A NOTE ON FUNCTIONS WITH A SPECTRAL GAP

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In this note we shall give a few refinements of theorems on functions with a spectral gap given by H. S. Shapiro in [4]. Shapiro has stated our Theorem 3 without proof and Theorem 2 under "uniform" estimates in $x$ for the inequality (2). He has kindly pointed out to us that Theorem 2 was known to him. Theorem 1 is probably new. In any case our proofs seem novel.

We employ standard vector notations; $\mathbb{R}$ denotes the real line and $\mathbb{C}$ denotes the complex plane. $t = (t_1, t_2, \ldots, t_n)$ and $x = (x_1, x_2, \ldots, x_n)$ are points in $\mathbb{R}^n$ and $(t, x)$ denotes $t_1x_1 + t_2x_2 + \ldots + t_nx_n; |x| = (x, x)^{1/2}$, and $dt$ denotes Lebesgue measure on $\mathbb{R}^n$. $\hat{f}$ or $\hat{f}$ denotes the Fourier transform of a function $f$ or a tempered distribution, i.e., if $f$ is a summable function, $\hat{f}(t)$ is given by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{-i(x,t)} dx.$$ 

$B(x; a)$ denotes the open ball in $\mathbb{R}^n$ with center $x$ and radius $a$. The spectrum of a tempered distribution is the distributional support of its Fourier transform. A gap in a distribution is a nonvoid open ball disjoint from its support. A spectral gap in a tempered distribution is a gap in its Fourier transform. $\mathcal{S}(\mathbb{R}^n)$ denotes the set of all rapidly decreasing infinitely differentiable functions in $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ denotes its dual, i.e., the set of all tempered distributions. $\langle , \rangle$ denotes the dual form for $\mathcal{S}'$ and $\mathcal{S}$.

1.

Our main results are as follows;

THEOREM 1. Let $f$ be a locally integrable function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} (1 + x^2)^{-1}|f(x)| dx < \infty.$$ 

Let $a > 0$. If, for two distinct real numbers $b_1, b_2$, the Poisson integral

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\[ u(x,y) = \pi^{-1} \int_{-\infty}^{\infty} f(t) \frac{y}{(x-t)^2 + y^2} dt \]

of \( f \) satisfies, for every \( \varepsilon > 0 \),

\[ |u(b_j, y)| \leq A(\varepsilon, j)e^{-\alpha - \varepsilon y}, \quad y > 1, \quad j = 1, 2, \]

for constants \( A(\varepsilon, j) \) depending only on \( \varepsilon \) and \( j \), then the spectrum of \( f \) is disjoint from \( B(0; \delta) \), where \( \delta = \min\{a, \pi/b_1-b_2\} \). If (1) holds for three rationally linearly independent real numbers, then \( B(0; \delta) \) can be replaced by \( B(0; a) \).

In the \( n \)-dimensional case we have

**Theorem 2.** Let \( f \) be a locally integrable function on \( \mathbb{R}^n \) such that

\[ \int_{\mathbb{R}^n} (1 + |x|^2)^{-(n+1)/2} |f(x)| dx < \infty. \]

Let \( a > 0 \). If the Poisson integral

\[ u(x,y) = c_n \int_{\mathbb{R}^n} f(t) \frac{y}{(|x-t|^2 + y^2)^{(n+1)/2}} dt \]

of \( f \) satisfies, for every \( \varepsilon > 0 \),

\[ |u(x,y)| \leq A(\varepsilon, a)e^{-\alpha - \varepsilon y}, \quad y > 1, \]

for all \( x \in E(\varepsilon) \), where \( E(\varepsilon) \) is any dense set in \( \mathbb{R}^n \) depending only on \( \varepsilon \) and \( A(\varepsilon, a) \) are constants depending only on \( \varepsilon \) and \( x \), then the spectrum of \( f \) is disjoint from \( B(0; a) \).

2.

To prove the theorems above we need the Theorem 3 below and for it we recall some definitions.

**Definition 1.** A function on \( \mathbb{R}^n \) is said to be radial if it depends only on radii \(|x|\). A locally integrable function on \( \mathbb{R}^n \) is said to be anti-radial if its integral over the open ball in \( \mathbb{R}^n \) with center 0 and radius \( r \) vanishes for every \( r > 0 \). Every locally integrable function admits an essentially unique decomposition into a radial and an anti-radial part.

To state Theorem 3 for other kernels than the Poisson kernel we recall the notion of radial tempered distributions.

