ON MULTIVALENT FUNCTIONS OF LARGE GROWTH IN TWO DIRECTIONS

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1.

Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an arreally mean \( p \)-valent function in \( |z| < 1 \) normalized so that \( \max_{0 \leq n \leq [p]} |a_n| = 1 \). If \( z_1 = \varrho_1 e^{i\theta_1} \), \( z_2 = \varrho_2 e^{i\theta_2} \) are two distinct points in \( |z| < 1 \) and \( |f(z_1)| \geq |f(z_2)| \) then Lucas [3] has proved that

\[
|f(z_1)|^{1/2p}|f(z_2)|^{(p^2+2\gamma)/2p} < A(p, \gamma)(1-\varrho_1)^{-1}(1-\varrho_2)^{-\gamma^2}|z_1-z_2|^{-2\gamma},
\]

where \( \gamma \) is positive and \( A(p, \gamma) \) is a positive constant depending only on \( p, \gamma \). (Pommerenke [4] established an analogous result for \( k \) points and univalent \( f \).) In [3] it was observed that (1) contains all inequalities of the type

\[
|f(z_1)|^a|f(z_2)|^b < A(p, a, b, c, d, e)(1-\varrho_1)^{-c}(1-\varrho_2)^{-d}|z_1-z_2|^{-e},
\]

which hold subject to \( |f(z_1)| \geq |f(z_2)| \geq 1 \) and \( |z_1-z_2| \geq \frac{1}{2} \max(1-\varrho_1, 1-\varrho_2) \).

It is possible to prove that (1) remains sharp (for appropriate choice of \( \gamma \)) even under the additional restriction that \( |z_1| = |z_2| \) (unpublished).

We shall assume that \( |z_1| = |z_2| = \varrho_1 \) and that \( 0 < \delta < |\theta_1 - \theta_2| < 2\pi - \delta \). If we assume further that

\[
|f(z_2)| > A(1-\varrho_1)^{-p\beta}
\]

for some \( \beta \in (0, 1] \), then, taking \( \gamma = \beta/(2 - \beta) \) in (1) we find

\[
|f(z_1)| < A(p, \delta, \beta)(1-\varrho_1)^{-p\alpha(\beta)}, \quad \alpha(\beta) = (4 - 2\beta - \beta^2)/(2 - \beta).
\]

Note that \( \alpha(1) = 1 \) and \( \alpha(\beta) \uparrow 2 \) as \( \beta \downarrow 0 \) as expected.

In the same way, the assumption that

\[
|f(z_1)| > A(1-\varrho_1)^{-p\alpha(\beta)}
\]

leads to

\[
|f(z_2)| < A(p, \delta, \beta)(1-\varrho_1)^{-p\beta},
\]

a remark we shall need in Section 4.

Received December 20, 1974.

1 Throughout \( A \) will denote some positive absolute constant not necessarily the same at each occurrence.
2.

Given \( \beta \in (0, 1) \), it is natural for a fixed positive \( p \) to introduce a class \( C(\beta) \) of areally mean \( p \)-valent functions \( f \) in the unit disk for which it is possible to find sequences \( \{r_n\}, \{\theta_{n1}\}, \{\theta_{n2}\} \) with

\[
    r_n \uparrow 1 \quad (n \to \infty), \quad \delta < |\theta_{n1} - \theta_{n2}| < 2\pi - \delta \quad \text{(all } n) \]

and having the properties

\[
    (2) \quad |f(r_n e^{i\theta_{n1}})| > A(1 - r_n)^{-p\beta}, \quad |f(r_n e^{i\theta_{n2}})| > A(1 - r_n)^{-p\beta} \quad \text{(all } n) .
\]

In [1] this was done for \( \beta = 1 \) and certain smoothness criteria obtained for the growth of \( f, f' \) and the Taylor coefficients of \( f \). We shall prove

