DETERMINATION OF A CLASS OF POINCARÉ SERIES

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1. Introduction.

CONVENTION. A ring is a ring with unit element, a local ring is a commutative Noetherian local ring.

In his proof of the fact that a local ring $Q$ is regular if and only if $\text{gl.dim.} Q < \infty$ (in [7]), Serre showed that the Poincaré series of a regular local ring is $(1 + Z)^n$ where $n$ is the Krull dimension of $Q$, the Poincaré series of a local ring $(Q, m)$ being defined as the formal power series

$$B^Q(Z) = \sum_{i \geq 0} b_i Z^i$$

with $b_i = \dim_k \text{Tor}_i^Q(k, k)$, $k = Q/m$. This was done by using the Koszul complex for $Q$. Since then it has been shown for certain other classes of local rings that the Poincaré series is rational. The most important of these classes are:

1. Complete intersections (in [12]).
2. Golod rings (defined by the vanishing of certain homology operators on the Koszul complex of the ring) (see [1]).
   (These two classes cover the case, where $m$ is generated by two elements ([6]).)
3. Regular local rings modulo an ideal generated by two elements ([9], [2]).

Other results are given in [13], [10], [11] and [4]. Most of the results mentioned can be found in [3], which we also give as a general reference on the subject.

In all cases we know of, the Poincaré series has the form $(1 + Z)^n/p(Z)$ where $p(Z)$ is a polynomial with integer coefficients and $n$ the embedding dimension of $Q$. In case 1 above $p(Z) = (1 - Z^2)^{n-m}$ ($m$ the Krull dimension of $Q$), in case 2

$$p(Z) = 1 - \sum_{i \geq 1} c_i Z^{i+1}$$

($c_i = \dim_k H_i(K)$, $K$ the Koszul complex of $Q$) and in case 3 $p(Z) = 1 - 2Z^2 - Z^3$ if $Q$ is not a complete intersection.

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The purpose of this paper is to show the rationality of the Poincaré series for the quotient of a regular local ring modulo an ideal, generated by any set of monomials of degree two in a minimal system of generators for the maximal ideal. For such rings we obtain a formula for the Poincaré series of the form \((1+Z)^n/p(Z)\) with

\[ p(Z) = 1 + \sum c_{ij}(-1)^jZ^{i+j}, \]

where \(c_{ij} = \dim_k H^{i+j}_i(K)\), the upper index indicating a graduation on \(H(K)\), induced by a filtration of the Koszul complex \(K\) to \(Q\) (see Section 5).

Our method consists in constructing a minimal \(Q\)-resolution of \(k = Q/m\), by means of a generalization of the Koszul complex. This resolution is closely related to a \(G(Q)\)-resolution of \(K\) where \(G(Q)\) is the graded ring associated with the \(m\)-filtration of \(Q\). \(G(Q)\) is a commutative \(k\)-algebra, but the proper scope of our method is a certain class of non-commutative \(k\)-algebras. So we shall start by studying such rings in Section 2, deriving the "associated" resolutions in Section 3 and determine their "Poincaré series" in Section 4. Finally, in Section 5, we pass to local rings.

2. Certain quotient rings of polynomial rings in non-commuting variables.

Notations. If \(R\) is a ring (commutative or not), we let \(R\langle X_1, \ldots, X_n \rangle\) denote the polynomial ring in \(n\) non-commuting variables over \(R\) (the variables commute with the elements of \(R\)).

