THE NORMAL DECOMPOSITION OF LATTICES AND
THE KRULL-SCHMIDT THEOREM

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The concept of divisibility of modules is essential for many considerations concerning lattices over orders. This concept was introduced by Roiter [3] and can be formulated as follows (see [2, Chapter IX, (4.8), (4,9)]): We say that a \( \Lambda \)-lattice \( M \) covers (divides) a \( \Lambda \)-lattice \( N \) if there is an epimorphism \( M^{(r)} \to N \to 0 \) for some natural number \( r \). We note this by \( M \succ N \). We say that a \( \Lambda \)-lattice \( M \) is normally decomposable if \( M \cong M_1 \oplus M_2 \) where \( M_1, M_2 \) are \( \Lambda \)-lattices, \( M_i \neq 0 \) and \( M_1 \succ M_2 \). We shall consider normal decompositions \( M = \bigoplus M_i \) such that \( M_i \succ M_j \) for \( i < j \) and the lattices \( M_i \) are not normally decomposable (i.e. they are normally indecomposable). It is natural to ask when the normal decompositions are unique in the following sense: if

\[
\bigoplus_{i=1}^r M_i \cong \bigoplus_{j=1}^s N_j
\]

are two normal decompositions with \( M_i, N_j \) normally indecomposable, then \( r = s \) and \( M_i \cong N_j \). This problem was suggested by H. Jacobinski. The aim of the paper is to give an answer to this question.

We know the other important decomposition in the category of lattices over an order — the decomposition of lattices into indecomposable lattices that is, \( M = \bigoplus M_i \) where the lattices \( M_i \) can not be represented as a direct sum of \( \Lambda \)-lattices. We say that the Krull-Schmidt theorem is valid for \( \Lambda \)-lattices if the last decomposition is unique up to isomorphism and permutation of the direct summands. We shall show that if \( \Lambda \) is an order over a Dedekind ring \( R \) in a separable \( K \)-algebra \( A \), where \( K \) is the field of fractions of \( R \), and the Krull-Schmidt theorem is valid for \( \Lambda \)-lattices, then the normal decomposition of lattices over \( \Lambda \) is unique (Theorem 1). At the end of the paper we shall construct two examples: the first shows that the converse implication is not true and the second shows that there are orders for which the normal decomposition is not unique even in the case of the hereditary orders over local rings.

Received February 18, 1975.
We shall use the following notations: \( R \) is a Dedekind ring, \( K \) its field of fractions, \( A \) a separable \( K \)-algebra and \( A \) an \( R \)-order in \( A \). By a \( \Lambda \)-lattice we shall mean always a left finitely-generated \( A \)-module projective over \( R \). We say that \( M \) and \( N \) are normally associated if \( M \gg N \) and \( N \gg M \). \( \Lambda \) is an order with cancelation if \( X \oplus M \cong X \oplus N \) implies \( M \cong N \) for any \( \Lambda \)-lattices \( X, M, N \).

**Proposition 1.** If \( \Lambda \) is an order with cancelation then the normal decomposition of \( \Lambda \)-lattices is unique if and only if two normally indecomposable and normally associated lattices are \( \Lambda \)-isomorphic.

**Proof.** If \( M, N \) are normally indecomposable and normally associated then \( M \oplus N \cong N \oplus M \) implies \( M \cong N \) if the normal decomposition is unique. On the other hand if we have two decompositions:

\[
M_1 \oplus \ldots \oplus M_r \cong N_1 \oplus \ldots \oplus N_s
\]

where \( M_i, N_j \) are normally indecomposable and \( M_i \gg M_{i+1}, N_j \gg N_{j+1} \), then \( M_1 \gg N_1 \) and \( N_1 \gg M_1 \). Hence \( M_1 \cong N_1 \). Since the cancelation is valid for \( \Lambda \)-lattices we can proceed by induction and we get \( r = s, M_i \cong N_i \).

We shall prove now two results which we need to go from the local case to the global case and conversely.

**Proposition 2.** If for every prime ideal \( \mathfrak{p} \) in \( R \) the normal decomposition of the \( \Lambda_\mathfrak{p} \)-lattices is unique then a \( \Lambda \)-lattice \( M \) is normally indecomposable if and only if \( M_\mathfrak{p} \) is normally indecomposable over \( \Lambda_\mathfrak{p} \) for every \( \mathfrak{p} \).

