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SUMS OF ROOTS OF UNITY

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1. Introduction.

Let β be a cyclotomic integer lying with its conjugates in a certain circle $|z|<R$. Classically β can be represented as a sum of roots of unity. If R is small, it is quite natural to suppose that β can be given as a sum of only a few roots of unity. Indeed, according to a theorem of J. W. S. Cassels [1], if $R^2=5.01$ then β can be represented as the sum of at most two roots of unity excluding some exceptional cases.

If β lies with its conjugates in $|z|<R$, then the same is true of any conjugate of β multiplied by any root of unity. Hence, for our problem, we can consider two cyclotomic integers α and β as equivalent $(\alpha \sim \beta)$ if α is a root of unity times a conjugate of β. We denote as usual by $\overline{|\beta|}$ the maximum of the absolute values $|\beta'|$ of the conjugates β' of β.

The object of this paper is the following:

THEOREM 1.1. *Let $R^2<6$. If β is a cyclotomic integer with $\overline{|\beta|}<R$, then one of the following conditions is satisfied:*

I. *β can be expressed as the sum of at most 3 roots of unity.*

II. *β is equivalent to one of the numbers*

$$(\zeta_5+\zeta_5^4)+(\zeta_5^2+\zeta_5^3)\varrho$$
$$1+(1+\zeta_7+\zeta_7^3)\varrho$$
$$1+(1+\zeta_{30}+\zeta_{30}^{12})\varrho$$

for some root ϱ of unity, where $\zeta_N=\exp(2\pi i/N)$.

III. *β is equivalent to an element of a certain finite set $E(R)$.*

J. W. S. Cassels [1] has proved this theorem for $R^2=5.01$. In this case the two latter possibilities in II do not occur. By taking into account the results of A. J. Jones [2] concerning the sums of three roots of unity, we have immediately

THEOREM 1.2. *Let $R_1^2<(1+\sqrt{2})^2=5.8284\ldots$ Suppose that β is a cyclotomic integer. A necessary and sufficient condition that $\overline{|\beta|}<R_1$ is that one of the following conditions is satisfied:*

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I. *β can be expressed as the sum of at most 2 roots of unity.*

II. *β is equivalent to one of the numbers*

$$1+\varrho-\varrho^{-1}$$
$$(\zeta_5+\zeta_5^4)+(\zeta_5^2+\zeta_5^3)\varrho$$

for some root ϱ of unity.

III. *β is equivalent to an element of a certain finite set $E(R_1)$.*

This corresponds to the conjecture 2 of R. M. Robinson [3] when $R_1^2=5$. According to Cassels [1] this theorem is correct for $R_1^2=5.01$.

On the other hand we shall prove that Theorems 1.1 and 1.2 do not hold if $R^2=6$ and $R_1^2=(1+\sqrt{2})^2$.

In the proof of Theorem 1.1, methods similar to those in [1] will be used. I am grateful to Professor Cassels for suggesting that Minkowski's linear forms theorem could be used in the proof of Theorem 1.1. This will considerably shorten the proof. I am also very grateful to Professor Ennola for many helpful comments.

2. Preliminaries.

The proofs of Lemmas 2.1–2.8 and 2.11 can be found in Cassels' paper [1].

For any algebraic number α we shall denote by $\mathscr{M}(\alpha)$ the mean of $|\alpha'|^2$ taken over all the conjugates α' of α. The function $\mathscr{M}$ is particularly easy to handle in a cyclotomic field, since then the conjugates of the algebraic number $|\alpha|^2$ are just the $|\alpha'|^2$ where α' runs through the conjugates of α and where each conjugate of $|\alpha|^2$ occurs the same number of times.

In the proof of Theorem 1.1 we can suppose that $\mathscr{M}(\beta)<R^2$ since

$$\overline{|\beta|}^2 \geqq \mathscr{M}(\beta)\,.$$

If α is a non-zero algebraic integer, its norm is at least 1 in absolute value and so

$$\mathscr{M}(\alpha) \geqq 1\,.$$

For any integer $P\geqq 1$ we denote by $\mathsf{Q}(P)$ the cyclotomic field $\mathsf{Q}(\zeta_P)$ obtained by adjoining $\zeta_P=\exp(2\pi i/P)$ to the rational field Q. We consider two cases.

FIRST CASE. Suppose, first, that $P=pP_1$ where p is a prime and $p\nmid P_1$. Let ξ be a primitive pth root of unity. Then any $\beta\in\mathsf{Q}(P)$ may be written in the shape

$$\beta = \textstyle\sum_{j=0}^{p-1}\alpha_j\xi^j \quad (\alpha_j\in\mathsf{Q}(P_1))\,. \tag{2.1}$$

This representation is not unique since the sum of the ξ^j vanishes. If β is an integer the coefficients α_j can be chosen as integers. The conjugates of β over $Q(P_1)$ are obtained from β by letting ξ run through all the primitive pth roots of unity in the representation (2.1).

The Corollary of Lemma 1 in [1] is as follows:

LEMMA 2.1. *If $\beta \in Q(P)$ is an integer with*

$$\mathscr{M}(\beta) < \tfrac{1}{4}(p+3),$$

then there is a representation

$$\beta = \textstyle\sum_{j=1}^{X} \gamma_j \xi^{r_j}$$

where $0 \leqq r_1 < r_2 < \ldots < r_X \leqq p-1$ and the $\gamma_j \neq 0$ are integers in $Q(P_1)$ and particularly

$$X \leqq \tfrac{1}{2}(p-1).$$

According to the calculations in [1] we have

LEMMA 2.2. *If β has a representation*

$$\beta = \textstyle\sum_{j=1}^{X} \gamma_j \xi^{r_j} \quad (\gamma_j \in Q(P_1),\ \gamma_j \neq 0)$$

where the r_j are incongruent mod p, *then*

$$\overline{|\beta|}^2 \geqq ((p-X)/(p-1)) \textstyle\sum_{j=1}^{X} |\gamma_j|^2$$

and

$$\mathscr{M}(\beta) \geqq ((p-X)/(p-1)) \textstyle\sum_{j=1}^{X} \mathscr{M}(\gamma_j).$$

In particular,

$$\mathscr{M}(\beta) \geqq ((p-X)/(p-1))X$$

if β is an integer.