The rings we are going to study in this section are quotients of

\[ K\langle X_1, \ldots, X_n \rangle \langle Y_1, \ldots, Y_n \rangle \]

\((K\) a commutative field) modulo certain systems of \(n^2\) relations, the images of \(X_i\) and \(Y_i\) being denoted \(x_i\) and \(y_i\) respectively. The systems of relations considered can be described as follows:

1. For every \(i\) \((1 \leq i \leq n)\) one of the relations \(X_i^2 = 0\) and \(Y_i^2 = 0\)

For every pair \((i, k)\) where \(i \neq k\) one of the following five types of relations, where \(c_{ik} \in K - \{0\}\):

2. \(X_iX_k + c_{ik}X_kX_i = Y_iY_k - c_{ik}Y_kY_i = 0\)

3. \(X_iX_k = X_kX_i = 0\)
\[ Y_i Y_k = Y_k Y_i = 0 \]
\[ X_i X_k = Y_k Y_i = 0 \]
\[ X_k X_i = Y_i Y_k = 0 \]

In case (2) we say that the pair \((i, k)\) commutes.

On a ring \(R\) of the described type, we will define \(K\)-linear maps \(d_i\) and \(D_i\) \((i = 1, \ldots, n)\). We call an element
\[ x_{i_1} \cdots x_{i_r} y_{j_1} \cdots y_{j_s} = x^{(\omega)} y^{(\nu)} = m \neq 0 \]
in \(R\) a monomial. It suffices to define \(d_i\) and \(D_i\) on monomials in \(R\), and then extend them \(K\)-linearly to \(R\). We say that \(x_i\) can be factored out to the right in a monomial \(m\) (respectively \(y_i\) can be factored out to the left in \(m\)) if
\[ m = c_i x^{(\omega)} x_i y^{(\nu)} \text{ for some } c_i \in K \]
(respectively \(m = c_i' x^{(\omega)} y_i y^{(\nu)} \text{ for some } c_i' \in K\)).

**Definitions.**

\[ D_i(m) = c_i x^{(\omega)} y_i y^{(\nu)} \text{ if } m = x^{(\omega)} y^{(\nu)} = c_i x^{(\omega)} x_i y^{(\nu)}, \]
\[ D_i(m) = 0 \text{ if } x_i \text{ cannot be factored out to the right in } m. \]
\[ d_i(m) = c_i' x^{(\omega)} x_i y^{(\nu)} \text{ if } m = x^{(\omega)} y^{(\nu)} = c_i' x^{(\omega)} y_i y^{(\nu)}, \]
\[ d_i(m) = 0 \text{ if } y_i \text{ cannot be factored out to the left in } m. \]

**Lemma 1.**

\(d_i^2 = 0\).

\(d_i d_k + d_k d_i = 0 \quad \text{if } i \neq k.\)

\(d_i D_k + D_k d_i = 0 \quad \text{if } i \neq k.\)

(10) If \(m\) is a monomial with \(d_i(m) \neq 0\), then \(D_i(m) = 0\).

(11) If \(m\) is a monomial with \(d_i(m) \neq 0\), then \(D_i d_i(m) = m\).

(For symmetry reasons the analogues of the above statements, obtained by interchanging \(d_i\) and \(D_i\) are also true.)

**Proof.** (7) follows directly from (1). To prove (8), let \(m\) be a monomial. If none of \(y_i\) and \(y_k\) can be factored out to the left in \(m\), then \(d_i(m) = \ldots \)
$d_k(m) = 0$, and thus (8) follows in this case. Now suppose that $y_i$ but not $y_k$ can be factored out to the left in $m$, so $d_k(m) = 0$. If now $y_k$ cannot be factored out to the left in $d_i(m)$, we have $d_kd_i(m) = 0$. If $y_k$ can be factored out to the left in $d_i(m)$ one of the relations (3) or (5) must hold for $(i, k)$, but then $x_i x_k = 0$ which gives $d_kd_i(m) = 0$, so (8) follows also in this case. Finally, if both $y_i$ and $y_k$ can be factored out to the left in $m$, relation (2) must hold for $(i, k)$, and that relation implies $d_kd_i(m) = -d_i d_k(m)$, so (8) is proved in full. (9) is proved similarly, (10) and (11) are fairly obvious from the definitions.

