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A FOLIATION OF TEICHMÜLLER SPACE BY TWIST INVARIANT DISKS

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To Werner Fenchel on his 70th birthday.

Let γ be a non-trivial simple loop on a surface S_0 of genus $g>1$, let T_g denote the $3g-3$ dimensional Teichmüller space with origin $(S_0, \mathrm{id.})$ and $T[\gamma]$ the biholomorphic automorphism of T_g resulting from a Dehn twist about γ. A Teichmüller disk is a one dimensional totally geodesic (in the Teichmüller metric) submanifold of T_g. Denote by $\partial_\gamma T_g$ the $3g-4$ dimensional boundary space resulting from pinching γ to a point. For a description of this space, its relation to T_g, and a topology we refer to [2]. Let $D=\{z \in \mathbf{C} : |z|<1\}$. The purpose of this paper is to prove the following result.

THEOREM. *There is a parabolic transformation T in D and a one-dimensional complex foliation of T_g given by a homeomorphism F: $\partial_\gamma T_g \times D \to T_g$ with the following properties. For fixed $P \in \partial_\gamma T_g$, $F(P,\cdot)$ is a holomorphic map of D onto a Teichmüller disk D' which is invariant under $T[\gamma]$. $T[\gamma]$ acts as a parabolic transformation on D' and the geodesic rays of D' extending towards the fixed point of $T[\gamma]$ approach P in $T_g \cup \partial_\gamma T_g$. The action of $T[\gamma]$ on D' is given by $F(P,\cdot) \mapsto F(P, T(\cdot))$.*

The Theorem has an exact analogue for the Teichmüller space $T(g,n)$ of an n-punctured compact surface of genus g if $(g,n) \neq (1,0)$, $(0,1)$, $(0,2)$, $(0,3)$. This leads to the following result.

COROLLARY. *Fix $(S_0, \mathrm{id.}) \in T(g,n)$ and a set $\{\gamma_i\}$, $1 \leqq i \leqq N = 3g+n-3$, of mutually disjoint simple loops which divide S_0 into triply connected regions. There exists a homeomorphism G: $D^N \to T(g,n)$ such that for fixed values of the remaining variables, G is a holomorphic map of the k'th factor D onto a Teichmüller disk D' in one of the factors of the Teichmüller space resulting from pinching $\gamma_1, \ldots, \gamma_{k-1}$ to points. Furthermore D' is invariant under the Dehn twist about γ_k.*

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What makes the Theorem possible is that one can locate the extremal Teichmüller mapping in the homotopy class of a Dehn twist about $\gamma \subset S_0$. This is so because of a result of Jenkins [5] (see also Strebel [11, 13]) that identifies the quadratic differential, here called the Jenkins differential, which gives the annulus of largest possible modulus in the free homotopy class of γ that fits on S_0. The details of this are carried out in sections 1, 2.

Given a point $Q \in T_g$, the Jenkins differential determines a Teichmüller disk with origin at Q. In section 3 we discuss this disk and show that $T[\gamma]$ acts on it as a parabolic transformation. Then we apply the recent results of [8] to identify the fixed point of $T[\gamma]$ as a point on $\partial_\gamma T_g$.

In section 5 the foliation is constructed. Here the development is based on work of Strebel [12, 13, 14] (see also Jenkins [5]) characterizing by their extremal properties certain quadratic differentials which can be obtained by degenerating Jenkins' differentials. These properties, which are discussed in section 4, have to do with the classical notion of mapping radius. We show that these differentials have certain uniformity properties on the entire space $\partial_\gamma T_g$. Each disk of the foliation is then indexed by the fixed point of $T[\gamma]$ where it is "tangent" to $\partial_\gamma T_g$.

The map F is unlikely to be very smooth. It certainly is not holomorphic. For if it were, there would be a continuous family of biholomorphic automorphisms of T_g in contradiction to Royden's theorem [9].

With respect to the Corollary, more explicit and no doubt smoother geometric parameters, given in terms of fuchsian groups, are originally due to Fricke (see Keen [6]) and Fenchel-Nielsen [4]. The ones suggested by the Corollary, on the other hand, depend directly on the N loops.