SECOND CASE. Suppose that

$$P = p^N P_2, \quad p \nmid P_2$$

where p is a prime and $N \geqq 2$. Let L be an integer with

$$2L \leqq N.$$

Put

$$P_1 = p^{N-L} P_2$$

and let ξ be a primitive p^Nth root of unity. Then every $\beta \in Q(P)$ is uniquely of the shape

$$\beta = \textstyle\sum_{j=0}^{p^L-1} \alpha_j \xi^j \quad (\alpha_j \in Q(P_1)) \tag{2.2}$$

where the α_j are integers if β is. The conjugates of β over $\mathsf{Q}(P_1)$ are obtained from β by replacing ξ with the primitive p^Nth roots of unity

$$\xi^{kp^{N-L}+1} \quad (k=1,2,\ldots,p^L) .$$

This time we have

LEMMA 2.3. *If β is as above, then*

$$\overline{|\beta|}^2 \geqq \sum_j |\alpha_j|^2$$

and

$$\mathscr{M}(\beta) = \sum_j \mathscr{M}(\alpha_j) .$$

In particular,

$$\mathscr{M}(\beta) \geqq X$$

if β is an integer and X of the α_j are non-zero.

Let β be a cyclotomic integer, so $\beta \in \mathsf{Q}(P)$ for some $P \geqq 1$, where P is chosen as small as possible. We shall call β a *minimal cyclotomic integer* if

$$\alpha \sim \beta,\ \alpha \in \mathsf{Q}(P') \Rightarrow\ P' \geqq P .$$

Since every cyclotomic integer is equivalent to a minimal cyclotomic integer, we can restrict our attention to the minimal ones. If β is a minimal cyclotomic integer and $\mathsf{Q}(P)$ is the least cyclotomic field containing β, then at least two of the α_j are non-zero in any representation of the type (2.1) or (2.2).

In the following we shall express some estimates of $\overline{|\beta|}$ and $\mathscr{M}(\beta)$ when β is a cyclotomic integer. On taking into account the proof of Lemma 2 in [1] we have

LEMMA 2.4. *Suppose that $\beta \neq 0$ is a cyclotomic integer but not a root of unity. Then*

$$\mathscr{M}(\beta) = \tfrac{3}{2} \quad \text{if } \beta \sim 1+\zeta_5 ,$$
$$\mathscr{M}(\beta) \geqq \tfrac{5}{3} \quad \text{otherwise} .$$

According to Lemma 3 we have

LEMMA 2.5. *Suppose that β is a cyclotomic integer which is neither a root of unity nor representable as a sum of two roots of unity. Then*

$$\mathscr{M}(\beta) \geqq 2 .$$

The following Lemma and its Corollary originates from Robinson [3].

LEMMA 2.6. *Suppose that β is a cyclotomic integer with*

$$\overline{|\beta|}^2 \leqq 4 .$$

Then

$$\overline{|\beta|}^2 = 2+2\cos(2\pi/N)$$

for some integer N.

COROLLARY. *Suppose that $\beta \neq 0$ is a cyclotomic integer but not a root of unity. Then*

$$\overline{|\beta|}^2 \geqq 2 .$$

Lemma 5 in [1] can immediately be generalized into the following form as Cassels has stated.

LEMMA 2.7. *Let N be an odd integer. If β is a cyclotomic integer with $\overline{|\beta|}^2 = 2+2\cos(2\pi/N)$, then*

$$\beta \sim 1+\zeta_N .$$

Lemma 6 in [1] implies

LEMMA 2.8. *Suppose that β is a cyclotomic integer with $|\beta|^2 = 2$. Then β is equivalent to one of the numbers*

$$1+i$$
$$1+\zeta_7+\zeta_7^3 \sim (1+i\sqrt{7})/2$$
$$1+\zeta_{30}+\zeta_{30}^{12} \sim (\sqrt{5}+i\sqrt{3})/2 .$$

LEMMA 2.9. *Suppose that β is a cyclotomic unit but not equivalent to any of the numbers 1, $1+\zeta_5$, $1+\zeta_7$, $1+\zeta_9$, $1+\zeta_{11}$. Then*

$$\overline{|\beta|}^2 \geqq 2+\sqrt{3} .$$

PROOF. According to Lemma 2.6 and its corollary it is enough to show that none of the possibilities

$$\overline{|\beta|}^2 = 2+2\cos(2\pi/N) \quad (4 \leqq N \leqq 11)$$

can occur. By Lemma 2.7, N cannot be 5, 7, 9 or 11. On the other hand, N cannot be 4, 6, 8 or 10 since none of the numbers 2, 3, $2+\sqrt{2}$ and $(5+\sqrt{5})/2$ are units.

LEMMA 2.10. *If $\eta \neq -1$ is a root of unity, then*

$$\mathcal{M}(3+\eta) \geqq 7 ,$$
$$\mathcal{M}(3+2\eta) \geqq 7 .$$

PROOF. If $\eta = 1$ then the statement is correct. Let η be a primitive kth root of unity with $\varphi(k) \geqq 2$, where φ is the Euler φ-function. If $\cos\alpha_j + i\sin\alpha_j$, $(j,k)=1$, are the conjugates of η, then

$$\begin{aligned}\mathcal{M}(3+\eta) &= (1/\varphi(k)) \textstyle\sum_j \{(3+\cos\alpha_j)^2 + (\sin\alpha_j)^2\} \\ &= 10 + (6/\varphi(k)) \textstyle\sum_j \cos\alpha_j \geqq 10 + (6/2)(-1) = 7\end{aligned}$$

since the sum of all the primitive kth roots of unity equals the value $\mu(k)$ of the Moebius μ-function. The latter assertion is proved in a similar way.

The following lemma introduced by Cassels is very useful when we estimate the value of $\overline{|\beta|}$.

LEMMA 2.11. *Let $\{c_j \mid 1 \leqq j \leqq T\}$ be a finite set of nonnegative real numbers with mean μ and variance σ^2. Then*

$$\max c_j \geqq \mu + \sigma^2/\mu .$$

3. Lemma.

Let β be a minimal cyclotomic integer and $\mathrm{Q}(P)$ the least cyclotomic field containing β. In this section we are going to prove

LEMMA 3.1. *Let $R^2 < 6$. There is an n_0 with the following property. Suppose that p is a prime and either*

$$P = pP_1,\ (p,P_1) = 1,\ p \geqq n_0$$

or

$$P = P^N P_2,\ (p,P_2) = 1,\ 2L \leqq N,\ p^L \geqq n_0,\ P_1 = p^{N-L}P_2$$

where the notations are similar to those in Section 2. Then one of the following conditions is satisfied:

I. *β is representable as the sum of at most three roots of unity.*
II. *β is equivalent to one of the following numbers*

$$(\zeta_5 + \zeta_5{}^4) + (\zeta_5{}^2 + \zeta_5{}^3)\varrho$$
$$1 + (1 + \zeta_7 + \zeta_7{}^3)\varrho$$
$$1 + (1 + \zeta_{30} + \zeta_{30}{}^{12})\varrho$$

for some root ϱ of unity, where $\zeta_N = \exp(2\pi i/N)$.