**Definition.** If $m$ is a monomial in $R$, we let

$$\text{Index}(m) = \{i \mid d_i(m) \neq 0 \text{ or } D_i(m) = 0\}.$$ 

**Note.** Index(1) is always empty but Index($m$) could be empty also if $m$ is a monomial of positive degree. An example is: Let $n = 4$, only (2, 3) commutes, $X_1X_3 = X_3X_4 = Y_1Y_2 = Y_2Y_4 = 0$ (the other relations chosen arbitrarily). Then Index($x_1x_2y_3y_4$) is empty. But in the cases of interest for us, Index($m$) is always nonempty for monomials of positive degree (see Lemma 4 below).

It is now easy to prove that $(d_iD_i + D_i d_i)(m) = n_m m$ for some integer $n_m > 0$ if every monomial of positive degree has non-empty Index. This would be sufficient for our needs if char($K$) = 0 (see the Theorem in next section), but to cover also the case char($K$) > 0, we must argue further.

**Lemma 2.** If $i, k \in \text{Index}(m)$, then $(i, k)$ commutes.

**Proof.** If $d_i(m) \neq 0$ and $d_k(m) \neq 0$ (or if $D_i(m) \neq 0$ and $D_k(m) \neq 0$), $(i, k)$ must commute, so we can assume that $d_i(m) \neq 0$ and $D_i(m) \neq 0$. But then

$$m = c x^{(a)} x_i y_k y^{(a)}.$$ 

Now if $(i, k)$ does not commute, either $x_i x_k = 0$ or $y_i y_k = 0$, which contradicts the assumption.

**Lemma 3.** If $d_k(m) \neq 0$, then Index($d_k(m)$) = Index($m$).

If $D_k(m') \neq 0$, then Index($D_k(m')$) = Index($m'$).

**Proof.** It suffices to show the first statement. Since $d_k(m) \neq 0$, $D_k d_k(m) = m \neq 0$ so $k \in \text{Index}(d_k(m))$. If $i \neq k$ and $i, k \in \text{Index}(m)$, then $i \in \text{Index}(d_k(m))$ since $(i, k)$ commutes according to Lemma 2. But then

$$\text{Index}(m) \subset \text{Index}(d_k(m)) \subset \text{Index}(D_k d_k(m)) = \text{Index}(m).$$
DEFINITIONS.

\[ i(m) = \min(\text{Index}(m)) \quad \text{and} \quad S(m) = D_{i(m)}(m) \]

where we assume \( \text{Index}(m) \) to be non-empty.

THEOREM.

(12) \quad \text{If} \ d = d_1 + \ldots + d_n \ \text{then} \ d^2 = 0 \\
and (13) \quad (Sd + dS)(m) = m \\
if \( m \) is a monomial with \( \text{Index}(m) \) non-empty.

PROOF. (12) follows from (7) and (8). On a monomial \( m \), with \( \text{Index}(m) \) non-empty, we have

\[ Sd + dS = D_{i(m)}d + dD_{i(m)} = D_{i(m)}(d_1 + \ldots + d_n) + \\
+ (d_1 + \ldots + d_n)D_{i(m)} = D_{i(m)}d_{i(m)} + d_{i(m)}D_{i(m)} = 1 \]

according to (9) and (11).

We single out the types of rings of special interest for us in the following lemma.

LEMMA 4. Every monomial of positive degree has non-empty Index if:

A'. No pair is commutative.

B'. For some pairs \( i + k \), \( (i, k) \) commutes, and for all other pairs \( (i + k) \)

\[ X_iX_k = X_kX_i = 0. \]

PROOF. A'. If \( m = x^{a_i}x_iy_ky^{a_k} \), either \( d_k(m) \neq 0 \) or \( D_i(m) \neq 0 \).

B'. If \( m = y_ky^{a_k} \), then \( d_k(m) = 0 \). If \( m \) contains \( x_i \) and if \( D_i(m) = 0 \), we must have \( y_i^2 = 0 \), but then \( x_i^2 = 0 \) so \( d_i(m) \neq 0 \).