**Definition 2.** A tempered distribution \( T \) on \( \mathbb{R}^n \) is said to be radial (anti-radial) if it holds \( \langle T, \varphi \rangle = 0 \) for all antiradial (radial) \( \varphi \in \mathcal{S}(\mathbb{R}^n) \)
(respectively). If \( T \) is a locally integrable function, these two definitions coincide.

**Lemma 1.** Every tempered distribution admits a unique decomposition into a radial and an anti-radial part.

In fact, every tempered distribution \( T \) can be represented in the form

\[
T = (1 - \Delta)^k \sum_{|\alpha| \leq m} (ix)^\alpha f(x) \equiv (1 - \Delta)^k g
\]

for some integers \( k, m \geq 0 \) and some \( f \in L^2(\mathbb{R}^n) \), where

\[
\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}.
\]

By decomposing \( g \) into a radial part \( g_1 \) and an anti-radial part \( g_2 \) we have

\[
T = (1 - \Delta)^k g_1 + (1 - \Delta)^k g_2 \equiv T_1 + T_2.
\]

One can easily see that \( T_1 \) is radial and \( T_2 \) is anti-radial. The uniqueness is obvious.

Now if we assume (2) in Theorem 2 holds for a single value of \( x \), we obtain the following information about the spectrum of \( f \).

**Theorem 3.** Let \( f \) be a locally integrable function on \( \mathbb{R}^n \) such that

\[
\int_{\mathbb{R}^n} (1 + |x|^2)^{-\frac{n+1}{2}} |f(x)| \, dx < \infty.
\]

If one has, for some \( a > 0 \),

\[
(3) \quad u(0, y) = c_n \int_{\mathbb{R}^n} f(x) \left( \frac{y}{y^2 + |x|^2} \right)^{(n+1)/2} \, dx = O(e^{-ay}),
\]

as \( y \to \infty \), then the radial part of \( f \) has spectrum disjoint from \( B(0; a) \).

**Proof.** We may assume \( f \) radial, since the anti-radial part contributes nothing to \( u(0, y) \).

(i) The case \( n \) is odd

Set \( (n+1)/2 = l \) and let \( \varepsilon > 0 \) be given. We have

\[
(4) \quad \frac{u(0, y)}{c_n} = \int f(x) \left[ \frac{y(1 + \varepsilon|x|^2) - \varepsilon(y^2 + |x|^2)^l}{(y^2 + |x|^2)^l} + \varepsilon y \right] \, dx.
\]

Further we have

\[
(y^2 + |x|^2)^l - |x|^2 = \sum_{k=1}^l \binom{l}{k} g^{2k} |x|^{2l-2k}
\]

and
$$\Delta^k e^{-v|t|} = \sum_{j=1}^{2^k} p_j(|t|) y^j e^{-v|t|} \quad (1 \leq k < l),$$

for some functions $p_j$, where $r^{2k-1} p_j(r)$ are polynomials in $r$. Hence we see that

$$\Delta^k e^{-v|t|} \in L^1(\mathbb{R}^n) \quad \text{for} \quad k < l$$

and we have for some constants $C_k$

$$((-\Delta)^k e^{-v|t|})^\wedge = C_k |x|^{2k} y(y^2 + |x|^2)^{-1} \quad \text{for} \quad k < l,$$

since $(e^{-v|t|})^\wedge = \text{const.} y(y^2 + |x|^2)^{-1}$. Set $G(t) = (f(1 + \epsilon |x|^2)^{-1})^\wedge$. Then (4) can be written by Parseval’s equality in the form

$$\mu(0, y)/c_n = \int G(t)(1 + \epsilon P(y, |t|)) e^{-v|t|} dt + \epsilon y G(0),$$

where $r^{2l-3} P(y, r)$ is a polynomial in $y$ and $r$. Since one gets by simple calculation

$$\int_{|t| < a} G(t)(1 + \epsilon P(y, |t|)) e^{-v|t|} dt = O(y^{2l} e^{-av}), \quad \text{as} \quad y \to \infty,$$

we have, taking account of the assumption,

$$\int_{|t| < a} G(t)(1 + \epsilon P(y, |t|)) e^{-v|t|} dt + \epsilon y G(0) = O(y^{2l} e^{-av}).$$

Put

$$H(z) = \int_{|t| < a} G(t)(1 + \epsilon P(z, |t|)) e^{-z|t| - a} dt + \epsilon z G(0) e^{az}.$$

Then $H$ is an entire function in $z$ and we have

$$|H(z)| \leq \text{const.} (1 + |z|^2) e^{a|z|} \quad \text{for} \quad z \in \mathbb{C},$$

and

$$|H(z)| \leq \text{const.} (1 + |z|^{2l}) \quad \text{for} \quad z \in \mathbb{R} \quad \text{and} \quad iz \in \mathbb{R}.$$

