THE CENTRALIZER UNDER TENSOR PRODUCT

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Abstract.

Let $M$ denote the fixed points of the modular automorphism group $t \to \sigma_t^\varphi$ of a faithful, normal state, $\varphi$, on a von Neumann algebra $M$. We calculate $(M \otimes N)_\varphi \otimes \psi$ when $\varphi$ and $\psi$ are periodic. In general we show when $(M \otimes N)_\varphi \otimes \psi = M \otimes N$. We also give a discussion of eigenoperators for a modular automorphism group.

1. Introduction.

In [1] and [7] Araki and Takesaki have given an analysis of "periodic", faithful, normal states viz, states for which there exists a smallest positive number $T$ such that the corresponding modular automorphism group at $T$ is the identity. If $\varphi$ and $\psi$ are two such states on $M$, $N$ respectively then (Theorem 1) $(M \otimes N)_\varphi \otimes \psi = M \otimes N$ if and only if $T_\varphi / T_\psi$ is irrational. We then give a general condition for this to hold without assuming periodicity.

It is shown that unitary operators cannot occur as "eigenoperators", [5], of modular automorphism group although they can (and do) occur for other automorphism groups. This observation leads to an alternative proof of Erling Størmer’s result, [5], that a compact abelian group acts ergodically on a von Neumann algebra $M$, only if $M$ is finite.

2.

Let $\varphi$ be a faithful, normal, periodic state on a von Neumann algebra $M$. If $T_\varphi$ is its period we set $\kappa = e^{-2\pi i / T_\varphi}$ and recall, [7], that

$$\epsilon_n(x) = (1/T_\varphi) \int_0^{T_\varphi} \kappa^{-int} \sigma_t^\varphi(x) dt$$

(1)

defines a normal projection of $M$ onto the subspace

$$M_n = \{ x : \sigma_t^\varphi(x) = \kappa^{int} x \}$$

(2)

of $M$.

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Assuming \( \varphi(x) = (x \xi_\varphi | \xi_\varphi) \) with \( \xi_\varphi \) cyclic and separating we have

(3) \[ \varepsilon_n \circ \varepsilon_m = \delta_{nm} \varepsilon_n \]

(4) \[ \varepsilon_n(axb) = a \varepsilon_n(x)b, \quad a, b \in M_\varphi \equiv M_0 \]

(5) \[ x \xi_\varphi = \sum_{n \in \mathbb{Z}} \varepsilon_n(x) \xi_\varphi. \]

Suppose then that \( M \) acts on a Hilbert Space \( \mathcal{H} \) and let \( N \) be another von Neumann algebra acting on \( \mathcal{H} \) with faithful, normal, periodic state \( \psi(\cdot) = (\cdot \xi_\varphi | \xi_\varphi) \).

Let \( \varepsilon_0 \) denote the unique projection of \( M \otimes N \) onto \( (M \otimes N)_\varphi \otimes \psi \) characterized by \( (\varphi \otimes \psi)(\varepsilon_0(w)) = \varphi \otimes \psi(w) \) for each \( w \in M \otimes N \).

**Theorem 1.** Suppose \( M \) and \( N \) are given with faithful, normal, periodic states \( \varphi \) and \( \psi \), having periods \( T_\varphi \) and \( T_\psi \) respectively. Then

(i) If \( T_\varphi/T_\psi \) is irrational then

a) \( \varepsilon_0 = \varepsilon_0^\varphi \otimes \varepsilon_0^\psi \)
equivalently

b) \( (M \otimes N)_{\varphi \otimes \psi} = M_\varphi \otimes N_\psi \).

(ii) If \( T_\varphi/T_\psi \) is rational then

a) \( \varepsilon_0 = \sum_{j \in I} \varepsilon_{-k_j} \otimes \varepsilon_{l_j} \)
equivalently (with some abuse of notation)

b) \( (M \otimes N)_{\varphi \otimes \psi} = \sum_{j \in I} (M_{-k_j} \otimes N_{l_j}) \).

(The direct sum is in the sense of the pre-Hilbert Space structure induced by \( \varphi \otimes \psi \) and the indices run over \( k_j \) and \( l_j \) for which \( k_j/l_j = T_\varphi/T_\psi \) or \( k_j = l_j = 0 \).)