III. $$\overline{|\beta|}^2 \geqq R^2 .$$

PROOF. According to Section 2, β has a representation

$$\beta = \sum_{j=1}^{X} \gamma_j \xi^{r_j}$$

where ξ is a primitive pth or p^Nth root of unity and the $\gamma_j \neq 0$ are integers in $Q(P_1)$. By minimality we have $X \geqq 2$. Suppose that $X \geqq 6$. In the case of the simple factor Lemmas 2.1 and 2.2 imply

$$\overline{|\beta|}^2 \geqq \mathscr{M}(\beta) \geqq ((p-X)/(p-1))\, X \geqq ((p-6)/(p-1))\cdot 6 > R^2$$

if p is large enough. In the case of the multiple factor Lemma 2.3 implies $\mathscr{M}(\beta) \geqq 6$ and we are finished. We shall discuss the remaining possibilities $X = 2, 3, 4, 5$ one by one and all these cases are divided into several subcases.

FIRST CASE. $X = 2$ so that $\beta = \gamma_1 \xi^{r_1} + \gamma_2 \xi^{r_2}$. By minimality $r_1 \not\equiv r_2$ modulo p, thus on multiplying β by an appropriate power of ξ we may suppose that

$$\beta = \gamma_0 + \gamma_1 \xi$$

where ξ is a new primitive pth or p^Nth root of unity.

First subcase. $\gamma_0\gamma_1$ is not a root of unity. By the corollary of Lemma 2.6 we have $\overline{|\gamma_0\gamma_1|}^2 \geqq 2$.

1A. If $\overline{|\gamma_0\gamma_1|}^2 > 2$ then Lemma 2.6 implies

$$\overline{|\gamma_0\gamma_1|}^2 \geqq 2 + 2\cos(2\pi/5) > 2.6 .$$

On applying a suitable automorphism of $Q(P)/Q$ we may suppose that $|\gamma_0\gamma_1|^2 > 2.6$ and

$$(|\gamma_0| + |\gamma_1|)^2 \geqq 4|\gamma_0\gamma_1| > 6.4 .$$

After a suitable automorphism of $Q(P)/Q(P_1)$ we have without restrictions

$$|\arg(\gamma_0^{-1}\gamma_1\xi)| \leqq 2\pi/n_0 .$$

Hence

$$|\gamma_0 + \gamma_1\xi|^2 > 6 > R^2$$

if n_0 is large enough.

1B. If $\overline{|\gamma_0\gamma_1|}^2 = 2$ then both of the γ_j cannot be roots of unity. We can suppose that $\overline{|\gamma_0|}^2 \geqq 2$.

a) If $\overline{|\gamma_0|}^2 = 2$ then according to Lemma 2.8 γ_0 is equivalent to one of the numbers $1+i$, $1+\zeta_7+\zeta_7^3$ and $1+\zeta_{30}+\zeta_{30}^{12}$. Thus the condition I or II of Lemma 3.1 is satisfied since, on the other hand, γ_1 must be a root of unity.

b) If $\overline{|\gamma_0|}^2 > 2$ then $\overline{|\gamma_0|}^2 > 2.6$ and, without restrictions,

$$(|\gamma_0|+|\gamma_1|)^2 = |\gamma_0|^2+2\sqrt{2}+2|\gamma_0|^{-2} > 6.1 .$$

The assertion is proved just as in the case 1A.

Second subcase. $\gamma_0\gamma_1$ is a root of unity. If either γ_0 or γ_1 is a root of unity, then so is the other and β is the sum of two roots of unity.

Suppose now that γ_0 and γ_1 are not roots of unity. Since γ_1 is a unit then according to Lemma 2.9 γ_1 is equivalent to one of the numbers $1+\zeta_5$, $1+\zeta_7$, $1+\zeta_9$, $1+\zeta_{11}$ or $\overline{|\gamma_1|}^2 \geqq 2+\sqrt{3}$.

If $\gamma_1 \sim 1+\zeta_5$ we have the first possibility in condition II.

If $\gamma_1 \sim 1+\zeta_7$, $1+\zeta_9$, $1+\zeta_{11}$ then $\overline{|\gamma_0|}^2 = \overline{|\gamma_1^{-1}|}^2 > 2+\sqrt{3}$. Thus we can suppose that

$$(|\gamma_0|+|\gamma_1|)^2 \geqq (2+\sqrt{3})+2+(2+\sqrt{3})^{-1} = 6$$

and the assertion is proved just as above.

SECOND CASE. $X=3$ so that $\beta = \gamma_0+\gamma_1\xi^{r_1}+\gamma_2\xi^{r_2}$.

According to Minkowski's linear forms theorem there exists a triplet $(u, v_1, v_2) \neq (0,0,0)$ of integers satisfying the inequalities

$$|r_i u - p^L v_i| < \tfrac{1}{4}p^L \quad (i=1,2)$$
$$|u| < 4^2+1$$

in the case of the multiple factor. Here $u \neq 0$. If p^k is the exact power of p dividing u then $p^k < 4^2+1$ and we can suppose that p^{L-k} is large enough. Let

$$u = p^k u'$$
$$r_i = k_i p^{L-k} + r_i' \quad (i=1,2)$$

where $0 \leqq r_1', r_2' < p^{L-k}$. On replacing the representation of β by a new one with $L-k$ instead of L we have

$$\beta = \gamma_0 + (\gamma_1\xi^{k_1 p^{L-k}})\xi^{r_1'} + (\gamma_2\xi^{k_2 p^{L-k}})\xi^{r_2'} .$$

We may suppose that $0 \neq r_1' \neq r_2' \neq 0$ since otherwise we are reduced to the case $X=2$. Moreover,

$$|r_i'u' - p^{L-k}(v_i - k_i u')| < \tfrac{1}{4}p^{L-k} \qquad (i=1,2)$$
$$|u'| < 4^2+1 .$$

Hence we may suppose $(u,p)=1$ originally if we change the notations when needed. Thus we can apply the automorphism $\xi \to \xi^u$ of $\mathsf{Q}(P)/\mathsf{Q}(P_2)$ and after multiplying β by an appropriate power of ξ we may suppose that

$$\beta = \gamma_0\xi^{r_0}+\gamma_1\xi^{r_1}+\gamma_2\xi^{r_2}, \qquad 0=r_0<r_1<r_2<\tfrac{1}{2}p^L .$$

In the case of the simple factor we can draw conclusions of the same kind. Again $(u,p)=1$ if p is large enough. Consequently we may suppose that the exponents satisfy the condition $0=r_0<r_1<r_2<\frac{1}{2}p$.