**Proof.** Let \( M \) be a \( \Lambda \)-lattice. We shall assume that \( M \) and all its localizations \( M_\mathfrak{p} \) are contained in a \( K \)-module \( V \). If \( \mathfrak{p} \) is a prime ideal in \( R \) then \( M_\mathfrak{p} = X^\mathfrak{p} \oplus X_1^\mathfrak{p} \) where \( X^\mathfrak{p} \) is a \( \Lambda_\mathfrak{p} \)-lattice normally indecomposable and \( X^\mathfrak{p} \gg M_\mathfrak{p} \). The lattice \( X^\mathfrak{p} \) is uniquely determined by \( M_\mathfrak{p} \) since if \( Y^\mathfrak{p} \) is another lattice with these properties then \( X^\mathfrak{p} \gg Y^\mathfrak{p} \) and \( Y^\mathfrak{p} \gg X^\mathfrak{p} \). Hence by the uniqueness of the normal decomposition over \( \Lambda_\mathfrak{p} \) we get that \( X^\mathfrak{p} \cong Y^\mathfrak{p} \). Let

\[
e_\mathfrak{p} : M_\mathfrak{p} \rightarrow M_\mathfrak{p}
\]

be the idempotent \( \Lambda_\mathfrak{p} \)-homomorphism such that \( X^\mathfrak{p} = M_\mathfrak{p} e_\mathfrak{p} \). Since the homomorphism \( e_\mathfrak{p} \) is defined over some open neighbourhood of \( \mathfrak{p} \in \text{Spec}(R) \) we can choose such neighbourhood \( U \) and a direct summand \( X^U \) of \( M_U = \bigcap_{\mathfrak{q} \in U} M_\mathfrak{q} \) (defined by the extended \( e_\mathfrak{p} \)) which is normally
indecomposable over $\Lambda_U = \bigcap_{q \in U} \Lambda_q$ and covers $M_U$. Since $X^p$ is defined by $M_p$ up to an isomorphism we can define

$$f : \text{Spec}(R) \to Z$$

to be the function such that $f(p) =$ the rank of $X^p$ over $R$. Since this function is continuous it must be constant. Now if $X'$ is a $\Lambda$-lattice such that $X'_q = X^q$ for $q \in U$, then there is only a finite number of points $p$ outside $U$ and we can choose a $\Lambda$-lattice $X$ such that $X'_q = X^q$ for every $q \in \text{Spec}(R)$. The lattice $X$ is normally indecomposable since its localizations have this property. Since $X_p$ is a direct summand of $M_p$ for every prime ideal $p$ there is a direct summand $M'$ of $M$ such that $X$ and $M'$ belong to the same genus over $\Lambda$ (see [4, Corollary 6.13]). Hence $X^{(r)} \cong M'^{(r)}$ for some natural number $r$. Since $X_p \supset M_p$ for every $p$ we have $X \supset M$. Hence $M' \supset M$. Now if $M$ is normally indecomposable over $\Lambda$ then $M \cong M'$ and the isomorphisms $M'_p \cong X'_p$ for every $p$ show that $M_p$ is normally indecomposable for every $p$. The converse implication ($M_p$ normally indecomposable for every $p$ implies $M$ normally indecomposable) is trivial.

**Proposition 3.** (a) If the normal decomposition of lattices is unique for $\Lambda$ then it is unique for $\Lambda_p$ for every prime ideal $p$ in $R$.

(b) If the Krull–Schmidt theorem is valid for $\Lambda$ then it is valid for $\Lambda_p$ for every prime ideal $p$ in $R$.

**Proof.** (a) Let $X_1, \ldots, X_n$ be all simple non-isomorphic $\Lambda$-modules and let $X_i = \Lambda e_i$ where $e_i$ are primitive idempotents in $\Lambda$. Let

$$M_1^p \oplus \cdots \oplus M_r^p \cong N_1^p \oplus \cdots \oplus N_s^p$$

be two normal decompositions over $\Lambda_p$ where $M_i^p, N_j^p$ are normally indecomposable over $\Lambda_p$ and $M_i^p \succ M_{i+1}^p, N_j^p \succ N_{j+1}^p$. If

$$KM_i^p = X_1^{(a_{1i})} \oplus \cdots \oplus X_n^{(a_{ni})}, \quad a_{ik} \geq 0,$$