Hence by the Phragmén–Lindelöf theorem, we see that $H(z)$ is a polynomial of the form

$$H(z) = \sum_{j=0}^{2l} a_j z^j.$$

Hence we obtain

$$(5) \quad \int_{|t| < a} G(t)(1 + \epsilon P(iy, |t|)) e^{-iv|t|} dt + \epsilon y G(0) = \sum_{j=0}^{2l} a_j (iy)^j e^{-iav}.$$

Now let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function such that support $\hat{\varphi} \subset B(0; a)$. Set $\xi = \psi_1(|t|)$, $y(\xi) = \psi_1(|\xi|)$ for $\xi \in \mathbb{R}$ and $\Phi(y) = (2\pi)^{-1} \int \psi(\xi) e^{iy \xi} d\xi$.

Then $\psi(y), \Phi(y) \in \mathcal{S}(\mathbb{R})$ and support $\psi \subset \{y \in \mathbb{R} ; \ |y| < a \}$. Multiplying the both sides of (5) by $\Phi(y)$ and integrating them with respect to $y$, we have via Fubini’s theorem

$$\int_{|t| < a} G(t) \hat{\varphi}(t) dt + \epsilon \int_{|t| < a} G(t) g(t) dt + i\epsilon G(0) \int \Phi(y) dy = \sum_{j=0}^{2l} a_j \psi^j(a) = 0,$$

where $g(t) = \int \Phi(y \xi) |t| \Phi(y) e^{-i\xi|t|} dy \in L^1(\mathbb{R}^n)$ and support $g \subset B(0; a)$. By Parseval’s equality we have thus
\[ (6) \int \frac{f(x)}{1 + \varepsilon |x|^2} \varphi(x) \, dx + \varepsilon \int \frac{f(x)}{1 + \varepsilon |x|^2} \hat{g}(x) \, dx + i\varepsilon \int \sigma \varphi(y) \, dy \int \frac{f(x)}{1 + \varepsilon |x|^2} \, dx = 0. \]

Since \( \varepsilon (1 + \varepsilon |x|^2)^{-1} \leq (1 + |x|^2)^{-1}, (0 < \varepsilon < 1) \), we have
\[ \varepsilon \int \frac{|f(x)|}{1 + \varepsilon |x|^2} \, dx \to 0 \quad \text{as} \ \varepsilon \to 0. \]

Hence by (6) we have letting \( \varepsilon \to 0 \)
\[ \int f(x) \varphi(x) \, dx = 0. \]

Since \( f \) is radial, we have thus the desired conclusion.

(ii) The case \( n \) is even.

Let \( \alpha = (n + 1)/2 \). We have, for every \( \varepsilon > 0 \),
\[ u(0,y)/c_n = \int \frac{f(x)}{1 + \varepsilon |x|^{n+2}} \frac{y}{(y^2 + |x|^2)} \, dx + \varepsilon \sum_{j=1}^{n} \int \frac{y_j f(x)}{1 + \varepsilon |x|^{n+2}} \frac{|y_j x_j|^n}{(y^2 + |x|^2)^{\alpha}} \, dx. \]
By calculation one can see that the Fourier transform of the function \( y_j x_j |x|^n (y^2 + |x|^2)^{-\alpha} - c y x_j |x|^n (1 + |x|^2)^{-\alpha} \) is of the form
\[ (7) \quad P_j(y, |t|) e^{-|t|} + c y_t |t|^{-(n+1)} (e^{-|t|} - e^{-|t|}) - c^2 y P_j(1, |t|) e^{-|t|}, \]
for some constant \( c \) and functions \( P_j \), where \( r^{n-1} P_j(y, r) \) are polynomials in \( y \) and \( r \). Hence each term in (7) is integrable on \( \mathbb{R}^n \). Therefore, modifying the proof of the first part one can easily show the desired conclusion.

3.

To prove our main theorems we need two more lemmas.

**Lemma 2.** Let \( T \) be a tempered distribution on \( \mathbb{R}^n \). If for an \( a > 0 \),
\[ \langle T, \hat{\varphi}(x-b) \rangle = 0 \]
for all \( b \in \mathbb{R}^n \) and all radial \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) with support in \( B(0; a) \), then
\[ \langle T, \psi \rangle = 0 \]
for all \( \psi \in \mathcal{S}(\mathbb{R}^n) \) with support in \( B(0; a) \).