**Theorem.** \( C(\beta) \) is empty if \( 0 < \beta < 1 \).

The reason for this situation is something like this. In order that the left hand side and right hand side in (1) should have the same order of magnitude when \( \varphi_1, \varphi_2 \) are near 1 and \( |z_1 - z_2| \geq \delta \), it is necessary that the area of the image of \( w = f(z) \) lying in an annulus \( R < |w| < CR \), where \( C \) is a large constant, arises roughly in a fixed proportion from points near \( z_1 \) and points near \( z_2 \), if \( R \) is large, and \( |f(z_1)| > CR \). On the other hand, if \( |f(z_2)| < R < CR < |f(z_1)| \) the corresponding area must arise almost entirely from points near \( z_1 \). This leads to a contradiction if there is a second such pair \( z_1', z_2' \) with

\[
    |f(z_1')| > C|f(z_2')| ,
\]

and if

\[
    |f(z_2')| > C|f(z_1)| .
\]

These conditions are satisfied in the class \( C(\beta) \) if \( \beta < 1 \). If \( \beta = 1 \), we may have \( |f(z_1)| = |f(z_2)| \) so that the contradiction fails. We now proceed to give details of the proof.

3.

We can assume that \( \theta_{n1} \to \varphi_1, \theta_{n2} \to \varphi_2 \) \( (n \to \infty) \) where \( \delta \leq |\varphi_2 - \varphi_1| \leq 2\pi - \delta \). If this is not so we extract appropriate subsequences and re-label. Let \( \Lambda_1, \Lambda_2 \) be disjoint open sectors in \( |z| < 1 \) having the origin as vertex and being symmetric about \( \text{arg} z = \varphi_1, \varphi_2 \) respectively.

If \( n(w) \) is the number of solutions of \( f(z) = w \) in \( |z| < 1 \) counted according to multiplicity, we write

\[
    p(R) = \frac{1}{2\pi} \int_0^{2\pi} n(R e^{i\vartheta}) \, d\vartheta \quad \text{for } R > 0 .
\]
Let \( p_1(R), p_2(R) \) be the analogous functions relating to \( \Lambda_1, \Lambda_2 \) respectively so that

\[
(3) \quad p_1(R) + p_2(R) \leq p(R), \quad (R > 0).
\]

We denote by \( M_k(r) \) \((k = 1, 2)\) the supremum of \(|f(z)|\) for \( |z| = r, \; z \in \Lambda_k \) and consider, in the definition of \( C(\beta) \), only those \( n \) for which \( M_k(r) \) is attained nearer to \( \text{arg} z = \varphi_k \) than to the boundary of \( \Lambda_k \) \((k = 1, 2)\). Then \([2, \text{Theorem 2.4}]\) indicates that

\[
(4) \quad \int_{R_0}^{M_1(r)} \frac{dR}{Rp_1(R)} < 2 \log(1-r)^{-1} + A(\delta, p),
\]

\[
(5) \quad \int_{R_0}^{M_2(r)} \frac{dR}{Rp_2(R)} < 2 \log(1-r)^{-1} + A(\delta, p),
\]

where \( R_0 \) is any suitable fixed number. In fact an intermediate map of \( \Lambda_k \) onto the unit disk is needed to obtain \((4), (5)\) from \([2, \text{Theorem 2.4}]\). Since such a map possesses an angular derivative at \( e^{i\varphi_k} \) the application is legitimate provided we modify the additive constant in \([2, \text{Theorem 2.4}]\) from \( A(p) \) to \( A(p, \delta) \). We also need \([2, \text{Lemma 2.1}]\) which says

\[
(6) \quad \int_{R_0}^{R_2} \frac{dR}{Rp(R)} \geq p^{-1} \log (R_2/R_0) - \frac{1}{2},
\]

for \( R_2 > R_0 > |f(0)| = |a_0| \).