1. Twists in an annulus.

1.1 Consider the annulus $A = \{z \in \mathbf{C} :\ 1 < |z| < R\}$ with the usual orientation induced from that of the plane. Orient the bounding circles so that A lies to their left. *A Dehn twist of order n* is a homeomorphism resulting from holding the inner contour of A fixed and rotating the outer n times in the positive (negative) direction if $n > 0$ ($n < 0$). In polar coordinates such a map is given by

$$(r, \theta) \mapsto \left(r, \theta + \frac{r-1}{R-1}\, 2\pi n\right).$$

We are actually only interested in the homotopy class of the twist. Here homotopy is used in the sense that f is homotopic to g if f can be continuously deformed to g while keeping ∂A pointwise fixed. Twists of different orders lie in different homotopy classes and any homeomorphism of A which is the identity on ∂A is homotopic to some twist.

1.2. We are looking for extremal quasiconformal maps in the homotopy class of a twist. That is, we are looking for maps that minimize the maximal dilatation. The situation is as follows.

LEMMA 1.1. (Strebel [10]). *There exists a unique extremal quasiconformal map $\mathcal{T}_n$ in the homotopy class of the Dehn twist of the annulus A of order n. It is given by the expression*

$$\mathcal{T}_n(z) = z|z|^{in/M}, \quad M = (\log R)/2\pi .$$

The complex dilatation $\mu(z)$ of $\mathcal{T}_n(z)$ is

$$\mu(z) = \frac{in/2M}{1+(in/2M)} \frac{\bar{z}^{-2}}{|z|^{-2}} = k_n e^{i\tau_n} \frac{\bar{z}^{-2}}{|z|^{-2}}$$

where

$$k_n = \frac{|n|/2M}{(1+(n/2M)^2)^{\frac{1}{2}}}, \quad \tan \tau_n = 2M/n, \quad -\pi/2 < \tau_n < \pi/2 .$$

Furthermore

$$\mathcal{T}_n = (\mathcal{T}_1)^n .$$

PROOF. The basic fact is that an affine map of a parallel strip is the unique extremal map for its boundary values [10]. The affine map in question is the map $(w=u+iv)$ $v \mapsto v$ and $u \mapsto u-v(n/M)$ of the parallel strip $\{0 \leqq v \leqq M\}$. The function $\exp(-2\pi i w)$ carries this strip to our annulus A.

Note that as $|n| \to \infty$, $k_n \to 1$ and $\tau_n \to 0$.

1.3. Actually we can twist A through any angle θ and the corresponding extremal map for the boundary values in the homotopy class is given by $\mathcal{T}_\alpha$ where $\alpha = \theta/2\pi$ and the same formulas as above with α replacing n. In particular τ_α lies between $\pm\pi/2$.

The twist maps $\mathcal{T}_\alpha$ are special cases of the map

$$Q(a,b): z \mapsto z|z|^{a+ib}, \quad a,b \text{ real},$$

which is an extremal quasiconformal map when $a > -1$ and has the complex dilatation

$$\frac{a+ib}{2+a+ib} \frac{\bar{z}^{-2}}{|z|^{-2}}.$$

$Q(a,b)$ maps A onto itself if and only if $a=0$ in which case it is a twist $\mathscr{T}_\alpha$ for $\alpha = Mb$. In the case $b=0$ it is a radial stretch ($a>0$) or contraction ($-1<a<0$) T_a of A onto $\{1<|z|<R^{1+a}\}$. In the general case, with respect to the appropriate annuli,

$$Q(a,b) = T_a \circ \mathscr{T}_\alpha = \mathscr{T}_\alpha \circ T_a .$$

1.4. The quadratic differential on A

$$\varphi dz^2 = -dz^2/z^2$$

plays a special role. Its (horizontal) trajectories are the circles $\{|z| = r\}$ in A, $1 \leqq r \leqq R$; that is $\varphi dz^2 > 0$ along these concentric circles. All of the extremal maps considered above are of the form

$$k\overline{(e^{i\theta}\varphi)}/|e^{i\theta}\varphi| = ke^{-i\theta}\, \bar{\varphi}/|\varphi| .$$

This means they are Teichmüller maps associated with the quadratic differential $e^{i\theta}\varphi dz^2$. In particular $\mathscr{T}_n$ is associated with $e^{-i(\tau_n+\pi)}\varphi dz^2$.