If the γ_j are roots of unity we are finished. In the following at least one of the γ_j is not a root of unity.

First subcase. All the r_i-r_j are distinct modulo p (or p^L).

1A. Exactly one of the γ_j is not a root of unity. Without restrictions this coefficient is at least $\sqrt{2}$ in absolute value. As

$$|\beta|^2-3 = \textstyle\sum_j |\gamma_j|^2-3+\sum_{i\neq j}\gamma_i\bar{\gamma}_j\xi^{r_i-r_j} ,$$

we have according to Lemma 2.3

$$\overline{\left||\beta|^2-3\right|^2} \geqq (\textstyle\sum_j |\gamma_j|^2-3)^2+\sum_{i\neq j}|\gamma_i\bar{\gamma}_j|^2 \geqq 1+(4\cdot 2+2\cdot 1) = 11$$

in the case of the multiple factor. Hence

$$\overline{|\beta|^2} \geqq 3+\sqrt{11} > 6$$

so that III of the enunciation of Lemma 3.1 is satisfied. In the case of the simple factor we have a slightly weaker estimate due to the difference between Lemmas 2.2 and 2.3. In the sequel the calculations will be carried out only in the case of the multiple factor.

Let μ and σ^2 denote the mean and the variance of the $|\beta'|^2$ where β' runs through the conjugates of β, thus

$$\mu = \mathscr{M}(\beta)$$
$$\sigma^2 = \mathscr{M}(|\beta|^2-\mu) .$$

1B. Exactly one of the γ_j is a root of unity. Lemma 2.4 implies that

$$\mu = \sum_j \mathscr{M}(\gamma_j) \geqq 1+2\cdot 1.5 = 4$$
$$\sigma^2 \geqq \sum_{i\neq j} \mathscr{M}(\gamma_i\bar{\gamma}_j) \geqq 4\cdot 1.5+2\cdot 1 = 8$$

and on applying Lemma 2.11 we have

$$\overline{|\beta|}^2 \geqq \mu+\sigma^2/\mu \geqq 6\,.$$

In the case of the simple factor $\overline{|\beta|}^2 \geqq R^2$ if p is large enough.

1C. None of the γ_j are roots of unity. Now $\mu \geqq 4.5$, $\sigma^2 \geqq 2\cdot 1.5+4\cdot 1=7$ since all the $\gamma_i\bar{\gamma}_j$ cannot be roots of unity. The assertion is clear.

Second subcase. All the r_i-r_j are not distinct modulo p (or p^L). The only possible congruence is $r_2-r_1\equiv r_1-r_0$, that is to say $r_2\equiv 2r_1$ modulo p (or p^L). By minimality r_1 is prime to p and so we can suppose that

$$\beta = \gamma_0+\gamma_1\xi+\gamma_2\xi^2\,.$$

In the following we shall show that

$$A = (|\gamma_0|+|\gamma_2|)^2+|\gamma_1|^2 \geqq 6 \tag{3.1}$$

after applying a suitable automorphism of $\mathsf{Q}(P)/\mathsf{Q}$.

2A. If $\gamma_0\gamma_1\gamma_2$ is not a root of unity, we can suppose that $|\gamma_0\gamma_1\gamma_2|^2 \geqq 2$ and so

$$A \geqq 4|\gamma_0\gamma_2|+2|\gamma_0\gamma_2|^{-2} \geqq 6\,.$$

2B. If $\gamma_0\gamma_1\gamma_2$ is a root of unity but $\gamma_0\gamma_2$ is not, we may assume that $|\gamma_0\gamma_2|^2 \geqq 2$ and so

$$A \geqq 4|\gamma_0\gamma_2|+|\gamma_0\gamma_2|^{-2} > 6\,.$$

2C. If $\gamma_0\gamma_1\gamma_2$ and $\gamma_0\gamma_2$ are roots of unity but γ_0 is not, we may suppose $|\gamma_0|^2 \geqq 2+2\cos(2\pi/5) = ((\sqrt{5}+1)/2)^2$ according to Lemma 2.9. Hence

$$A \geqq (|\gamma_0|+|\gamma_0|^{-1})^2+1 \geqq 6\,.$$

There are no other possibilities since at least one of the γ_j is not a root of unity.

Once more a suitable automorphism of $\mathsf{Q}(P)/\mathsf{Q}(P_1)$ is applied so that

$$|\arg(\bar{\gamma}_0\gamma_2\xi^2)| \leqq 4\pi/n_0, \quad |\arg(\bar{\gamma}_0\gamma_1\xi)| \leqq \pi/2\,. \tag{3.2}$$

It is geometrically obvious that according to the inequalities (3.1) and (3.2) we have

$$|\beta|^2 \geqq R^2$$

if n_0 is large enough.

THIRD CASE. $X=4$. Just as at the beginning of the case $X=3$ we have

$$\beta = \gamma_0+\gamma_1\xi^s+\gamma_2\xi^{s+t}+\gamma_3\xi^{s+t+u}$$

where s, t, $u>0$ and $s+t+u<\frac{1}{4}p$ (or $\frac{1}{4}p^L$). Hence

$$\begin{aligned}|\beta|^2 &= A_0+\bar{\gamma}_0\gamma_1\xi^s+\bar{\gamma}_1\gamma_2\xi^t+\bar{\gamma}_2\gamma_3\xi^u+\bar{\gamma}_3\gamma_0\xi^{-s-t-u}+\\ &\quad +\bar{\gamma}_0\gamma_2\xi^{s+t}+\bar{\gamma}_1\gamma_3\xi^{t+u}+\bar{\gamma}_2\gamma_0\xi^{-s-t}+\bar{\gamma}_3\gamma_1\xi^{-t-u}+\\ &\quad +\bar{\gamma}_0\gamma_3\xi^{s+t+u}+\bar{\gamma}_1\gamma_0\xi^{-s}+\bar{\gamma}_2\gamma_1\xi^{-t}+\bar{\gamma}_3\gamma_2\xi^{-u}\\ &= A_0+\sum_{\substack{1\leqq i\leqq s+t+u\\ A(i)\neq 0}} A(i)\xi^i+\sum_{\substack{1\leqq i\leqq s+t+u\\ A(i)\neq 0}} \bar{A}(i)\xi^{-i}\end{aligned}$$

where the $A(i)$ are integers in $\mathsf{Q}(P_1)$. In the following we shall consider only the terms $A(i)\xi^i$; similar results hold for the terms of the latter sum.