**Proof.** Let \( \psi \in \mathcal{S}(\mathbb{R}^n) \) with support in \( B(0; a) \) and \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) be radial, with support in \( B(0; a) \) and \( \varphi(t) = 1 \) on the support of \( \psi \). Then one can easily see that
0 = \int \hat{\varphi}(-b)\langle T, \hat{\varphi}(x+b) \rangle db = \langle T, \int \hat{\varphi}(-b)\hat{\varphi}(x+b)db \rangle.

Now we have by Parseval’s equality

\[(2\pi)^{-n} \int \hat{\varphi}(-b)\hat{\varphi}(x+b)db = \int \varphi(t)\varphi(t)e^{-it\xi,0}dt = \int \varphi(t)e^{-it\xi,0}dt = \hat{\varphi}(\xi).\]

The third equation follows from the assumption \(\varphi(t) = 1\) on the support of \(\varphi\). Hence we have \(\langle T, \hat{\varphi} \rangle = 0\).

In the one dimensional case we have a stronger result.

**Lemma 3.** Let \(a > 0\) and \(T\) be a tempered distribution on \(\mathbb{R}\) such that or two distinct points \(b_1, b_2 \in \mathbb{R}\)

\[(8) \quad \langle T, \hat{\varphi}(x+b_j) \rangle = 0 \quad (j = 1, 2)\]

for all even functions \(\varphi \in \mathcal{S}(\mathbb{R})\) with support in \(B(0; a)\). Then one has

\[\langle T, \hat{\varphi} \rangle = 0\]

for all \(\varphi \in \mathcal{S}(\mathbb{R})\) with support in \(B(0, \delta)\), where \(\delta = \min(a, \pi/|b_1 - b_2|)\).

If, in particular, one assumes (8) holds for three rationally linearly independent points in \(\mathbb{R}\), then \(B(0; \delta)\) can be replaced by \(B(0; a)\).

**Proof.** Let \(\varphi \in \mathcal{S}(\mathbb{R})\) with support in \(B(0; a)\). Then

\[\varphi(x) = e^{-ib_1x}\varphi(x) + e^{ib_1x}\varphi(-x)\]

is even, in \(\mathcal{S}(\mathbb{R})\) and has support in \(B(0; a)\). Further we have

\[\langle \hat{\varphi}(t + b_1) \rangle(-x) = \varphi(x)e^{-2ib_1x} + \varphi(-x)\].

Hence by assumption

\[\langle \hat{T}, e^{-2ib_1x}\varphi(x) + \varphi(-x) \rangle = 0\.

This equation holds also for \(b_2\). Hence we have

\[\langle \hat{T}, (e^{-2ib_1x} - e^{-2ib_2x})\varphi(x) \rangle = 0\.

This implies \(\langle \hat{T}, \Phi \rangle = 0\) for all \(\Phi \in \mathcal{S}(\mathbb{R})\) with support in \(B(0; \delta) \setminus \{0\}\). Therefore there exists a polynomial \(P(x)\) such that

\[\langle (T - P(x))^{\wedge}, \Phi \rangle = 0\]

for all \(\Phi \in \mathcal{S}(\mathbb{R})\) with support in \(B(0; \delta)\). Hence by direct calculation we get

\[\langle T - P, k_{\varphi}(x+b_j) \rangle = O(e^{-a|x|^2/2}) \quad (j = 1, 2),\]
where $k_y(x) = y^{-1}e^{-x^2/y^2}$. By assumption one can get in a similar way the same estimates for $T$, since $k_y$ is an even function in $\mathcal{S}(\mathbb{R})$. Hence we have

$$\langle \hat{P}, e^{-y^{2a-jb_j}} \rangle = \text{const.} \langle P, k_y(x+b_j) \rangle = O(e^{-a y^{2j/2}}), \quad j = 1, 2,$$

which implies however $P(x) = 0$. Hence we have the first assertion. The last one is then obtained by modifying the above discussion somewhat.

4.

Proofs of Theorems 1 and 2. Theorem 1 follows immediately from Theorem 3 and Lemma 3. Theorem 2 follows from Theorem 3 and Lemma 2.

5.

Remark. By a similar method one can show the same results as Theorems 1, 2 and 3, when one replaces the Poisson kernel by the kernel $k_y(x) = y^{-n}e^{-|x|^2/y^2}$ and $f$ by any tempered distribution. In this case one does not need to introduce $\varepsilon$ in the proof of Theorem 3, since the kernel is in $\mathcal{S}(\mathbb{R}^n)$ and one can use Lemma 1.

REFERENCE


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