**Proof.** Let \( E_0 = \{ \eta \in \mathcal{H} \otimes \mathcal{H} : (A_\varphi^\mu \otimes A_\psi^\mu) \eta = \eta \} \). Then since \( \xi_\varphi \otimes \xi_\psi \) is cyclic and separating for \( M \otimes N \) we have

\[ \varepsilon_0(x \otimes y)(\xi_\varphi \otimes \xi_\psi) = E_0(x \otimes y)(\xi_\varphi \otimes \xi_\psi) = E_0(x \xi_\varphi \otimes y \xi_\psi) \]
for arbitrary \( x, y \) belonging to \( M, N \) respectively.

Now according to [7],

\[ x \xi_\varphi = \sum_{n \in \mathbb{Z}} x(n) \xi_\varphi \quad x(n) = \varepsilon_n^\varphi(x) \]
\[ y \xi_\varphi = \sum_{n \in \mathbb{Z}} y(n) \xi_\varphi \quad y(n) = \varepsilon_n^\psi(x). \]

Thus

\[ E_0(x \xi_\varphi \otimes y \xi_\varphi) = E_0(\sum_{n,m} x(n) \xi_\varphi \otimes y(m) \xi_\varphi) \]
\[ = \sum_{n,m} E_0((x(n) \otimes y(m))(\xi_\varphi \otimes \xi_\psi)) \]
\[ = \sum_{n,m} \varepsilon_0(x(n) \otimes y(m))(\xi_\varphi \otimes \xi_\psi). \]
We recall that

$$
\varepsilon_0(w) = \text{wk.-lim}_{T \to \infty} (1/2T) \int_{-T}^{T} \sigma^\varphi \otimes \varphi(w) dt.
$$

By the orthogonality of distinct characters on $\mathbb{R}$ under Wiener mean, one sees that if the periods, $T_\varphi$ and $T_\psi$, are irrationally related, we have

$$
\varepsilon_0(x \otimes y) = \varepsilon_0^\varphi(x) \otimes \varepsilon_0^\psi(y),
$$

using $\sigma_i^\varphi \otimes \varphi = \sigma_i^\varphi \otimes \sigma_i^\psi$. Since elements $x \otimes y$ generate $M \otimes \bar{N}$, we have that $(M \otimes \bar{N})_{\varphi} \otimes \varphi \subseteq M_{\varphi} \otimes \bar{N}_{\psi}$, hence equality and (i) is proven.

The second statement is now clear, since by the above mentioned orthogonality, the only elements contributing to $\varepsilon_0(x \otimes y)$ are those specified in the statement.

We now consider the first part of Theorem 1 for general, i.e. not necessarily periodic, states.

If $\mu$ is a finite Borel measure on $\mathbb{R}$, then it is known [8] that

$$
\lim_{T \to \infty} (1/2T) \int_{-T}^{T} \hat{\mu}(t) dt = \mu(\{0\}),
$$

where $\hat{\mu}(t)$ is the Fourier transform of $\mu$.

Consider now the expression (setting as above, save for periodicity)

$$
((\Delta_\varphi^\mu \otimes \Delta_\varphi^\mu)(x \otimes y) \xi_\varphi \otimes \xi_\varphi | \eta_1 \otimes \eta_2)
$$

where $\eta_1, \eta_2$ are arbitrary vectors in $\mathcal{H}, \mathcal{K}$ respectively. This equals

$$
(\Delta_\varphi^\mu x_\varphi | \eta_1)(\Delta_\varphi^\mu \xi_\varphi | \eta_2)
= (\int e^{itd}(e(\lambda)\xi_\varphi | \eta_1)) (\int e^{ityd}(f(\gamma)\xi_\varphi | \eta_2))
= \int e^{itd}d\mu(\beta).
$$

Here $e(\lambda), f(\gamma)$ are the spectral measures corresponding to $\log \Delta_\varphi, \log \Delta_\varphi$ respectively and $\mu$ represents the convolution of the two measures $d(e(\lambda)\xi_\varphi | \eta_1)$ and $d(f(\gamma)\xi_\varphi | \eta_2)$.

As in Theorem 1 we are interested in finding when $\varepsilon_0 = \varepsilon_0^\varphi \otimes \varepsilon_0^\psi$. Thus we must find $\mu(\{0\})$.

Remembering, that the continuous measures are a two sided ideal in $M(\mathbb{R})$, [2, Chapter V], we see by the above remarks that a calculation of $\mu(\{0\})$ involves only the convolution of the discrete parts of the two aforementioned measures.