$$KN_j^p = X_1^{(b_{j1})} \oplus \cdots \oplus X_n^{(b_{jm})}, \quad b_{jk} \geq 0,$$

then we can define a $\Lambda$-lattice $M_i$ in the following way:

$$(M_i)_q = (\Lambda e_1)_q^{(a_{1i})} \oplus \cdots \oplus (\Lambda e_n)_q^{(a_{ni})}$$

for $q \neq p$ and

$$(M_i)_p = M_i^p.$$  

Let $N_j$ be a $\Lambda$-lattice defined analogical with $b_{jk}, N_j^p$ in place of $a_{ik}, M_i^p$. The existence of $M_i, N_j$ follows from the fact that

$$(\Lambda e_1)^{(a_{1i})} \oplus \cdots \oplus (\Lambda e_n)^{(a_{ni})} \quad \text{and} \quad (\Lambda e_1)^{(b_{j1})} \oplus \cdots \oplus (\Lambda e_n)^{(b_{jm})}$$
are \( \Lambda \)-lattices and the modification in a finite number of points of \( \text{Spec}(R) \) (in this case in one point) is possible. Since

\[
K(M_1^p \oplus \ldots \oplus M_r^p) \cong K(N_1^p \oplus \ldots \oplus N_s^p)
\]

we get that

\[
\sum_{i=1}^r a_{ik} = \sum_{j=1}^s b_{jk}
\]

This means that

\[
M = M_1 \oplus \ldots \oplus M_r \quad \text{and} \quad N = N_1 \oplus \ldots \oplus N_s
\]

belong to the same genus over \( \Lambda \) since \( M_q = N_q \) for \( q \neq p \) by the definition and \( M_p = N_p \) by the assumption. This implies that there is a natural \( t \) such that \( M^{(t)} \cong N^{(t)} \).

Now if

\[
X_1^{(a_1)} \oplus \ldots \oplus X_n^{(a_n)} > X_1^{(a_1')} \oplus \ldots \oplus X_n^{(a_n')}
\]

over \( \Lambda \) then \( a_i' \neq 0 \) implies \( a_i = 0 \). Hence \( M_i^p > M_{i+1}^p, N_j^p > N_{j+1}^p \) and the definition of \( M_i, N_j \) imply that \( M_i > M_{i+1} \) and \( N_j > N_{j+1} \) (see [2, Chapter 9, (4.14)]). Moreover \( M_i, N_j \) are normally indecomposable over \( \Lambda \) since \( M_i^p, N_j^p \) have this property over \( \Lambda_p \). The isomorphism \( M^{(t)} \cong N^{(t)} \) implies now that

\[
M_1 \oplus \ldots \oplus M_1 \oplus \ldots \oplus M_r \oplus \ldots \oplus M_r \cong N_1 \oplus \ldots \oplus N_1 \oplus \ldots \\
\oplus N_s \oplus \ldots \oplus N_s
\]

are two normal decompositions of the same module over \( \Lambda \). By the assumption that the normal decompositions are unique we get \( r = s \) and \( M_i \cong N_i \) for \( i = 1, \ldots, r \). Hence \( M_i^p \cong N_i^p \) and the normal decomposition over \( \Lambda_p \) is unique.

(b) Exactly the same idea of the proof applies (but we need not think about the covering relation).

We shall assume now that \( R \) is a discrete valuation ring and \( \Lambda \) is a simple \( K \)-algebra where \( K \) is the field of fractions of \( R \). Let \( \Gamma \) be a maximal \( R \)-order in \( \Lambda \) that contains the order \( \Lambda \). We shall denote by \( R^* \) the completion of \( R \) and by \( A^*, \Gamma^*, \Lambda^* \) the completions of \( A, \Gamma, \Lambda \). If \( X \) is a module over \( A, \Gamma \) or \( \Lambda \) then \( X^* \) will denote its completion as a module over \( A^*, \Gamma^* \) or \( \Lambda^* \) respectively.

If \( l \) is a simple \( \Lambda \)-module and \( A^* = B_1 \oplus \ldots \oplus B_k \), then

\[
l^* = L_1^{(a_1)} \oplus \ldots \oplus L_k^{(a_k)}
\]

where \( B_i \) are simple \( K^* \)-algebras and \( L_i \) is a simple \( B_i \)-module. If \( M \) is a \( \Lambda^* \)-lattice then we shall write

\[
\text{sgn}(M) = (b_1, \ldots, b_k) = (b_i)_{1 \leq i \leq k}
\]
if \( K^* M = L_1^{(b_1)} \oplus \ldots \oplus L_k^{(b_k)} \). \( \text{sgn}(M) \) is called the signature of \( M \) (see [2, Chapter 9, (5.5)]).