3. Resolutions.

We shall construct resolutions of the residue class field \( K \) for rings of the following two types:

A. \( K\langle X_1, \ldots, X_n \rangle/I \) where \( K \) is a commutative field and \( I \) is an ideal generated by any set of monomials of degree 2 in \( \{X_i\} \).

B. \( K\langle X_1, \ldots, X_n \rangle/I \) where \( K \) is a commutative field and \( I \) is an ideal generated by any set of monomials of degree 2 in \( \{X_i\} \) and furthermore by one element \( X_iX_k + c_{ki}X_kX_i \) for each \( i \neq k \), where \( c_{ki} \in K - \{0\} \).

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If $R$ is a ring of type A (respectively B), there is exactly one ring $R'$ of type A' (respectively B') (definition in Lemma 4 in Section 2), with the same number of $X$-variables and the same relations between these variables. This is so, because the relations between the $X$-variables determine the relations between the $Y$-variables in the rings examined in Section 2. We shall say that $R'$ belongs to $R$. Now $R'$ is in a natural way a graded algebra over $R$,

$$R' = \bigoplus_{i \geq 0} R'_i$$

($R'_i$ consists of the homogenous elements of degree $i$ in \{y_i\}). Clearly $d$ becomes a homogenous $R$-linear map of degree $-1$.

**Theorem.** If $R$ is a ring of type A (respectively B) and $R' \bigoplus_{i \geq 0} R'_i$ the ring of type A' (respectively B') belonging to $R$, then

$$\cdots \xrightarrow{d} R'_i \xrightarrow{d} \cdots \xrightarrow{d} R'_1 \xrightarrow{d} R \xrightarrow{n} K \rightarrow 0$$

is a resolution of $K$ ($n$ is the natural map.)

**Proof.** Lemma 4 shows that every monomial in $R'_i$ ($i > 0$) has non-empty Index. The Theorem preceding Lemma 4 then gives that we have a complex with a chain homotopy, i.e. an exact sequence ($n$ is obviously an augmentation map).

We illustrate the connection between $R$ and $R'$ with some simple examples, which in particular shed light on the obvious duality between the $X$- and $Y$-variables.

1a. If $R = K\langle X_1, \ldots, X_n \rangle$, then $R' = R\langle Y_1, \ldots, Y_n \rangle/M^2$ where $M = (Y_1, \ldots, Y_n)$.

1b. If $R = K\langle X_1, \ldots, X_n \rangle/m^2$ where $m = (X_1, \ldots, X_n)$, then $R' = R\langle Y_1, \ldots, Y_n \rangle$.

2a. If

$$R = K[X_1, \ldots, X_n] = K\langle X_1, \ldots, X_n \rangle/(\{X_iX_j - X_jX_i ; i \neq j\})$$

then

$$R' = R\langle Y_1, \ldots, Y_n \rangle/(\{Y_i^2, \{Y_iY_j + Y_jY_i ; i \neq j\})$$

(the Koszul complex of $R$).

2b. If

$$R = K\langle X_1, \ldots, X_n \rangle/(\{X_i^2, \{X_iX_j + X_jX_i ; i \neq j\})$$

($R$ "Koszul") then $R' = R[Y_1, \ldots, Y_n]$. 


NOTE. If \( R = \oplus_{i \geq 0} R_i \) is a graded ring with \( R_0 \) a field, then gl. dim. \( R = \text{h.d.} R_0 \) (see [5]). So a ring \( R \) of type \( A \) will have finite global dimension if \( R' \) has finite rank as an algebra over \( R \), i.e. if \( (y_1, \ldots, y_n) \) is nilpotent in \( R' \).

4. "Poincaré series".

**Notations.** If \( R' = \oplus_{i \geq 0} R'_i \) is a graded algebra over a ring \( R \), with \( R'_i \) free \( R \)-modules, we let

\[
H_R^{R'} = \sum_{i \geq 0} h_i Z^i, \quad \text{where} \ h_i = \text{rank}_R R'_i.
\]

If \( R = K\langle X_1, \ldots, X_n \rangle/\bar{I} \) is a ring of type \( A \), we study \( \bar{R} = K\langle X_1, \ldots, X_n \rangle/\bar{I} \), where \( \bar{I} \) is generated by monomials of degree two, such that \( X_i X_j \in \bar{I} \) if and only if \( X_i X_j \notin I \). We call \( \bar{R} \) the complement \( K \)-algebra to \( R \).