It is clear that $A(s+t+u)=\bar{\gamma}_0\gamma_3\neq 0$. In the sum $\sum A(i)\xi^i$ there may occur the following values of the index i:

$$\begin{array}{lll} s & t & u\\ s+t & t+u & \\ s+t+u\,. & & \end{array} \tag{3.3}$$

We say that two entries in a table such as (3.3) are *companions* if they are egual.

First subcase. Suppose, first, that $s+t$ has no companion in the table (3.3).

1A. If $t+u$ also is alone in the table, the same is true for one of the values s, t, u. Otherwise we should have $s=t=u$ contradicting $s+t\neq t+u$. In this case

$$\mu = \mathscr{M}(\beta) \geqq 4, \quad \sigma^2 = \mathscr{M}(|\beta|^2-\mu) \geqq 8$$

and the assertion is clear according to Lemma 2.11.

1B. If $t+u$ has a companion in the table (3.3), the only possibility is $s=t+u$. If $t\neq u$, we have $\sigma^2\geqq 8$ and we are finished. Hence we can suppose that $t=u$ and without restrictions

$$\beta = \gamma_0+\gamma_2\xi^2+\gamma_3\xi^3+\gamma_4\xi^4\,.$$

In order to calculate the variance σ^2 we must consider the representation

$$|\beta|^2 = \eta_0+\eta_1\xi+\bar{\eta}_1\xi^{-1}+\ldots+\eta_4\xi^4+\bar{\eta}_4\xi^{-4}$$

where

$$\begin{aligned} \eta_0 &= |\gamma_0|^2+|\gamma_2|^2+|\gamma_3|^2+|\gamma_4|^2 \\ \eta_1 &= \bar{\gamma}_2\gamma_3+\bar{\gamma}_3\gamma_4 \\ \eta_2 &= \bar{\gamma}_0\gamma_2+\bar{\gamma}_2\gamma_4 \\ \eta_3 &= \bar{\gamma}_0\gamma_3 \neq 0 \\ \eta_4 &= \bar{\gamma}_0\gamma_4 \neq 0\,. \end{aligned}$$

a) If at most one of the γ_j is a root of unity, then the assertion is clear since $\mu \geqq 5.5$ and $\sigma^2 \geqq 4$.

b) If exactly two of the γ_j are roots of unity, then η_3 or η_4 cannot be a root of unity since either one or two of γ_0, γ_3, γ_4 are roots of unity. Hence $\sigma^2 \geqq 5$ and on the other hand $\mu \geqq 5$ implying $\overline{|\beta|}^2 \geqq 6$.

c) If exactly three of the γ_j are roots of unity then $\mu \geqq 4.5$.

If either γ_0 or γ_4 is not a root of unity, then $\eta_2 \neq 0$ and η_4 is not a root of unity. Hence $\sigma^2 \geqq 7$ and we are finished.

If γ_2 is not a root of unity we can suppose that η_1 is a root of unity since otherwise $\sigma^2 \geqq 7$. Furthermore we can suppose that $|\gamma_2|^2 \geqq 2$ which implies

$$\overline{\left||\beta|^2-3\right|}^2 \geqq (\eta_0-3)^2+2(|\eta_1|^2+|\eta_3|^2+|\eta_4|^2) \geqq 10\,.$$

Hence

$$\overline{|\beta|}^2 \geqq 3+\sqrt{10} > 6\,.$$

If γ_3 is not a root of unity we have without restrictions $|\gamma_3|^2 \geqq 2$ and again

$$\overline{\left||\beta|^2-3\right|}^2 \geqq (\eta_0-3)^2+2(|\eta_3|^2+|\eta_4|^2) \geqq 10\,.$$

d) In the remaining case all the γ_j are roots of unity. We can suppose that at least one of the coefficients η_1, η_2 equals zero since otherwise $\mu = 4$ and $\sigma^2 \geqq 8$.

In the sequel we shall consider the algebraic integer

$$\theta = |\beta|^2(|\beta|^2-5)\,.$$

The conjugates θ' of θ are

$$\theta' = |\beta'|^2(|\beta'|^2-5)$$

where β' runs through the conjugates of β. By means of Lemma 2.11 we shall show that

$$\max\theta' \geqq 6$$

which implies

$$\overline{|\beta|}^2 \geqq 6 .$$

The $\theta' + 25/4$ are nonnegative and have the mean

$$\mu^* = \sigma^2 + \mu^2 - 5\mu + 25/4 . \tag{3.4}$$

This formula is needed later, too. As $\mu = 4$ and $\sigma^2 \geqq 4$ we have now $\mu^* \geqq 25/4$. In order to calculate the variance of the $\theta' + 25/4$ we consider

$$\theta + 25/4 - \mu^* = A_0 + A_1\xi + \bar{A}_1\xi^{-1} + \ldots + A_8\xi^8 + \bar{A}_8\xi^{-8}$$

where e.g.

$$\begin{aligned} A_2 &= 2\bar{\eta}_2\eta_4 + 2\bar{\eta}_1\eta_3 + 3\eta_2 + \eta_1{}^2 \\ A_3 &= 2\bar{\eta}_1\eta_4 + 3\eta_3 + 2\eta_1\eta_2 \\ A_4 &= 3\eta_4 + 2\eta_1\eta_3 + \eta_2{}^2 \\ A_7 &= 2\eta_3\eta_4 . \end{aligned}$$

If $\eta_2 = 0$ then

$$\begin{aligned} A_3 &= 5\eta_3 + 2\bar{\gamma}_0\gamma_2\bar{\gamma}_3\gamma_4 \\ A_4 &= 5\eta_4 + 2\bar{\gamma}_0\bar{\gamma}_2\gamma_3{}^2 \end{aligned}$$

and

$$\begin{aligned} \sigma^{*2} &= \mathscr{M}(\theta + 25/4 - \mu^*) \geqq 2(\mathscr{M}(A_3) + \mathscr{M}(A_4) + \mathscr{M}(A_7)) \\ &\geqq 2(9+9+4) = 44 . \end{aligned}$$

On the other hand, if $\eta_1 = 0$, $\eta_2 \neq 0$, then $A_2 = 5\eta_2$ implying $\sigma^{*2} \geqq 50$. As $\sigma^{*2} \geqq 44$ we have

$$\max\{\theta' + 25/4) \geqq 25/4 + \frac{44}{25/4}$$

and $\max\theta' > 6$.

Second subcase. We now suppose that $s+t$ has a companion in the table (3.3). There are the possibilities $s+t = u$, $t+u$.