If \( X \) is a \( \Lambda^* \)-lattice then there is an uniquely determined (up to an isomorphism) \( \Lambda^* \)-lattice \( c(X) \) such that \( X \oplus c(X) \cong M^* \) where \( M \) is a \( \Lambda \)-lattice and \( c(X) \) is a direct summand in every \( \Lambda^* \)-lattice \( Y \) such that \( X \oplus Y \cong M'^* \) for some \( \Lambda \)-lattice \( M' \) (see [1]).

**Lemma 1.** If the Krull–Schmidt theorem is valid for \( \Lambda \) then we can choose a module \( L_{i_0} \) such that for every \( \Lambda^* \)-lattice \( M \) that does not have a \( \Lambda^* \)-lattice as direct summand

\[
\text{sgn}(M) = (b_1, \ldots, b_k) = (b_j)_{1 \leq j \leq k}
\]

where \( a_{i_0} \mid b_{i_0} \) and \( q_M = b_{i_0}/a_{i_0} = \max([b_j/a_j]) \) ([x] denotes the integral part of x). We can and we shall assume that \( i_0 = 1 \).

**Proof.** Let \( M \) be a \( \Lambda \)-lattice and let \( b_j = a_j q_j + r_j \) where \( 0 \leq r_j < a_j \) and \( q = \max(q_j) \) (\( q_j = [b_j/a_j] \)). If \( t > 0 \) is an integer then

\[
\text{sgn}(M^t) = (tb_j)_{1 \leq j \leq k}
\]

and if \( [b_i/a_i] > [b_j/a_j] \) then \( [tb_i/a_i] > [tb_j/a_j] \). This shows that if

\[
I_t = \{ i : [tb_i/a_i] = \max([tb_j/a_j]) \}
\]

then \( I_t \subseteq I_t \) for \( t | t' \). Since \( I_t \) are finite, non-empty sets and \( I_t \cap I_{t'} \supseteq I_{t'} \), we get that

\[
I = \bigcap_{t=1}^{\infty} I_t
\]

is not empty (one can use also the argument that the inverse limit of finite and non-empty sets is non-empty). Let \( i \in I \), that is,

\[
[tb_i/a_i] = \max([tb_j/a_j]) \quad \text{for every } t > 0 .
\]

We shall show that \( r_i = 0 \), that is, \( a_i \mid b_i \). By Proposition 1 in [1]

\[
c(M^t) = c(M)^t
\]

if \( M \) satisfies the assumptions of the Lemma. It is easy to see that if \( i \in I \) and \( r_i \neq 0 \) then

\[
\text{sgn}(c(M)) = ((q - q_j)a_j + a_j - r_j)_{1 \leq j \leq k}
\]

Let \( t = \prod_{i \in I} a_i \). Then

\[
tb_i = (a_i q_i + r_i)(t/a_i) a_i
\]
and by our assumption \( tb_j/a_j = \max ([tb_j/a_j]) \). This means that all \( i \)-coordinates where \( i \in I \) in \( \text{sgn}(M^0) \) are 0. But (1) shows that

\[
\text{sgn}(c(M^0)) = t \text{sgn}(c(M)).
\]

Therefore if \( r_i \neq 0 \) the \( i \)-coordinate is \( t(a_i - r_i) \neq 0 \) and we get a contradiction. Now let us note that \( I = I_1 \). Indeed, if \( i \in I \) then \( q_i = b_i/a_i = \max(q_j) \). Hence if \( i_1 \in I_1 \), that is, \( q_{i_1} = \max(q_j) \) then

\[
q_{i_1} = [b_{i_1}/a_{i_1}] = q_i
\]

and

\[
tb_{i_1}/a_{i_1} \geq tq_i = \max ([tb_j/a_j])
\]

for every \( t > 0 \). This shows that

\[
[tb_{i_1}/a_{i_1}] = \max ([tb_j/a_j])
\]

for every \( t > 0 \), that is, \( i_1 \in I \). We shall denote \( \max(q_j) \) by \( q_M \).