2. Dehn twists on a surface.

2.1. Let S be a closed Riemann surface of genus $g \geqq 2$ and $\gamma \subset S$ a simple loop not contractible to a point. Fix an annular neighborhood $\tilde{A}$ about γ. The orientation of S determines an orientation of $\tilde{A}$ and consequently of $\partial\tilde{A}$. *A* Dehn twist of order n in $\tilde{A}$ can be extended to $S-\tilde{A}$ by setting it equal to the identity there. This is a Dehn twist of order n on S. We are only interested in the homotopy class of this map which does not depend on the choice of γ in its free homotopy class or the particular choice of $\tilde{A}$. Twists of different orders belong to different homotopy classes and a classical theorem of Dehn says that a general (orientation preserving) homeomorphism of S onto itself is homotopic to a product of Dehn twists.

2.2. According to Jenkins [5] (see also Strebel [11, 13]) there is a holomorphic quadratic differential $Jd\zeta^2 \equiv J[\gamma]d\zeta^2$ on S uniquely determined up to a positive multiplicative constant by the following property. Let S' denote the result of deleting from S the critical trajectories of $Jd\zeta^2$, namely those that pass through the $4g-4$ zeros with multiplicities of $Jd\zeta^2$. Then S' is an open annulus swept out by a one parameter family of closed trajectories of $Jd\zeta^2$ which are freely homotopic to γ and S–S' is the union of a finite number of analytic arcs. The trajectories sweeping out S' have equal length in the metric $|J|^{\frac{1}{2}}|d\zeta|$. Normalize $Jd\zeta^2$ so that this length is 2π. See [5, 11, 13] for details.

The constant C can be chosen so that

$$z = p(\zeta) = C \exp(i\textstyle\int^{\zeta} \sqrt{J} d\zeta)$$

is a conformal map of S' onto $A = \{1 < |z| < R\}$ for some $R > 1$.

In terms of this change in coordinates,

$$-dz^2/z^2 = J d\zeta^2 .$$

2.3. We are now ready to find the extremal quasiconformal map of S in the homotopy class of a Dehn twist of order n. Let $\mathscr{T}_n$ denote as in section 1.2 the extremal twist of order n in A. The quasiconformal homeomorphism $p^{-1}\mathscr{T}_n p : S' \to S'$ can be extended to a homeomorphism $S \to S$ by setting it equal to the identity on the finite number of analytic arcs composing S–S'. A standard removable singularity theorem says that $p^{-1}\mathscr{T}_n p : S \to S$ is quasiconformal.

In terms of the local parameter ζ in S' we find that the complex dilatation $\mu(\zeta)$ of $p^{-1}\mathscr{T}_n p$ is

$$\mu(\zeta) = -k_n e^{i\tau_n} \bar{J}/|J|$$

where k_n, τ_n are defined in section 1.2. It follows that $p^{-1}\mathscr{T}_n p$ is a Teichmüller map on S associated with the differential $\exp -i(\tau_n+\pi)$ $J d\zeta^2$. By a famous theorem of Teichmüller (see [1]), such maps are unique in their homotopy class. Consequently we have proved the following:

LEMMA. 2.1. *The unique extremal quasiconformal map in the homotopy class of a Dehn twist of order n about γ is the Teichmüller map associated with the quadratic differential* $\exp -i(\tau_n+\pi) J[\gamma] d\zeta^2$.

2.4. There are two generalizations of Lemma 2.1 that we will describe briefly. The first is that everything can be done on a finitely punctured compact surface S. For this case one takes a simple loop γ which is not contractible to one of the punctures. The corresponding Jenkins differential has at most simple poles at the punctures. In any case the punctures lie on the critical trajectories which when removed from S leave a single annulus belonging to the free homotopy class of γ. All the formulas above hold without change.