2A. If $u = s+t$, in the sum $\sum A(i)\xi^i$ there may occur the following values of the index i:

$$\begin{array}{lll} s & t & s+t \\ s+t & 2t+s & \\ 2s+2t . & & \end{array}$$

If $s \neq t$, then $\mu \geqq 4$, $\sigma^2 \geqq 8$ and we are finished. If $s = t$ we can suppose that

$$\beta = \gamma_0 + \gamma_1\xi + \gamma_2\xi^2 + \gamma_4\xi^4 \sim \gamma_4 + \gamma_2(\xi^{-1})^2 + \gamma_1(\xi^{-1})^3 + \gamma_0(\xi^{-1})^4 .$$

This corresponds to the case 1B.

2B. If $s+t=t+u$ the index i may be

$$\begin{array}{lll} s & t & s \\ s+t & t+s & \\ 2s+t\,. & & \end{array}$$

a) If $t=s$ we can suppose

$$\beta = \gamma_0+\gamma_1\xi+\gamma_2\xi^2+\gamma_3\xi^3\,.$$

Hence

$$|\beta|^2 = \eta_0+\eta_1\xi+\bar{\eta}_1\xi^{-1}+\ldots+\eta_3\xi^3+\bar{\eta}_3\xi^{-3}$$

where

$$\begin{aligned} \eta_0 &= |\gamma_0|^2+|\gamma_1|^2+|\gamma_2|^2+|\gamma_3|^2 \\ \eta_1 &= \bar{\gamma}_0\gamma_1+\bar{\gamma}_1\gamma_2+\bar{\gamma}_2\gamma_3 \\ \eta_2 &= \bar{\gamma}_0\gamma_2+\bar{\gamma}_1\gamma_3 \\ \eta_3 &= \bar{\gamma}_0\gamma_3 \neq 0\,. \end{aligned}$$

(i) If none of the γ_j are roots of unity then $\overline{|\beta|^2}\geqq\mu\geqq 6$.

(ii) If exactly one of the γ_j is a root of unity then $\mu\geqq 5.5$. We can suppose that $|\beta|^2=\eta_0+\eta_3\xi^3+\bar{\eta}_3\xi^{-3}$, where η_3 is a root of unity, since otherwise $\sigma^2\geqq 3$. We first apply a suitable automorphism of $\mathsf{Q}(P)/\mathsf{Q}$ in order to have

$$\eta_0 \geqq \mathscr{M}(\gamma_0)+\mathscr{M}(\gamma_1)+\mathscr{M}(\gamma_2)+\mathscr{M}(\gamma_3) \geqq 5.5\,.$$

After applying once again an automorphism of $\mathsf{Q}(P)/\mathsf{Q}(P_1)$ we can, furthermore, suppose that $|\arg(\eta_3\xi^3)|\approx 0$ if n_0 is large enough. The assertion is clear again.

(iii) If exactly two of the γ_j are roots of unity then $\mu\geqq 5$. Without restrictions $\sigma^2<5$ so that we have to consider the following three cases:

$$\begin{aligned} \eta_1 &= \eta_2 = 0 \\ \eta_1 &= 0;\ \eta_2,\ \eta_3 \text{ roots of unity} \\ \eta_2 &= 0;\ \eta_1,\ \eta_3 \text{ roots of unity}\,. \end{aligned}$$

Let

$$|\beta|^2 = \eta_0+\eta_3\xi^3+\bar{\eta}_3\xi^{-3}\,.$$

If η_3 is a root of unity we can suppose that $\eta_0\geqq 5$, $|\arg(\eta_3\xi^3)|\approx 0$ and we are finished. If η_3 is not a root of unity we can suppose that $|\eta_3|^2\geqq 2$ and $|\arg(\eta_3\xi^3)|\approx 0$. Now we have

$$\eta_0 \geqq 2|\eta_3|+|\gamma_1|^2+|\gamma_2|^2 > 2\sqrt{2}+1\,,$$

since at least one of γ_1,γ_2 is a root of unity. The assertion is clear.

Let

$$|\beta|^2 = \eta_0+\eta_2\xi^2+\bar{\eta}_2\xi^{-2}+\eta_3\xi^3+\bar{\eta}_3\xi^{-3}\,.$$

Once again we can suppose that

$$(3.5) \qquad \eta_0 \geqq 5,\ |\arg(\eta_3\xi^3)| \approx 0,\ |\arg(\eta_2\xi^2)| \leqq \pi/3$$

which implies $\overline{|\beta|}^2 > 6$.

The last case is proved likewise.

(iv) If exactly three of the γ_j are roots of unity then $\mu \geqq 4.5$ and η_2, $\eta_3 \neq 0$. It suffices to consider the following two possibilities with $\sigma^2 < 7$.

If

$$|\beta|^2 = \eta_0 + \eta_1\xi + \bar{\eta}_1\xi^{-1} + \eta_2\xi^2 + \bar{\eta}_2\xi^{-2} + \eta_3\xi^3 + \bar{\eta}_3\xi^{-3}$$

where η_1, η_2, η_3 are roots of unity, then we can suppose that (3.5) is satisfied. Hence $|\beta|^2 \geqq R^2$ if n_0 is large enough.

If $\eta_1 = 0$ we consider such a conjugate of β that $|\eta_3|$ is as large as possible and (3.5) is satisfied.

(v) If all the γ_j are roots of unity then $\mu = 4$. As $\gamma_2\eta_1 - \gamma_1\eta_2 = \bar{\gamma}_1\gamma_2{}^2 \neq 0$, η_1 and η_2 cannot both vanish. Let

$$\theta = |\beta|^2(|\beta|^2 - 5) = A_0 + A_1\xi + \bar{A}_1\xi^{-1} + \ldots + A_6\xi^6 + \bar{A}_6\xi^{-6}$$

where

$$\begin{aligned}
A_1 &= 2\bar{\eta}_2\eta_3 + 2\bar{\eta}_1\eta_2 + 3\eta_1 = 7\eta_1 + 2\gamma_0\bar{\gamma}_1{}^2\gamma_3 + 2\bar{\gamma}_0\gamma_2{}^2\bar{\gamma}_3 \\
A_2 &= 2\bar{\eta}_1\eta_3 + 3\eta_2 + \eta_1{}^2 = 5\eta_2 + \eta_1{}^2 + 2\bar{\gamma}_0\gamma_1\bar{\gamma}_2\gamma_3 \\
A_3 &= 3\eta_3 + 2\eta_1\eta_2 \\
A_4 &= 2\eta_1\eta_3 + \eta_2{}^2 \\
A_5 &= 2\eta_2\eta_3 \\
A_6 &= \eta_3{}^2 .
\end{aligned}$$

According to (3.4) the $\theta' + 25/4$ have the mean $\mu^* \geqq 25/4$. It suffices to show that the variance σ^{*2} of the same numbers satisfies the condition $\sigma^{*2} \geqq 37.5$.