Now let \( N \) be another \( \Lambda^* \)-lattice that does not have a \( \Gamma^* \)-lattice as direct summand and let \( \text{sgn}(N) = (c_j)_{1 \leq j \leq k} \). By Proposition 1 in [1]

(2)

\[
c(M \oplus N) = c(M) \oplus c(N).
\]

Let \( p = \max ([c_j/a_j]) \) and \( c_j = p_ja_j + s_j \), \( 0 \leq s_j < a_j \). The first part of our considerations shows that

\[
\text{sgn}(c(M)) = ((q - q_j - 1)a_j + a_j - r_j)_{1 \leq j \leq k}
\]

and

\[
\text{sgn}(c(N)) = ((p - p_j - 1)a_j + a_j - s_j)_{1 \leq j \leq k}.
\]

By (2)

(3)

\[
\text{sgn}(c(M \oplus N)) = ((p + q - p_j - q_j - 2)a_j + 2a_j - (r_j + s_j))_{1 \leq j \leq k}.
\]

But

(4)

\[
\text{sgn}(M \oplus N) = ((p_j + q_j)a_j + (r_j + s_j))_{1 \leq j \leq k}.
\]

Let

\[
e_j = \begin{cases} 0 & \text{if } r_j + s_j < a_j, \\ 1 & \text{if } r_j + s_j \geq a_j,
\end{cases}
\]

\[
f_j = \begin{cases} 1 & \text{if } r_j + s_j \leq a_j, \\ 0 & \text{if } r_j + s_j > a_j
\end{cases}
\]

Now (4) and the first part of our considerations show that

\[
\text{sgn}(c(M \oplus N)) = ((t - p_j - q_j - e_j - 1)a_j + (a_j - t_j))_{1 \leq j \leq k}
\]

where \( t = \max (p_j + q_j + e_j) \) and \( t_j = r_j + s_j - e_ja_j \). Hence (3) and (4) give

\[
t - p_j - q_j - e_j - 1 = p + q - p_j - q_j - 2 + f_j,
\]

that is,

(5)

\[
\max(p_j + q_j + e_j) = \max(p_j) + \max(q_j) + (e_j + f_j - 1).
\]
We want to show that there is an \( i_0 \) such that
\[
 p_{i_0} = \max(p_j) \quad \text{and} \quad q_{i_0} = \max(q_j) .
\]
Let us suppose that this is not true, that is, if \( p_{i_0} = \max(p_j) \) and \( q_{i_0'} = \max(q_j) \) then \( i_0 + i_0' \). We have three cases:

— If \( e_j = 0 \) then \( f_j = 1 \) and (5) shows that
\[
 p_j + q_j + e_j = p_j + q_j < p_{i_0} + q_{i_0'} = \max(p_j) + \max(q_j) + (e_j + f_j - 1) .
\]

— If \( e_i = 1 \), \( r_j + s_j = a_j \), \( f_j = 1 \) then
\[
 p_j + q_j + e_j = p_j + q_j + 1 < p_{i_0} + q_{i_0'} + 1 = \max(p_j) + \max(q_j) + (e_j + f_j - 1) .
\]

— If \( e_j = 1 \), \( f_j = 0 \) (that is \( r_j + s_j > a_j \)) then \( r_j + 0 > s_j \), that is, \( p_j < p_{i_0} \), \( q_j < q_{i_0'} \) and
\[
 p_j + q_j + e_j = p_j + q_j + 1 < p_{i_0} + q_{i_0'} = \max(p_j) + \max(q_j) + (e_j + f_j - 1) .
\]

In every case we get a contradiction with (5). Hence there is a common coordinate \( i_0 \) such that \( p_{i_0} = \max(p_j) \) and \( q_{i_0} = \max(q_j) \). Of course we can assume that \( i_0 = 1 \). This proves the Lemma.

**Remark.** If \( A \) is a separable \( K \)-algebra then we can generalize Lemma 1 in the following way. If \( A = A_1 \oplus \ldots \oplus A_r \) where \( A_i \) are simple algebras and \( \Lambda \) is an \( R \)-order in \( A \) such that the Krull–Schmidt theorem is valid for \( A \) then for every \( A^* \)-lattice \( M \) which does not have a \( \Gamma^* \)-lattice as direct summand
\[
 \text{sgn}(M) = (\text{sgn}_1(M), \ldots, \text{sgn}_r(M))
\]
has the properties described in Lemma 1 on every component \( \text{sgn}_p(M) \) (\( \text{sgn}_p(M) \) is defined by the decompositions of \( K^*M \) and \( l_p^* \) over \( A^* \) where \( l_p \) is a simple \( A_p^* \)-module). We shall always assume that the first coordinate in \( \text{sgn}_p(M) \) has the maximal property described in Lemma 1.