**Theorem.** If \( R \) is a ring of type \( A \), \( \bar{R} \) the complement \( K \)-algebra to \( R \), then

\[
H_K^R(-Z) \cdot H_K^{\bar{R}}(Z) = 1.
\]

**Proof.** Let

\[
H_K^R(Z) = \sum_{i \geq 0} h_i Z^i \quad \text{and} \quad H_K^{\bar{R}}(Z) = \sum_{i \geq 0} \bar{h}_i Z^i.
\]

We shall show, that \( \bar{h}_0 \cdot h_0 = 1 \) (clear) and that

\[
\bar{h}_n \cdot h_0 - \bar{h}_{n-1} \cdot h_1 + \ldots + (-1)^n \bar{h}_0 \cdot h_n = 0 \quad \text{if} \ n > 0.
\]

Now \( \bar{h}_r \) is the number of different monomials of degree \( r \) in \( \{X_i\} \), such that their images in \( \bar{R} \) is \( \pm 0 \). We say that a monomial \( X_{i_1} \ldots X_{i_n} \) can be divided after \( X_{i_r} \), if \( X_{i_1} \cdot \ldots \cdot X_{i_r} \) has an image \( \pm 0 \) in \( \bar{R} \), at the same time as \( X_{i_{r+1}} \cdot \ldots \cdot X_{i_n} \) has an image \( \pm 0 \) in \( R \). If a monomial can be divided in this sense at some place, then it can be divided at exactly two places (besides after \( X_{i_r} \) also after \( X_{i_{r-1}} \) or \( X_{i_{r+1}} \), depending on whether \( X_{i_r} X_{i_{r+1}} \) has an image \( \pm 0 \) in \( R \) or in \( \bar{R} \)). This shows, that there are just as many monomials, that can be divided after an even number of steps, as after an odd number of steps. As the total number of monomials of degree \( n \), that can be divided after \( r \) steps is \( \bar{h}_r \cdot h_{n-r} \), this shows that

\[
\bar{h}_n \cdot h_0 + \bar{h}_{n-2} \cdot h_2 + \ldots = \bar{h}_{n-1} \cdot h_1 + \bar{h}_{n-3} \cdot h_3 + \ldots
\]
Corollary 1. If $R$ is a ring of type A, $R'$ the ring of type A' belonging to $R$, then
\[ H^R_K(-Z) \cdot H^{R'}_R(Z) = 1. \]

Proof. $H^{R'}_R(Z) = H^R_R(Z)$, where $R$ is the complement $K$-algebra to $R$.

Corollary 2. If $R$ is a ring of type B, $R'$ the ring of type B' belonging to $R$, then
\[ H^R_K(-Z) \cdot H^{R'}_R(Z) = 1. \]

Proof. Let
\[ R = K\langle X_1, \ldots, X_n \rangle/I \quad \text{and} \quad R' = R\langle Y_1, \ldots, Y_n \rangle/I'. \]

Now $h_i = \text{rank}_{R_i}^R R'$ is the maximum number of linearly independent elements in $R_i$. But since $R_i'$ can be generated by monomials in $\{y_i\}$, $h_i$ is the maximum number of linearly independent monomials of degree $i$ in $R_i$, that is the number of different monomials in $R_i$, when represented in lowest possible lexicographical way. So $H^{R'}_R(Z) = H^S_R(Z)$, where $S = R\langle Y_1, \ldots, Y_n \rangle/J$, and $J$ is generated by the following monomials of degree two:
\[ \{Y_i Y_j; \ i > j \text{ and } (i,j) \text{ commutes} \} \quad \text{and} \quad \{Y^2_i; \ Y^2_i \in I' \}. \]