If either η_1 or η_2 vanishes, it is easy to see that $\sigma^{*2} \geqq 40$. Thus we may suppose that $\eta_1, \eta_2 \neq 0$.

If η_1 is not a root of unity then $\mathscr{M}(A_1) \geqq (7\sqrt{1.5} - 4)^2 > 20$. Hence $\sigma^{*2} > 40$ as we required.

Let η_1 be a root of unity. Then $\mathscr{M}(A_1) \geqq 9$. We subdivide cases according as the nonzero number η_2 is or is not a root of unity. In each case it is easy to verify the assertion.

b) Let $t \neq s$. As

$$\beta = \gamma_0 + \gamma_1\xi^s + \gamma_2\xi^{s+t} + \gamma_3\xi^{2s+t},$$

we have

$$\begin{aligned}
|\beta|^2 = \eta_0 &+ \eta_1\xi^s + \bar{\eta}_1\xi^{-s} + \eta_2\xi^t + \bar{\eta}_2\xi^{-t} + \\
&+ \eta_3\xi^{s+t} + \bar{\eta}_3\xi^{-s-t} + \eta_4\xi^{2s+t} + \bar{\eta}_4\xi^{-2s-t} .
\end{aligned}$$

In this representation the powers of ξ are distinct, since $4s+2t<\frac{1}{2}p$ (or $\frac{1}{2}p^L$). Moreover,

$$\begin{aligned}\eta_0 &= |\gamma_0|^2+|\gamma_1|^2+|\gamma_2|^2+|\gamma_3|^2\\ \eta_1 &= \bar\gamma_0\gamma_1+\bar\gamma_2\gamma_3\\ \eta_2 &= \bar\gamma_1\gamma_2 \neq 0\\ \eta_3 &= \bar\gamma_0\gamma_2+\bar\gamma_1\gamma_3\\ \eta_4 &= \bar\gamma_0\gamma_3 \neq 0\,.\end{aligned}$$

(i) If at most one of the γ_j is a root of unity then $\mu \geqq 5.5$ and $\sigma^2 \geqq 4$. Thus $\overline{|\beta|}^2 > 6$.

(ii) If exactly two of the γ_j are roots of unity then $\mu \geqq 5$.

Let e.g. γ_0 and γ_1 be roots of unity. Then $\sigma^2 \geqq 6$ and we are finished.

Let e.g. γ_0 and γ_3 be roots of unity. Then η_2 is not a root of unity or $\eta_1, \eta_3 \neq 0$. Hence $\sigma^2 \geqq 5$.

All the other possibilites correspond to one of the cases above.

(iii) If exactly three of the γ_j are roots of unity then $\eta_1, \eta_3 \neq 0$. Therefore, $\mu \geqq 4.5$ and $\sigma^2 \geqq 8$.

(iv) Let all the γ_j be roots of unity. Now $\mu = 4$. As $\eta_1 = \gamma_1\bar\gamma_2\eta_3$, we can suppose that $\eta_1 = \eta_3 = 0$ because otherwise $\sigma^2 \geqq 8$. Thus

$$|\beta|^2 = 4+\eta_2\xi^t+\bar\eta_2\xi^{-t}+\eta_4\xi^{2s+t}+\bar\eta_4\xi^{-2s-t}$$

where η_2 and η_4 are roots of unity. Consider the cyclotomic integer

$$\begin{aligned}\theta &= |\beta|^2(|\beta|^2-5)\\ &= \eta_4{}^2\xi^{4s+2t}+2\eta_2\eta_4\xi^{2s+2t}+\eta_2{}^2\xi^{2t}+3\eta_4\xi^{2s+t}+2\bar\eta_2\eta_4\xi^{2s}+3\eta_2\xi^t+\dots\,.\end{aligned}$$

If $t \neq 2s$, then all the powers of ξ in the representation of θ are distinct since $t \neq s$ and $8s+4t<p$ (or p^L). Then $\mu^* \geqq 25/4$ and $\sigma^{*2} \geqq 56$, which implies $\max\theta' > 6$.

Let $t = 2s$ so that

$$\theta = \eta_4{}^2\xi^{8s}+2\eta_2\eta_4\xi^{6s}+(\eta_2{}^2+3\eta_4)\xi^{4s}+(2\bar\eta_2\eta_4+3\eta_2)\xi^{2s}+\dots\,.$$

If $\eta_4 \neq -\eta_2{}^2$ then by Lemma 2.10 we have $\sigma^{*2} \geqq 38$ which implies $\max\theta' > 6$. Let $\eta_4 = -\eta_2{}^2$ so that

$$|\beta|^2 = 4+\eta_2\xi^t+\bar\eta_2\xi^{-t}-\eta_2{}^2\xi^{2t}-\bar\eta_2{}^2\xi^{-2t}\,.$$

We can suppose that $\arg(\eta_2\xi^t) \approx \pi/2$ and therefore $|\beta|^2 \geqq R^2$, if n_0 is large enough.

FOURTH CASE. $X = 5$. Just as at the beginning of the case $X = 3$ we have

$$\beta = \gamma_0+\gamma_1\xi^s+\gamma_2\xi^{s+t}+\gamma_3\xi^{s+t+u}+\gamma_4\xi^{s+t+u+v}$$

where $s, t, u, v > 0$ and $s+t+u+v < \frac{1}{4}p$ (or $\frac{1}{4}p^L$). Let

$$|\beta|^2 = A_0 + \sum_{\substack{1 \leqq i \leqq s+t+u+v \\ A(i) \neq 0}} A(i)\xi^i + \sum_{\substack{1 \leqq i \leqq s+t+u+v \\ A(i) \neq 0}} \bar{A}(i)\xi^{-i}$$

where the $A(i)$ are integers in $Q(P_1)$. Now $A(s+t+u+v) = \bar{\gamma}_0\gamma_4 \neq 0$.

First subcase. If at most three of the γ_j are roots of unity then $\overline{|\beta|}^2 \geqq \mu \geqq 6$.

Second subcase. If exactly four of the γ_j are roots of unity then $\mu \geqq 5.5$. We can assume that

$$|\beta|^2 = A_0 + C\xi^c + \bar{C}\xi^{-c} \tag{3.6}$$

where $C\xi^c = A(s+t+u+v)\xi^{s+t+u+v}$, since otherwise $\sigma^2 \geqq 4$. Moreover, we can suppose that

$$A_0 \geqq 5.5, \quad |C| \geqq 1, \quad |\arg(C\xi^c)| \leqq \pi/4$$

so that $|\beta|^2 > 6$.