**Corollary.** Let \( A \) be an arbitrary separable \( K \)-algebra, \( \Lambda \) an \( R \)-order such that the Krull–Schmidt theorem is valid for \( \Lambda \)-lattices and \( \Gamma \) a maximal \( R \)-order that contains \( \Lambda \). Let \( F \) be an indecomposable \( \Gamma^* \)-lattice such that the only not equal to 0 coordinate in \( \text{sgn}(F) \) is not the first coordinate in any \( \text{sgn}_p(F) \). If \( M \) is a \( \Lambda \)-lattice and \( \text{Hom}_{\Lambda^*}(M, F) = 0 \) then \( M \succ F \).

**Proof.** If \( f : M \to F \) and \( f \neq 0 \) then \( f(M) \) is an indecomposable \( \Lambda^* \)-lattice and by Lemma 1 it must be a \( \Gamma^* \)-lattice since \( \text{sgn}(f(M)) = \text{sgn}(F) \). Since \( F \) is \( \Gamma^* \)-indecomposable and \( \Gamma^* \) is maximal we get \( f(M) \simeq F \). Hence there is an epimorphism of \( M \) onto \( F \).
We shall prove now the main result of the paper.

**Theorem 1.** Let $A$ be an order in a separable $K$-algebra $A$, where $R$ is a Dedekind ring and $K$ is the field of fractions of $R$. If the Krull–Schmidt theorem is valid for $A$ then the normal decomposition of the lattices over $A$ is unique.

**Proof.** Let $R$ be a discrete valuation ring and let $\Gamma$ be a maximal order that contains $A$. Let $M$ be a normally indecomposable $A$-lattice and let

$$M \cong M_1 \oplus \ldots \oplus M_t \oplus F_1 \oplus \ldots \oplus F_s$$

where $M_i$ are indecomposable $A$-lattices which are not $\Gamma$-lattices and $F_j$ are indecomposable $\Gamma$-lattices. By the proof of Proposition 1 in [1]

(6) $$M_i^* \cong X_1 \oplus c(X_i)$$

where $X_1$ is an indecomposable $A^*$-lattice which is not a $\Gamma^*$-lattice and $c(X_i)$ is a $\Gamma^*$-lattice. Let

$$M^* \cong X \oplus Y$$

where $X$ is a $\Lambda^*$-lattice normally indecomposable and $X \succ Y$. We shall show that all $X_i$, $i=1,\ldots,r$, are direct summands of $X$.

We can assume that

$$X = X_1 \oplus \ldots \oplus X_t \oplus X'$$

where $X'$ is a $\Gamma^*$-lattice and $X \succ X_i$ for $i=1,\ldots,t$. Between the indecomposable $\Gamma^*$-lattices which are direct summands of $X'$ may appear the lattices of three types:

- direct summands of $c(X_i)$ for $i=1,\ldots,t$,
- direct summands of $F_j^*$ for $j=1,\ldots,s$,
- direct summands of $c(X_i)$ for $i=1,\ldots,t$.

Since $X \succ M^* \succ M_i^*$ we can complete $X$ by some $\Gamma^*$-lattices of the first two types in the way such that

(*) $$M_1^* \oplus \ldots \oplus M_t^* \oplus F_1^* \oplus \ldots \oplus F_s^* \oplus X'' > M_i^*$$

