A NOTE ON PG-MODULES

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In this note, we prove that PG-modules of finite G-dimension are projective.

$R$ denotes a commutative noetherian ring with unity. $R$-modules will be finitely generated and unitary. For an $R$-module $M$, $M^*$ denotes $\text{Hom}_R(M,R)$.

In [1], H-B. Foxby defines an $R$-module $M$ to be a PG-module if $\text{Hom}_R(M,M)$ is projective and $\text{Ext}_R^i(M,M) = 0$ for all $i > 0$. Let $M$ be a PG-module of finite G-dimension (For definition, see [2, § 3.2.2]. We shall prove that $M$ is projective.

We can assume $R$ to be local. If $x$ is a non-zero divisor for $R$ and $M, M/xM$ are both PG-module and of finite G-dimension as $R/x$-modules [1, Proposition 1.1 vii and 2, § 3.2.2, Lemma 4]. So, by an easy induction on depth $R$, we may also assume depth $R = 0$.

Since depth $M + G$-dim $M = \text{depth} R$, G-dim $M = 0$ [2, § 3.2, Theorem 2]. Hence $M$ is a reflexive $R$-module such that $\text{Ext}_R^i(M,R) = \text{Ext}_R^i(M^*,R) = 0$ for all $i > 0$.

Consider the two spectral sequences with the same limit

$$\text{Ext}_R^p(M, \text{Ext}_R^q(M^*,R)) \Rightarrow_p H^n$$

$$\text{Ext}_R^p(\text{Tor}_R^q(M, M^*), R) \Rightarrow_p H^n$$

By the assumptions on $M$, we get $H^n = 0$ for $n > 0$ from the first spectral sequence. The low term exact sequence for the second spectral sequence then yields

$$E_{2}^{1,0} = \text{Ext}_R^1(M \otimes_R M^*,R) = 0.$$  

Also, $\text{Hom}(M \otimes_R M^*,R) \cong \text{Hom}_R(M,M)$ is free. Let $K = M \otimes_R M^*$. If we prove $K$ is free, it easily follows that $M$ is free.

Let

$$0 \rightarrow T \rightarrow F \rightarrow K \rightarrow 0$$

be exact with $F$ finitely generated and free. Then taking duals, we get an exact sequence

$$0 \rightarrow K^* \rightarrow F^* \rightarrow T^* \rightarrow 0.$$  

Received February 10, 1975.
Since $K^*$ is free, $T^*$ is of finite homological dimension, hence free as depth $R=0$, and (2) splits. Therefore taking duals again, and combining with (1), we get a commutative diagram with exact rows.

\[
\begin{array}{cccccc}
0 & \rightarrow & T & \rightarrow & F & \rightarrow & K & \rightarrow & 0 \\
\downarrow & & \downarrow & \cong & \downarrow & & \downarrow \\
0 & \rightarrow & T^{**} & \rightarrow & F^{**} & \rightarrow & K^{**} & \rightarrow & 0
\end{array}
\]

The vertical arrows are the natural maps into the double dual and the middle one is an isomorphism. Hence by the snake lemma, $K \rightarrow K^{**}$ is surjective, and $K \cong K^{**} \oplus L$ for some module $L$. Taking duals again, $K^* \cong K^{***} \oplus L^*$. By rank considerations now, $L^* = 0$. Since depth $R=0$, $L=0$. So, $K$ is free and hence $M$ is free.

Hence, we get

**Proposition.** PG-modules of finite G-dimension are projective.

**References**


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