The second generalization makes use of Strebel differentials [11, 13] (see also Jenkins [5]). On a closed surface S of genus $g \geqq 2$ choose k mutually disjoint simple loops $\gamma_1, \ldots, \gamma_k$, $1 \leqq k \leqq 3g-3$, which represent distinct free homotopy classes other than the identity. Fix k positive numbers $M_1, M_2, \ldots, M_k$. Then there is a unique (up to a positive multiple) quadratic differential $\Gamma d\zeta^2$ with the following property. Let S' denote the result of removing the critical trajectories of $\Gamma d\zeta^2$ from S. Then S' is a union of annuli $\tilde{A}_1, \ldots, \tilde{A}_k$ which can be indexed so that for all j

a) γ_j is freely homotopic to a simple loop separating the components of $\partial \tilde{A}_j$,

and

b) for some constant $c > 0$, the modulus $\tilde{A}_j$ is cM_j.

Furthermore S–S' is the union of a finite number of analytic arcs, the smooth pieces of the critical trajectories.

Now suppose an integer $n_j \geqq 1$ is assigned to γ_j for each j. We want to find the extremal quasiconformal map homotopic to the product of Dehn twists $T_k \circ \ldots \circ T_1$ where T_j is the twist of order n_j about γ_j (order does not matter since these commute up to homotopy).

Set $M_j = n_j/2$ and let $\Gamma d\zeta$ be the corresponding Strebel differential. Then there is a $C_j > 0$ such that

$$z = p_j(\zeta) = C_j \exp(2\pi i L_j^{-1} \textstyle\int^{\zeta} \sqrt{\Gamma} d\zeta)$$

maps $\tilde{A}_j$ conformally onto $A_j = \{1 < |z| < R_j\}$ where $cM_j = (\log R_j)/2\pi$. Here L_j is the common length in the $|\Gamma|^{\frac{1}{2}} |d\zeta|$ metric of the trajectories of $\Gamma d\zeta^2$ which sweep out $\tilde{A}_j$.

The extremal twist $\mathscr{T}_{n_j}$ in A_j is a Teichmüller map associated with the differential $\exp -i(\tau_{n_j} + \pi)\varphi dz^2$ where $\varphi = -z^{-2}$ and multiplier k_{n_j} (see section 1.2). We find

$$k_{n_j} = (c^2 + 1)^{-\frac{1}{2}}, \quad \tan \tau_{n_j} = c\,.$$

The quasiconformal map $p_j^{-1} \mathscr{T}_{n_j} p_j : \tilde{A}_j \to \tilde{A}_j$ is thus a Teichmüller map associated with the differential

$$\exp -i(\tau_n + \pi)\Gamma d\zeta^2 = [(-1 + ic)/(1 + c^2)^{\frac{1}{2}}]\Gamma d\zeta^2$$

and the multiplier $(1 + c^2)^{-\frac{1}{2}}$ neither of which depend on j.

Define f on S' by setting it equal to $p_j \mathscr{T}_{n_j} p_j^{-1}$ on $\tilde{A}_j$. Extend f to a homeomorphism $S \to S$ by setting it equal to the identity on S–S'. By the removable singularity property f is quasiconformal on S. Thus we see that f is a Teichmüller map $S \to S$ which by Teichmüller's theorem is uniquely determined in its homotopy class.

Exactly as before this result has a direct extension to the case that S is a finitely punctured compact surface.

The case that $n_j \leqq -1$ for all j is solved in exactly the same manner. On the other hand we do not know the extremal map in the case $n_i \geqq 1$ for some indices i while $n_j \leqq -1$ for others.

3. Teichmüller disks.

3.1. Let T_g denote the Teichmüller space of closed surfaces of genus $g \geqq 2$. Points of T_g can be described as pairs (S,f) where f is a quasiconformal map of a fixed surface S_0 onto S, with the equivalence relation $(S_1,f_1) \equiv (S_2,f_2)$ if $f_2 f_1^{-1}: S_1 \to S_2$ is homotopic to a conformal map. The origin of T_g is taken as $(S_0, \mathrm{id.})$. We will make heavy use of the Teichmüller metric in T_g.

If $h: S_0 \to S_0$ is a homeomorphism not homotopic to the identity then h determines a non-trivial biholomorphic automorphism $(S,f) \mapsto (S,fh)$ of T_g which is also an isometry in the Teichmüller metric and conversely, by Royden's theorem [9], all automorphisms of T_g arise in this way. Such maps h, we recall, can be expressed as a product of Dehn twists.