where $X''$ is a direct summand of $X'$ and a direct sum of $\Gamma^*$-lattices of the third type which are not of the first two types. We shall show that $X''=0$. Let $Z$ be an indecomposable direct summand of $X''$ that is a direct summand in $c(X_i)$ for some $i=1,\ldots,t$. sgn$(Z)$ has exactly one coordinate equal to 1 (the only coordinate that is not 0). If this coordinate is in sgn$_p(Z)$ (see Remark), then it is not the first coordinate by Lemma 1 (Z completes $X_i$ and the first coordinate in sgn$_p(X_i)$ is maxi-
mal). The relation (*) shows that the first coordinate in $\text{sgn}_p(M_i^*)$ for some $i = 1, \ldots, t$ or in $\text{sgn}_p(F_j^*)$ for some $j = 1, \ldots, s$ is not 0 (we can look at this relation after the tensoring with $K^*$). But we must have the first possibility since $Z$ is not a direct summand of $F_j^*$ for $j = 1, \ldots, s$. Hence $\text{sgn}_p(M_i^*)$ for some $i = 1, \ldots, t$ and $\text{sgn}_p(Z)$ have a common coordinate $\neq 0$. This shows that $\text{Hom}_{\mathcal{A}^*}(M_i^*, Z) \neq 0$ and by the Corollary $M_i^* \succ Z$. Hence
\[ M_1^* \oplus \cdots \oplus M_t^* \oplus F_1^* \oplus \cdots \oplus F_s^* \succ M_i^*. \]

This contradicts the fact that $M$ is normally indecomposable over $\mathcal{A}$. Hence if $M$ is an indecomposable $\mathcal{A}$-lattice such that
\[ M^* = X \oplus Y \]
where $X$ is normally indecomposable and $X \succ Y$ then all $\mathcal{A}^*$-indecomposable direct summands of $M^*$ that are not $\mathcal{I}^*$-lattices are direct summands of $X$.

Let $M, N$ be two normally indecomposable and normally associated $\mathcal{A}$-lattices. To prove that the normal decomposition of $\mathcal{A}$-lattices is unique we have to prove that $M \simeq N$ (Proposition 1). By the assumption that $M$ and $N$ cover each other if one of these lattices is a $\mathcal{I}$-lattice then the other is too. In such a case the exact sequence
\[ M^0 \rightarrow N \rightarrow 0 \]
splits and by the Krull–Schmidt theorem (over $\mathcal{A}$ or $\mathcal{I}$) we get $M \simeq N$. Let us suppose that $M$ and $N$ are not lattices over $\mathcal{I}$. By Proposition 5 in [3] there is an $\mathcal{A}^*$-lattice $X$ such that
\[ M^* \simeq X \oplus Y_M, \quad N^* \simeq X \oplus Y_N \]
where $X$ is normally indecomposable over $\mathcal{A}^*$ and $X \succ Y_M, X \succ Y_N$. Let
\[ M^* \simeq M_A^* \oplus M_\mathcal{I}^*, \quad N^* \simeq N_A^* \oplus N_\mathcal{I}^* \]
where $M_A, N_A$ have not direct summands which are $\mathcal{I}$-lattices and $M_\mathcal{I}, N_\mathcal{I}$ are $\mathcal{I}$-lattices. Now if $M_i$ is an indecomposable direct summand of $M_A$ and
\[ M_i^* \simeq X_i \oplus c(X_i) \]
where $X_i$ is $\mathcal{A}^*$-indecomposable and $c(X_i)$ is a $\mathcal{I}^*$-lattice (see (6)), then $X_i$ is a direct summand of $X$ by the first part of the proof. The Krull–Schmidt theorem over $\mathcal{A}^*$ and the fact that $c(X_i)$ is uniquely determined by $X_i$ give that $M_i$ is isomorphic with a direct summand of $N_A$. Hence every direct summand of $M_A$ is isomorphic with some direct summand
of \( N_A \) (the direct summands \( M_i \) of \( M \) are not isomorphic since \( M \) is normally indecomposable). By symmetry we get \( M_A \cong N_A \). Now if \( F \) is an indecomposable \( \Gamma \)-lattice that is a direct summand of \( M_i \), and there is no direct summand of \( N \) which is isomorphic with \( F \), then

\[
\text{Hom}_A(N_F,F) = 0
\]

and the relation \( N > M \) implies that \( N_A > F \). Therefore \( M_A > F \) (since \( M_A \cong N_A \)) and we get a contradiction. Hence every \( \Gamma \)-indecomposable direct summand of \( M \) is isomorphic with a direct summand of \( N \) and conversely. This shows that also \( M_F \cong N_F \). Hence \( M \cong N \).

