q_n} S_n$. Since $\Delta u \in H$,

\[ \Delta u = \sum_{n=1}^\infty (a_n r^{p_n} + b_n r^{q_n}) S_n + a \log r + b. \]
Set
\[ u_0 = \sum_{n=1}^{\infty} (a_n u_n + b_n v_n) + a \tau(r) + b s(r). \]
Clearly \( \Delta(u - u_0) = 0 \). Thus \( u - u_0 \in H \) and
\[ u = u_0 + \sum_{n=1}^{\infty} (c_n r^{p_n} + d_n r^{q_n})S_n + c \log r + d \]
for some constants \( c_n, d_n, c, d \).

10. To show that \( E \in O_{H^2B}^N \), suppose there exists a \( u \in H^2B \). We make use of the inequality \(|(u, \varphi)| \leq \sup |u|(1, |\varphi|)\) for all \( \varphi \in L^1 \). In the expansion of \( u \), if \( a_n \neq 0 \) for some \( n \), let \( \varphi_1 = \varphi_1(r)S_n \), where \( \varphi_1 \) is a fixed continuous function, \( \varphi_1 \geq 0 \), \( \text{supp} \varphi_1 \subset (1, 2) \), and \( \varphi_t(r) = \varphi_1(r + 1 - t) \) for \( t \geq 1 \). By the orthogonality of \( \{S_n\} \), and \( \int_1^t \varphi_t(r) dr = \text{const} \) as \( t \to \infty \), we have
\[ (u, \varphi_t) \sim C \int_t^{t+1} r^{p_n-1} \log r \cdot \varphi_t(r) dr \sim C t^{p_n-1} \log t \]
and
\[ (1, |\varphi_t|) \sim C t^{-1}. \]
Therefore \( a_n = 0 \) for \( p_n - 1 \geq -1 \), that is, for all \( n \). That \( c_n = 0 \) for all \( n \) is concluded in the same manner.

If \( b_n \neq 0 \) for some \( n \), then we choose \( \varphi_t(r) = \varphi_1(r/t) \), with \( \varphi_1 \) and \( \varphi_t \) as above, and have
\[ (u, \varphi_t) \sim C t^{q_n} \log t, \quad (1, |\varphi_t|) \sim C, \]
as \( t \to 0 \). Clearly all \( n \) with \( q_n \leq 0 \) are ruled out, and we have \( b_n = 0 \) for all \( n \). Similarly all \( d_n = 0 \).

Thus the function \( u \) reduces to \( a \tau(r) + b s(r) + c \log r + d \). Since \( \tau, s, \log r \) are linearly independent and unbounded, we have \( a = b = c = 0 \), and \( u \) is a constant.

11. To show that \( E \in O_{H^2D}^N \), suppose there exists a \( u \in H^2D \). In its expansion, let \( u_n \) now signify the sum of the terms involving an \( S_n \), and denote by \( u_0 \) the radial part of the expansion of \( u \). Then
\[ u = \sum_{n=0}^{\infty} u_n. \]
By the Dirichlet orthogonality of the \( S_n \), we have \( D(u) \geq D(u_n) \) for every \( n \). A direct computation shows that \( D(u_n) = \infty \) for every nonconstant \( u_n \). Thus \( E \in O_{H^2D}^N \).

The proof of our theorem in Section 2 is herewith complete.
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