If $\nu_1, \nu_2 \in M_d(\mathbb{R})$, i.e. are discrete measures in $M(\mathbb{R})$ then

$$
\nu_1 * \nu_2(\{0\}) = \sum_{\lambda \in \text{supp } \nu_2} \nu_1(\{-\lambda\}) \nu_2(\{\lambda\})
$$
Thus one see that if for $\lambda \neq 0$ belonging to the support of $v_2$ we never have $-\lambda$ belonging to the support of $v_1$, then

$$v_1 * v_2(\{0\}) = v_1(\{0\}) v_2(\{0\}).$$

This motivates

**Definition.** For faithful, normal states $\varphi, \psi$ on $M, N$ respectively, let us say that $\sigma_t^\varphi$ and $\sigma_t^\psi$ are *disharmonic* if whenever the character $\chi(t)(\neq 1)$ is an eigenvalue of $\sigma_t^\varphi$ i.e. there is $x \neq 0$ such that $\sigma_t^\varphi(x) = \chi(t)x$, then $\chi(t)$ is *not* an eigenvalue of $\sigma_t^\psi$.

The preceding remarks and the arbitrary choice of $\eta_1 \in \mathfrak{H}$ and $\eta_2 \in \mathfrak{F}$ then yield

**Theorem 2.** Let $M, N$ be von Neumann algebras with faithful, normal states $\varphi, \psi$ respectively. If the corresponding modular automorphism groups are disharmonic then

$$(M \bar{\otimes} N)_{\varphi \otimes \psi} = M_{\varphi} \bar{\otimes} N_{\psi}.$$

3.

We referred above to eigenvalues and implicitly to what Størmer [5] has called eigenoperators viz, elements in $M$ such that $\sigma_t^\varphi(x) = \chi(t)x$ for some character $\chi(t)$. Indeed for any group $G$ acting ergodically on $M$ such eigenoperators are, for fixed $\chi \in \hat{G}$ multiples of a fixed unitary, [5, Lemma 2.1]. By ergodicity we mean that the only elements fixed by the group action, are multiples of the identity.

We make the following

**Remark.** In [7] it is shown that for periodic homogeneous states the subspaces $M_n$ (of Section 2) contain either isometries or coisometries. It is implicit in the calculations there, that for $n \neq 0$ no unitary can be in $M_n$. Indeed, for general $\varphi$, one cannot have $\sigma_t^\varphi(u) = \chi(t)u = e^{it\lambda}u$ (for some $\lambda \in \mathbb{R}$) with $u$ a unitary, unless $\sigma_t^\varphi(u) = u$. If such $u$ existed, then $u$ would clearly be analytic for $\sigma_t^\varphi$ so we apply the "distributional form" of the KMS condition obtaining, for all $x \in M$,

$$\varphi(uxu^*) = \varphi(\sigma_t(u^*)ux) = \varphi((\sigma_{-i}(u))^*ux) = e^{i\lambda} \varphi(x).$$

Setting $x = I$ one gets $\lambda = 0$ whence $\sigma_t^\varphi(u) = u$. 
This remark now leads to an alternate proof of

**Theorem 3.** (Størmer [5]). Let \( g \to \alpha_g \) be a strongly continuous representation of a compact abelian group, \( G \), as automorphisms of a von Neumann algebra \( M \). If \( G \) acts ergodically, then \( M \) is finite.

**Proof.** There is nothing sacrosanct in Section 2 about the interval \([0,T]\). In fact any compact abelian group will do once we recall that \( G \), has a faithful, normal, \( G \)-invariant, state, say \( \varphi \) (average any normal state over \( G \) using Haar measure. The new state is normal [4, Proposition 3] and faithful; the latter since the state and its support projection are invariant under \( G \). Alternatively the existence of such states follows from Størmer’s paper, [6]). Replacing the integers by \( G \), one has in analogy with the statements in Section 2, \( x\xi_\varphi = \sum_{x \in G} \xi_\varphi(x) \xi_\varphi \), where \( \varphi(x) = (x\xi_\varphi, \xi_\varphi) \) is the faithful, normal, \( G \)-invariant state alluded to above. As we mentioned Størmer show that \( \xi_\varphi(x) \) must be a multiple of a fixed unitary, that unitary being independent of \( x \). Let us work with the unitary, call it \( v_x \). Now since \( \varphi(\alpha_g(x)) = \varphi(x), \alpha_g \) and \( \sigma_t^\varphi \) commute [3] so that \( \sigma_t^\varphi \) takes \( M_x \to M_x \). Thus, \( \sigma_t^\varphi(v_x) = e^{it}v_x \). Our remark now yields \( \sigma_t^\varphi(v_x) = v_x \) so that \( \sigma_t^\varphi(\xi_\varphi(x)) = \xi_\varphi(x) \), yielding \( \sigma_t^\varphi(x) = x \) for all \( x \in M \), whence \( \varphi \) is a trace.

**REFERENCES**