Third subcase. Let all the γ_j be roots of unity. Now we have $\mu = 5$. If $|\beta|^2$ is of the form (3.6), we have $\overline{|\beta|}^2 > 6$. Therefore, we can suppose that

$$|\beta|^2 = 5 + B\xi^b + \bar{B}\xi^{-b} + C\xi^c + \bar{C}\xi^{-c}$$

where $0 < b < c < \frac{1}{4}p$ (or $\frac{1}{4}p^L$) and furthermore B and C are roots of unity, since otherwise $\sigma^2 \geqq 5$. Once again we consider

$$\begin{aligned} \theta &= |\beta|^2(|\beta|^2 - 5) \\ &= C^2\xi^{2c} + 2BC\xi^{b+c} + B^2\xi^{2b} + 5C\xi^c + 5B\xi^b + 2\bar{B}C\xi^{c-b} + \dots . \end{aligned}$$

If $c \neq 2b$, $3b$ then $\sigma^{*2} \geqq 120$. If $c = 2b$ or $3b$, in the representation of θ there are equal powers of ξ and we have only $\sigma^{*2} \geqq 60$. The assertion is clear since $\mu^* \geqq 4 + 25/4$.

This completes the proof of Lemma 3.1 because we showed at the beginning of the proof that it is enough to consider $X \leqq 5$.

4. Conclusion of the proofs.

Proof of Theorem 1.1. Let β be any cyclotomic integer with $\overline{|\beta|} < R$. Suppose that β does not satisfy any of the conditions I or II of Theorem 1.1. Let β be equivalent to a minimal cyclotomic integer β_0 in $Q(P)$ where P is as small as possible. Then $\overline{|\beta_0|} < R$ and β_0 does not satisfy any of the conditions I or II.

According to Lemma 3.1 there is no prime power $P^N \geqq n_0^3$ $(N \geqq 1)$ dividing P. Hence $P \mid P^*$ for some fixed integer P^* independent of β. It is trivial that β_0 is in $\mathsf{Q}(P^*)$. On the other hand, in $\mathsf{Q}(P^*)$ there are only finitely many integers β_0 satisfying $\overline{|\beta_0|} < R$. Hence the proof is complete.

Finally we are going to show that the values of R and R_1 in Theorems 1.1 and 1.2 are the best possible.

Theorem 4.1. *Theorem* 1.2 *does not hold if* $R_1 = 1+\sqrt{2}$.

Proof. Consider the cyclotomic integers

$$\beta_k = 1+i+\zeta_{2k+3} \quad (k=1,2,3,\ldots) .$$

We have

$$\sqrt{5} < \overline{|\beta_1|} < \overline{|\beta_2|} < \ldots < 1+\sqrt{2}$$

so that the β_k satisfy the condition $\overline{|\beta_k|} < 1+\sqrt{2}$. Furthermore the β_k are inequivalent and they cannot satisfy any of the conditions I or II in Theorem 1.2, since the integers mentioned in these conditions lie with their conjugates in the circle $|z| \leqq \sqrt{5}$.

Also the integers $1+(1+\zeta_7+\zeta_7^3)\varrho$ or $1+(1+\zeta_{30}+\zeta_{30}^{12})\varrho$ mentioned in Theorem 1.1 could be used instead of the integers $1+i+\varrho$ in the proof of Theorem 4.1.

Theorem 4.2. *Theorem* 1.1 *does not hold if* $R=\sqrt{6}$.

Proof. We have to show that there are infinitely many inequivalent cyclotomic integers which do not satisfy any of the conditions I or II in Theorem 1.1.

Let

$$\beta_k = -1+\sqrt{2}\zeta_{2k+4}+\zeta_{2k+4}^2 \quad (k=1,2,3,\ldots) .$$

It is easy to see that

$$1+\sqrt{2} < \overline{|\beta_1|} < \overline{|\beta_2|} < \ldots < \sqrt{6} .$$

Hence the β_k are inequivalent and they do not satisfy the condition II. Moreover, all the conjugates β_k' of the β_k satisfy the inequality

$$|\beta_k'| > \sqrt{2} .$$

We shall show that only finitely many of the β_k can be expressed as the sum of three roots of unity. The truth of Theorem 4.2 follows from this since all the sums of at most three roots of unity can be expressed as the sums of three roots of unity.

Suppose that infinitely many of the β_k are sums of three roots of unity. Then these numbers $\beta_{k'}$ satisfy the equivalence relations

$$\beta_{k'} \sim 1+\zeta_M{}^m+\zeta_N{}^n$$

where $(m,M)=(n,N)=1$, $M \geqq N$. The index M cannot be bounded since the $\beta_{k'}$ are inequivalent. In the following we consider such a $\beta_{k'}$ that the corresponding M is large enough. As $(m,M)=1$, we can find integers x and y satisfying $1=xm+yM$, $(x,M)=1$. Let $r=(M-1)/2$, $M/2-1$ or $M/2-2$ according as M is odd, $M \equiv 0 \bmod 4$ or $M \equiv 2 \bmod 4$. Then r is prime to M and r is about $M/2$. On applying an automorphism mapping ζ_M to $\zeta_M{}^{rx}$ we have

$$\beta_{k'} \sim 1+\zeta_M{}^m+\zeta_N{}^n \sim 1+\zeta_M{}^{rxm}+\varrho = 1+\zeta_M{}^r+\varrho$$

for some root ϱ of unity. As $\zeta_M{}^r$ is about -1 then $\beta_{k'}$ has a conjugate less than $\sqrt{2}$ in absolute value, which is a contradiction.

Theorem 1.1 is perhaps correct for some $R \geqq \sqrt{6}$ if the cyclotomic integers equivalent to the numbers $-1+\sqrt{2}\varrho+\varrho^2$ are excluded. In order to improve Theorem 1.2 it is necessary to exclude the numbers equivalent to $1+i+\varrho$, $1+(1+\zeta_7+\zeta_7{}^3)\varrho$ and $1+(1+\zeta_{30}+\zeta_{30}{}^{12})\varrho$ for some root ϱ of unity. But there may be other exceptional integers to be excluded, too.

REFERENCES

1. J. W. S. Cassels, *On a conjecture of R. M. Robinson about sums of roots of unity*, J. Reine Angew. Math. 238 (1969), 112–131.
2. A. J. Jones, *Sums of three roots of unity*, II, Proc. Cambridge Philos. Soc. 66 (1969), 43–59.
3. R. M. Robinson, *Some conjectures about cyclotomic integers*, Math. Comp. 19 (1965), 210–217.

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