3.2. The results of this section concerning Teichmüller maps are well known and are based on the following property. If $f: S_0 \to S$ is the map corresponding to the Beltrami coefficient $k\bar{\varphi}/|\varphi|$ for some quadratic differential φ on S_0 then in terms of suitable local coordinates on S_0 and S, f is an affine map except at the zeros of φ. We refer to Bers [1] for details (see also Kravetz [7]).

Suppose φ is a quadratic differential on S and consider the map of the unit disk $D = \{z \in \mathbf{C} : \ |z| < 1\}$

$$z \mapsto -z\bar{\varphi}/|\varphi| \, .$$

Each Beltrami coefficient $-z\bar{\varphi}/|\varphi|$ gives rise to a Teichmüller map $F_z: S \to S_z$ which is the unique extremal in its homotopy class. Define the *Teichmüller disk* with origin at $(S,f) \in T_g$ to be

$$D[\varphi] = \{(S_z, F_z f) : \ z \in D\} \, .$$

Note that $D[\varphi]$ depends not only on φ and S but also on the particular point (S,f). If we take the corresponding disk $D_1[\varphi]$ based at (S,f_1), the automorphism $f^{-1}f_1$ of S_0 determines a holomorphic automorphism of T_g that maps (S,f) to (S,f_1) and $D[\varphi]$ onto $D_1[\varphi]$.

The map

$$F: z \mapsto (S_z, F_z f) \in T_g$$

is a holomorphic injection of D into T_g with $F(0) = (S,f)$. Its image $D[\varphi]$ is totally geodesic in the Teichmüller metric and the pull-back to D of the restriction of this metric is the hyperbolic (Poincaré) metric.

LEMMA 3.1. *Suppose $x_1, x_2 \in T_g$ are distinct points. Suppose the Teichmüller disks D_1 and D_2 each pass through x_1 and x_2. Then $D_1 \equiv D_2$.*

PROOF. At $x_1 = (S, f)$ there exist quadratic differentials φ, ψ on S such that $D_1 = D[\varphi]$ and $D_2 = D[\psi]$. Then $x_2 \in D_1 \cap D_2$ is determined by a Teichmüller map associated with φ and also by one associated with ψ. But Teichmüller's uniqueness theorem implies that ψ must then be a scalar multiple of φ.

3.3 Suppose γ is a simple loop on S_0 not contractible to a point. Then a simple loop $f(\gamma) \subset S$ is determined at each point $(S, f) \in T_g$. Furthermore a Dehn twist $\mathscr{T}$ of order 1 about γ gives rise to the Dehn twist $f\mathscr{T}f^{-1}$ about $f(\gamma)$ and more generally induces a biholomorphic isometry $T[\gamma]$ of T_g,

$$T[\gamma]: (S, f) \to (S, f\mathscr{T}) .$$

$T[\gamma]$ maps one Teichmüller disk onto another.

Given a point $(S, f) \in T_g$ let $J \equiv J[f(\gamma)]$ denote the Jenkins differential on S corresponding to $f(\gamma)$. We are interested in the Teichmüller disk $D[J]$ with origin at (S, f).

LEMMA 3.2. *$D[J]$ is invariant under $T[\gamma]$. Furthermore the pull-back T of $T[\gamma]$ to D is a parabolic transformation with fixed point $z = 1$.*

PROOF. Using the formulas of section 1.2, $T[\gamma]^n$ sends $F(0) \in D[J]$ to $F(k_n e^{i\tau_n})$, $n = 0, \pm 1, \ldots$ In particular $D[J]$ and its $T[\gamma]$ image share the points $F(0)$, $F(k_1 e^{i\tau_1})$. Consequently by Lemma 3.1, $D[J]$ and its $T[\gamma]$ image coincide.