4. Let \( c \) be a real parameter. Schwarz’s inequality and \((3)\) give

\[
\left( \frac{1}{\sqrt{p_1}} \cdot \sqrt{p_1} + \frac{c}{\sqrt{p_2}} \cdot \sqrt{p_2} \right)^2 \leq \left( \frac{1}{p_1} + \frac{c^2}{p_2} \right) (p_1 + p_2) \leq \left( \frac{1}{p_1} + \frac{c^2}{p_2} \right) p.
\]

We integrate

\[
(7) \quad (1+c^2/Rp(R)) \leq 1/Rp_1(R) + c^2/Rp_2(R)
\]

to \( M_2(r_n) \) and use \((5), (6)\) and

\[
\log M_2(r_n) \geq \log |f(r_n e^{i\theta_n})| \geq p\beta \log (1-r_n)^{-1} + A
\]

to obtain

\[
\int_{R_0}^{M_2(r_n)} \frac{dR}{Rp_1(R)} \geq \{(1+c^2\beta - 2c^2) \log (1-r_n)^{-1} + A(\delta, p) \}. 
\]
Taking $c = \beta/(2 - \beta)$, this gives

$$
(8) \quad \int_{R_0}^{M_2(r_n)} \frac{dR}{R p_1(R)} \geq 2\beta/(2 - \beta) \log(1 - r_n)^{-1} + A(\delta, p).
$$

We rewrite (8) as

$$
(8)' \quad \int_{R_0}^{M_2(r_n)} \frac{dR}{R p_1(R)} = 2\beta/(2 - \beta) \log(1 - r_n)^{-1} + K_n,
$$

where $\{K_n\}$ is a sequence with finite infimum. We assume that the supremum is infinite and obtain a contradiction. Extract a subsequence which tends to $+\infty$ and re-label so that this subsequence is itself $\{K_n\}$. From (6),

$$
\int_{M_2(r_n)}^{M_1(r_n)} \frac{dR}{R p_1(R)} \geq \int_{M_2(r_n)}^{M_1(r_n)} \frac{dR}{R p(R)} \geq p^{-1} \log \{M_1(r_n)/M_2(r_n)\} - \frac{1}{4},
$$

and, adding this to (8)', we deduce

$$
(9) \quad p^{-1} \log M_1(r_n)
\leq \int_{R_0}^{M_1(r_n)} \frac{dR}{R p_1(R)} + \frac{1}{2} + p^{-1} \log M_2(r_n) - K_n - 2\beta/(2 - \beta) \log(1 - r_n)^{-1}.
$$

At the end of Section 1, it was remarked that

$$
M_1(r_n) > A(1 - r_n)^{-p_1(\beta)}
$$

implies

$$
M_2(r_n) < A(p, \beta, \delta)(1 - r_n)^{-p_2}.
$$

However, the assumption that

$$
M_1(r_n) > A(1 - r_n)^{-p_1(\beta)},
$$

combined with (4), (9) yields

$$
K_n + (\alpha(\beta) + 2\beta/(2 - \beta) - 2) \log(1 - r_n)^{-1} \leq p^{-1} \log M_2(r_n) + A(\delta, p),
$$

which contains the desired contradiction since $\alpha(\beta) + 2\beta/(2 - \beta) - 2 = \beta$. Consequently the sequence $\{K_n\}$ in (8)’ is bounded. The discussion also shows that

$$
\int_{M_2(r_n)}^{M_1(r_n)} \frac{1/p_1(R) - 1/p(R)}{R} \frac{dR}{R} = O(1) \quad (n \to \infty),
$$

whence

$$
(10) \quad \int_{M_2(r_n)}^{M_1(r_n)} \frac{p_2(R) dR}{R p^2(R)} = O(1) \quad (n \to \infty).
$$
Inequalities (4), (5) are used in proving that \( \{K_n\} \) is bounded and reasoning similar to the above establishes

(11) \[
\int_{R_0}^{M_1(r_n)} \frac{dR}{R_{p_1}(R)} = 2 \log(1 - r_n)^{-1} + O(1),
\]

(12) \[
\int_{R_0}^{M_2(r_n)} \frac{dR}{R_{p_2}(R)} = 2 \log(1 - r_n)^{-1} + O(1),
\]
as \( n \to \infty \).

5.