It is clear that $H^S_R(Z) = H^T_K(Z)$, where $T = K\langle Y_1, \ldots, Y_n \rangle/L$, and $L$ is generated by "the same" monomials as $J$. Representing the monomials in $R$ in the highest possible lexicographical way, one sees that $H^R_K(Z) = H^U_K(Z)$, where $U = K\langle X_1, \ldots, X_n \rangle/N$, $N$ generated by the following monomials of degree two:
\[ \{X_i X_j; \ i < j \} \quad \text{and} \quad \{X_i X_j; \ i \geq j \text{ and } X_i X_j \in I \}. \]

But then $y_i^2 = 0$ in $T$ if and only if $x_i^2 = 0$ in $U$, and $y_i y_j = 0$ in $T(i \neq j)$ if and only if $x_i x_j = 0$ in $U$. Hence $U$ is the complement $K$-algebra to $T$, and the result follows from the Theorem.

5. Local rings.

The Poincaré series for a local ring $Q$ with residue class field $k$ is defined as
\[ B^Q_k(Z) = \sum_{i \geq 0} b_i Z^i, \]
where $b_i = \dim_k \text{Tor}_i^Q(k,k)$. To determine $b_i$, one can construct a minimal resolution of $k$. A resolution
\[ \ldots Q^{d_2} \overset{d_2}{\rightarrow} Q^{d_1} \overset{d_1}{\rightarrow} Q \overset{d_0}{\rightarrow} k \rightarrow 0 \]
is minimal, if \( c_i \) has been chosen minimally, which means that
\[
d_i \otimes \text{id}_k : \mathbb{Q}^{c_i} \otimes k \to \text{Im} d_i \otimes k
\]
is an isomorphism for every \( i \) (Nakayama's lemma). Then \( c_i = b_i \). See e.g. [8].

We shall determine the Poincaré series for a ring \( Q \) of the following type: \( Q = \overline{Q}/\overline{I} \), where \( \overline{Q} \) is a regular local ring and \( \overline{I} \) is generated by a set of monomials of degree two in a minimal set of generators for the maximal ideal \( \overline{m} \) in \( \overline{Q} \). The associated graded ring
\[
R = G(Q) = \bigoplus_{i \geq 0} m^i/m^{i+1}
\]
to \( Q \) (\( m \) the maximal ideal of \( Q \)) is then a ring of type B. Let
\[
R' = R\langle Y_1, \ldots, Y_n \rangle/J'
\]
be the ring of type B' belonging to \( R \), and let
\[
Q' = Q\langle Y_1, \ldots, Y_n \rangle/I',
\]
where \( I' \) is generated by "the same" elements as \( J' \). On \( Q' = \bigoplus_{i \geq 0} Q'_i \) (graded by the degree in \( \{y_i\} \)), we define a map \( \partial \), in the same way as the differential \( d \) is defined at \( R' \).

**Theorem.**

\[
\ldots \xrightarrow{\partial} Q'_i \xrightarrow{\partial} \ldots \xrightarrow{\partial} Q'_1 \xrightarrow{\partial} Q \xrightarrow{n} k \rightarrow 0
\]
is a minimal resolution of \( k = Q/m \) (\( n \) is the natural quotient map).