Now the pull-back T of $T[\gamma]$ to D is a biholomorphic automorphism and hence a Möbius transformation. It has the property $T^n(0) = k_n e^{i\tau_n}$ where $k_n \to 1$ and $\tau_n \to 0$ as $n \to \pm\infty$. Hence T is parabolic with fixed point $z = 1$. The actual formula for T is as follows, writing $k = k_1$, $\tau = \tau_1$,

$$T(z) = \frac{(2ke^{i\tau} - 1)z - ke^{i\tau}}{ke^{i\tau}z - 1}.$$

COROLLARY 3.3 *The two Teichmüller disks formed by $J[f_1(\gamma)]$ at (S_1, f_1) and $J[f_2(\gamma)]$ at (S_2, f_2) either coincide or are disjoint.*

3.4 In the Teichmüller disk $D[J]$ with origin $F(0) = (S, f)$ consider the *cusp ray from* (S, f). This is the image under F of the ray $\{0 \leqq \operatorname{Re} z < 1, \operatorname{Im} z = 0\}$ in D and is given by the Beltrami coefficients $-k\bar{J}/|J|$, $0 \leqq k < 1$.

Closely following Masur [8] we will describe the surfaces which correspond to the points on this ray.

Start by deleting the critical trajectories of $J[f(\gamma)]$ from S as in section 2.2 to get a conformal map of the cut surface S' onto an annulus $A = \{1 < |z| < R\}$.

Let A_k be the annulus

$$A_k = \{1 < |z| < R^{1+a}\}, \quad a = 2k/(1-k)$$

which arises from A by the radial stretch $T_a\colon z \to z|z|^a$.

Form a new Riemann surface from A_k by identifying certain arcs on ∂A_k as follows. Let $\alpha_1{}^k$ denote the inner component of ∂A_k and $\alpha_2{}^k$ the outer. Let $B_1{}^k \subset A_k$, $B_2{}^k \subset A_k$ be annular neighborhoods of $\alpha_1{}^k$, $\alpha_2{}^k$ so thin that the image of $B_1{}^k$ under the map $g_1\colon z \to z$ of $B_1{}^k$ into A is disjoint from the image of $B_2{}^k$ under $g_2\colon z \to (R/R^{1+a})z$.

Now S with its conformal structure is formed from A by making certain identifications of ∂A. Form exactly the same identifications of ∂A_k by using the conformal maps g_1, g_2. We get a Riemann surface S_k with a natural conformal embedding $A_k \to S_k{}' \subset S_k$. The maps g_1, g_2 determine a conformal map of a neighborhood of $S_k - S_k{}'$ into S which does not have a conformal extension to S_k.

The differential $-dz^2/z^2$ in A_k lifts to a quadratic differential J_k on S_k. The only problem in proving this is near ∂A_k but here the situation is governed by the differential $-dz^2/z^2$ in A which by construction is known to lift.

The stretch $T_a\colon A \to A_k$ also lifts and determines a quasiconformal map $T_a{}^*\colon S \to S_k$.

However everything is tied together: $T_a{}^*$ is the Teichmüller map corresponding to $-k\bar{J}/|J|$ and J_k is the Jenkins differential corresponding to $T_a{}^*(f(\gamma))$.

Now let $k \to 1$. Using different normalizations we get A_k to converge respectively to

$$A_\infty = \{1 < |z| < \infty\}, \quad A_\infty{}' = \{0 < |z| < 1\};$$

in the former case we can take $\alpha_1{}^k$ to be $|z| = 1$ for all k and in the latter case $\alpha_2{}^k$. Use g_1 and g_2 to make A_∞ and $A_\infty{}'$ into one twice punctured or two once punctured closed surfaces S_c depending on whether or not γ divides. Again $-dz^2/z^2$ lifts to S_c to give one (if S_c is connected) or two differentials on S_c with double poles at the two punctures with the same leading terms.

3.5. With the help of section 1.3 the surfaces corresponding to each point of $D[J]$ also can be described. Given $z \in D$ set $w = 2z/(1-z) \in \{\operatorname{Re} w > -1\}$, $w = u + iv$. The surface S_z corresponding to $F(z) \in D[J]$ is obtained from S and its representation as the annulus A as follows. First apply the radial stretch T_u to A. Then apply the twist $\mathscr{T}_\nu$, $\nu = (v/2\pi) \log R$, to the resulting annulus A_z. To form the surface S_z make identifications of ∂A_z as dictated by conformal maps of thin neighborhoods of ∂A_z back to neighborhoods of ∂A as in section 3.4.