We return to (7) with \( c = \beta/(2 - \beta) \) and integrate to find

(13) \[
4 \int_{R_0}^{R'} \frac{dR}{R_{p}(R)} \leq (2 - \beta)^2 \int_{R_0}^{R'} \frac{dR}{R_{p_1}(R)} + \beta^2 \int_{R_0}^{R'} \frac{dR}{R_{p_2}(R)}.
\]

When \( R' = M_2(r_n) \), we can use (8)', (12), to show that the right hand side of (13) is equal to

\[
((2 - \beta)^2 \cdot 2\beta/(2 - \beta) + 2\beta^2) \log(1 + r_n)^{-1} + O(1) = 4\beta \log(1 - r_n)^{-1} + O(1) \quad \text{as } n \to \infty.
\]

But (6) with \( R_2 = M_2(r_n) \) indicates that the left hand side of (13) is at least

\[
4\beta \log(1 - r_n)^{-1} - p^{-1} \log R_0 - 2
\]
as \( n \to \infty \). Thus

(14) \[
(2 - \beta)^2 \int_{R_0}^{R'} \frac{dR}{R_{p_1}(R)} + \beta^2 \int_{R_0}^{R'} \frac{dR}{R_{p_2}(R)} - 4 \int_{R_0}^{R'} \frac{dR}{R_{p}(R)}
\]
is a non-negative function of \( R' \) which is bounded above on the sequence \( \{M_2(r_n)\} \).

Let \( E(R_0, R') \) denote the expression (14). In view of (7) (with \( c = \beta/(2 - \beta) \)) \( E(R_0, R') \) increases with \( R' \). Thus \( E(R_0, R') \) is a bounded function of \( R' \) on \( (R_0, +\infty) \).

Since (6) was used in the above discussion we have also shown that

(15) \[
\int_{R_0}^{R'} \frac{dR}{R_{p}(R)} = p^{-1} \log R' + O(1) \quad \text{as } R' \to \infty.
\]

6.

To complete the proof we use \( E(R_0, M_1(r_n)) = O(1) \) as \( n \to \infty \), which, with (11), (15) gives
\[ (16) \quad \beta^2 \int_{R_0}^{M_1(r_n)} \frac{dR}{Rp_2(R)} = 4p^{-1} \log M_1(r_n) - 2(2-\beta)^2 \log (1-r_n)^{-1} + O(1) \]
\[ = O(\log (1-r_n)^{-1}) \quad \text{as } n \to \infty, \]

since \( M_1(r_n) = O(1-r_n)^{-2p} \) ([2, Theorem 2.5]).

Now
\[ M_1(r_n) > A(1-r_n)^{-p_2}, \quad M_2(r_n) < A(1-r)^{-p_2} \]

and so (6), (10), (16) combine to give
\[ (\alpha(\beta) - \beta) \log (1-r_n)^{-1} - O(1) \leq \int_{M_1(r_n)}^{M_2(r_n)} \frac{dR}{Rp(R)} \]
\[ \leq \left( \int_{M_1(r_n)}^{M_2(r_n)} \frac{p_2(R)dR}{Rp_2(R)} \right)^{\frac{1}{2}} \left( \int_{M_1(r_n)}^{M_2(r_n)} \frac{dR}{Rp_2(R)} \right)^{\frac{1}{2}} \]
\[ = O(\log (1-r_n)^{-1}) \quad (n \to \infty) \]

and unless \( \alpha(\beta) = \beta \) (i.e. \( \beta = 1 \)) this produces a contradiction for large enough \( n \) and establishes that the class \( C(\beta) \) is empty if \( \beta \in (0, 1) \).

7.

Finally we remark that the following result can be established by arguing along the lines of the examples in [1].

Let \( \beta \in (0, 1) \) and suppose \( \mu(r) \) is a positive function on \( (0, 1) \) which decreases to 0 as \( r \uparrow 1 \). Then there is a univalent function \( f \) and a sequence \( \{r_n\} \) with \( r_n \uparrow 1 \) (\( n \to \infty \)) for which
\[ |f(r_n)| > A(1-r_n)^{-\alpha(\beta)}, \quad |f(-r_n)| > \mu(r_n)(1-r_n)^{-\beta} \quad (n = 1, 2, \ldots). \]

(The function \( \mu \) could be associated with the sequence \( \{r_n\} \) rather than with \( \{-r_n\} \).)

The author acknowledge the referee’s useful remarks.

REFERENCES


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