**Proof.** That \( \partial^2 = 0 \) is clear. Now let \( \partial(\Sigma \overline{d}_{i} m_{i,r}) = 0 \), where \( q_i \in Q \) and \( m_{i,r} \) are monomials in \( \{y_i\} \) of some degree \( r \). Let \( \overline{q}_i \) be the images of \( q_i \) in \( Q/m \). Then \( \overline{d}(\Sigma \overline{q}_i m_{i,r}) = 0 \) in \( R' \) (we identify the monomials in \( \{y_i\} \) in \( Q \) and \( G(Q) \)). Since \( (R',d) \) is a resolution of \( k \), we have \( \Sigma \overline{q}_i m_{i,r} = \overline{d}(r') \) for some \( r' \in R'_r \). Let \( q' \) be an inverse image of \( r' \) in \( Q'_r \). Then \( \Sigma \overline{q}_i m_{i,r} - \partial(r') \in Q_r \), so
\[
\text{Ker} \partial \subset \text{Im} \partial + mQ'.
\]
In the same way we see that \( \text{Ker} \partial \subset \text{Im} \partial + m^nQ' \) for all \( n \). But then
\[
\text{Ker} \partial \subset \bigcap_{n=1}^{\infty} (\text{Im} \partial + m^nQ') = \text{Im} \partial,
\]
the last equality by Artin-Rees' lemma applied to the various \( Q'_r \). The resolutions are clearly minimal.
COROLLARY 1. Let $\mathcal{Q}$ be a commutative regular Noetherian local ring, $\mathcal{I}$ an ideal in $\mathcal{Q}$ generated by monomials of degree two in a minimal system of generators for the maximal ideal. Let the maximal ideal and residue class field of $\mathcal{Q} = \mathcal{Q}/\mathcal{I}$ be $m$ and $k$. Let

$$H^0(Q)(Z) = \sum_{i \geq 0} h_i Z^i, \quad \text{where } h_i = \dim_k (m^i/m^{i+1})$$

(we call $H^0(Q)(Z)$ the Hilbert series for $Q$).

Then $H^0(-Z) \cdot B^0(Q)(Z) = 1$.

PROOF. If $R = G(Q)$ is the associated graded ring to $Q$, $R'$ the ring of type $B'$ belonging to $R$, then $B^0(Q)(Z) = H^0_R R'(Z)$ since the resolution is minimal. Since $H^0(Q)(Z) = H^0_k R'(Z)$ the result follows from Corollary 2 of Section 4.

Let $Q$ be a local ring with maximal ideal $m = (t_1, \ldots, t_n)$ (minimal set of generators), $K$ its Koszul complex in the variables $T_1, \ldots, T_n$. Let $c_{ij} = \dim_k H_i^{j+i}(K)$, where the upper index indicates the successive quotient in the filtration of $H_i(K)$, induced by the filtration

$$K^p = (t_1, \ldots, t_n, T_1, \ldots, T_n)^p K$$

of the Koszul complex. Then we can give the Hilbert series for $Q$ as

$$(1 + \sum c_{ij} (-1)^i Z^{i+j})/(1 - Z)^n$$

(see [8] chapitre IV. 3). Thus we get

COROLLARY 2. If $Q$ is as in Corollary 1, then with the notions above

$$B^0(Q)(Z) = (1 + Z)^n/(1 + \sum c_{ij} (-1)^i Z^{i+j}) \cdot$$

We conclude with a concrete example. If

$$Q = k[[X_1, X_2, X_3]]/(X_1^2, X_2^2, X_3^2, X_1 X_2, X_1 X_3),$$

then $H^0(Q)(Z) = 1 + 3Z + Z^3$, so

$$B^0(Q)(Z) = 1/(1 - 3Z + Z^2) = (1 + Z)^9/(1 - 5Z^2 - 5Z^3 + Z^5)$$

(cf. the Gorenstein ring

$$k[[X_1, X_2, X_3]]/(X_1^2, X_2^2, X_3^2, X_1 X_2 - X_2 X_3, X_1 X_3 - X_2 X_3)$$

with the same Poincaré series, see [13]).
Note. An example by Gunnar Sjödin shows that our formula of Corollary 1 is false in general if $I$ is generated by forms of degree two. Namely, let

$$Q = k\left([X_1, X_2, X_3, X_4] / (X_1^2 - X_2^2, X_2^2 - X_3^2, X_3^2 - X_4^2, X_1X_2, X_1X_3 - X_2X_4)\right).$$

Then $H^Q(Z) = 1 + 4Z + 5Z^2$, so $1/H^Q(-Z)$ will have negative coefficients.

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