Under a Dehn twist about $F_z f(\gamma)$ on S_z, the point $F(z)$ is sent to $(T[\gamma] \circ F)(z) = (F \circ T)(z)$ as deseribed in section 3.3.

From each point $F(z) \in D[J]$ there is a uniquely determined *cusp ray* which is the image under F of the geodesic ray in D from z to 1 (in the hyperbolic metric). This definition agrees with the geometric significance of the term as described in section 3.4, taking the surface corresponding to $F(z)$ as the base of operations rather than that associated with $F(0)$ (or alternatively, replacing $F(0)$ by $F(z)$ as the origin of $D[J]$). Any two cusp rays have zero asymptotic Teichmüller distance apart. Therefore, as shown in [8], all cusp rays of $D[J]$ determine the same surface S_c.

4. The differentials on the boundary space.

4.1. Let $D[\varphi]$ be a fixed Teichmüller disk with origin at $(S,f) \in T_g$ determined by the simple loop $\gamma \subset S_0$ and let S_c denote the surface or surfaces corresponding to the end point of the cusp ray eminating from $(S,f) \in D[\varphi]$ as constructed in section 3.4. The construction also gives a homeomorphism $f_c \colon S - f(\gamma) \to S_c$. Using $(S_c, \text{id.})$ as base point form the Teichmüller space $T(S_c) = \{(X,g)\}$ of complex dimension $3g-4$. Here the quasiconformal map $g \colon S_c \to X$ can be extended to the two punctures (the ideal boundary components). If S_c has two components, $T(S_c)$ splits into the product of the Teichmüller spaces of the individual components.

We will regard $T(S_c)$ as lying in ∂T_g (from the point of view of [2], $T(S_c)$ is called a boundary space). However the map $g f_c f \colon S_0 - \gamma \to X$, it should be noted, determines $f \colon S_0 \to S$ only up to a Dehn twist about γ: two homeomorphism $f_i \colon S_0 \to S$ taking γ to $f(\gamma)$ determine homotopic maps $g f_c f_i \colon S_0 - \gamma \to X$ if and only if $f_2^{-1} f_1$ is a power of the twist about γ (see [2]).

If S_c is connected fix an origin O and a set of simple loops α_1, α_2, β from O or if S_c has two components fix a pair of simple loops α_1, β; α_2, β' on each component. Take these so that α_i is retractable to the puncture

x_i of S_c and is so oriented that x_i lies to its left and assume β or β and β' are not freely homotopic to $\alpha_i^{\pm 1}$.

Suppose x is the puncture $g(x_i)$ of X, $(X,g) \in T(S_c)$. Represent the component Y of X associated with x as a fuchsian group Γ_x in the upper half plane U in such a way that

(*i*) a lift of $g(\alpha_i)$ determines the translation L: $\zeta \to \zeta + 1$ of Γ_x,

and

(*ii*) the element of Γ_x determined by the corresponding lift of $g(\beta)$ (or $g(\beta')$) has attractive fixed point at $\zeta = 1$.

In this way Y and x uniquely determine Γ_x.

It is known that the natural projection π: $U \to U/\Gamma_x = Y$ sends the half plane $\{\zeta : \operatorname{Im}\zeta > 1\}$ to a neighborhood N_x of x which is conformally equivalent to the once punctured disk.

In fact the map P: $\zeta \to z$ given by

$$z = P(\zeta) = \exp 2\pi i \zeta$$

is a conformal map $U/\{L\} \to D(1)$ where

$$D(r) = \{z : 0 < |z| < r\}.$$

Then $P \circ \pi^{-1}$ determines a conformal map $N_x \to D(\exp(-2\pi))$.

Note further that if $\mathscr{D}'$ is a simply connected region in $Y \cup \{x\}$ containing x and $\mathscr{D} = \mathscr{D}' - \{x\}$ then $P \circ \pi^{-1}$ determines a conformal map of $\mathscr{D}$ into $D(1)$.