Now let \( R \) be an arbitrary Dedekind ring. By Proposition 3 the Krull–Schmidt theorem is valid for every \( R_p \)-order \( A_p \), where \( p \) is a prime ideal in \( R \). The first part of the proof gives that the normal decomposition of \( A_p \)-lattices is unique for every \( p \). Hence if \( M, N \) are normally indecomposable and normally associated \( A \)-lattices then \( M_p, N_p \) are normally indecomposable (Proposition 2) and normally associated for every \( p \). By the uniqueness of the normal decomposition over \( A_p \) we get \( M_p \cong N_p \) for every \( p \). Hence \( M \) and \( N \) belong to the same genus over \( A \). Since the Krull–Schmidt theorem is valid for \( A \)-lattices this implies that \( M \cong N \) and the proof is complete.

We shall construct two examples:

1) An order such that the Krull–Schmidt theorem is not valid for it but the normal decomposition of lattices is unique.

2) A hereditary order over a discrete valuation ring such that the normal decomposition of lattices over it is not unique.

In both examples we use the considerations of Roggenkamp in [2, Chapter IX, (2.29)]. Let \( R \) be a discrete valuation ring and \( A = D \) a finite dimensional central skewfield over \( K \), where \( K \) is the field of fractions of \( R \). Let \( R^* \) be the completion of \( R \) and \( D^* = R^* \otimes_R D = (K^*)_r \), where \( K^* = R^* \otimes_R K \) (that is, \( K^* \) is a splitting field of \( D \)). Let \( A \) be a non-maximal hereditary \( R \)-order in \( D \). We shall show that if \( r = 2 \) then \( A \) has the properties mentioned in 1) and if \( r = 3 \) those mentioned in 2).

**Example 1.** That the Krull–Schmidt theorem is not valid for \( A \) follows from [2, Chapter IX, (2.29)]. We shall show that the normal decomposition of \( A \)-lattices is unique. Since \( A^* \) is hereditary but not maximal there are exact two classes of non-isomorphic and indecomposable \( A^* \)-lattices \( (r = 2) \). Let \( M_1^*, M_2^* \) represent these classes. Let \( M \) be a \( A \)-in-

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decomposable lattice. The completion $M^*$ is a $\Lambda^*$-lattice and is isomorphic
with a direct sum of some number of $M_1^*$ and $M_2^*$. But there are $\Lambda$-indecomposable
lattices $M_{11}, M_{12}, M_{22}$ such that

$$M_{11}^* \simeq M_1^* \oplus M_1^*, \quad M_{12}^* \simeq M_1^* \oplus M_2^*, \quad M_{22}^* \simeq M_2^* \oplus M_2^*$$

and we get that there are only three classes of $\Lambda$-indecomposable lattices. But at the same
time the lattices $M_{11}, M_{12}, M_{22}$ represent all classes of normally indecomposable lattices. To show this let $M$ be a normally
indecomposable $\Lambda$-lattice. Then $M$ is a direct sum of $\Lambda$-indecomposable
lattices and any two of direct summands of $M$ can be isomorphic. But

$$M_{12} \succ M_{11}, \quad M_{12} \succ M_{22}, \quad M_{11} \oplus M_{22} \simeq M_{12} \oplus M_{12}$$

since this is true for completions. This means that $M$ can not be a direct
sum of two (or more) indecomposable lattices. Now any of $M_{11}, M_{22}$
covers the other and they do not cover $M_{12}$ (since $\Lambda^*$ is hereditary
and the Krull–Schmidt theorem is valid for $\Lambda^*$) and by Proposition 1 we
get that the normal decomposition over $\Lambda$ is unique.

Example 2. The idea of this example is the same as in [2, Chapter IX,
(2.29)]. Let $M_1^*, M_2^*$ be two non-isomorphic and indecomposable $\Lambda^*$-
lattices. Then there are $\Lambda$-indecomposable (hence $\Lambda$-normally indecom-
posable) $\Lambda$-lattices $N_1, N_2, N_3$ such that

$$N_1^* \simeq M_1^* \oplus M_1^* \oplus M_2^*, \quad N_2^* \simeq M_1^* \oplus M_1^* \oplus M_1^*, \quad N_3^* \simeq M_1^* \oplus M_2^* \oplus M_2^*.$$  

We have

$$N_1 \oplus N_1 \simeq N_3 \oplus N_2$$

where $N_1 \succ N_1$ and $N_3 \succ N_2$ (since this is true for completions). But
$N_1 \cong N_3$ and we get two various normal decompositions of the same latt-

cice over $\Lambda$.

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