We can use z as a local coordinate in N_x. It will be called the *canonical local coordinate* and N_x the *canonical neighborhood* of x. The group Γ_x will be called the *fuchsian equivalent* of Y corresponding to x.

4.2 Denote the two punctures of S_c by x_1, x_2 and fix $(X,g) \in T(S_c)$. We will apply a result of Strebel [12] and an extension of this [14] which allows one to specify the leading coefficient at each $g(x_i)$ rather than the reduced modulus (see also Jenkins [5]). According to his results there is a quadratic differential $J_c d\zeta^2 \equiv J_c[\gamma] d\zeta^2$ on X uniquely determined by the following properties (if X has two components then the restriction of $J_c d\zeta^2$ to each component is a quadratic differential which is uniquely determined). Let X' denote the result of removing the critical trajectories of $J_c d\zeta^2$ from X. Then X' has two components X_1', X_2' each of which is conformally equivalent to the once punctured disk. A curve around the puncture of X_i' is retractable in X_i' to the puncture $g(x_i)$ of X'. Furthermore X_1', X_2' are swept out by the closed trajectories of $J_c d\zeta^2$

each of which has length 2π in the metric $|J_c|^{\frac{1}{2}}|d\zeta|$. Consequently if ζ is *any* local coordinate at $g(x_i)$ with $\zeta(g(x_i))=0$ then $J_c d\zeta^2$ has expansion near $g(x_i)$

$$J_c d\zeta^2 = (-1/\zeta^2 + \ldots)d\zeta^2 .$$

Consider the function defined in each X_i'

$$z = p_i(\zeta) = C_i \exp(-i \textstyle\int^\zeta \sqrt{J_c}\, d\zeta)$$

where C_i is chosen so that the derivative *with respect to the canonical coordinate* ζ at $g(x_i)$, $\zeta(g(x_i))=0$, satisfies $(dz/d\zeta)(0)=1$. Then $p_i(\zeta)$ is a conformal map of X_i' onto some punctured disk

$$D(r_i) = \{z:\ 0 < |z| < r_i\}$$

where $r_i = r_i(X)$ depends only on X, $i=1,2$.

In terms of the parameter z we have

$$J_c d\zeta^2 = -dz^2/z^2 .$$

Starting with section 5.1 we are going to show how the differentials $J_c d\zeta^2$ can be used to index Teichmüller disks $D[J]$ in T_g. Before doing this however we must prove these differentials have certain uniformity properties on $\partial_\gamma T_g$ with respect to the Poincaré metric on the surfaces. To do this we must first discuss their extremal properties.

4.3. Suppose D_1', D_2' are two simply connected, mutually disjoint regions in $X \cup \{g(x_1)\} \cup \{g(x_2)\}$ with $g(x_i) \in D_i'$, $(X,g) \in T(S_c)$. Set $D_i = D_i' - \{g(x_i)\}$. If ζ denotes the canonical local coordinate in N_x about $x = g(x_i)$, the *reduced modulus* $M(D)$ of $D = D_i$ with respect to x and the associated canonical coordinate is defined as

$$M(D) = \lim_{r\to 0} \left(M(r) + (\log r)/2\pi\right) .$$

Here $M(r)$ is the modulus of the annular region

$$D - \{p \in N_x :\ |\zeta(p)| < r\}$$

on X.

Alternatively let f be the conformal map of D onto the punctured disk $D(r_0)$ where r_0 and f are uniquely determined by the condition $(df/d\zeta)(0)=1$ in terms of the canonical parameter ζ at x, $\zeta(x)=0$. Then $M(D) = (\log r_0)/2\pi$ and r_0 is called the *mapping radius* of D with respect to x and ζ. Thus it is clear that for $D \subset D^*$, $M(D) \leqq M(D^*)$.

If π is the natural projection $U \to U/\Gamma_x$ and $P(\zeta)$ is as defined in section 4.1, $z = P\circ\pi^{-1}(\zeta)$ determines a conformal map of D into the once punctured unit disk $D(1)$ with $0 \to 0$ and $(dz/d\zeta)(0)=1$. Hence $M(D)$

$< M(D(1)) = 0$. On the other hand D can be chosen to contain N_x and in this case

$$(1) \qquad -1 = M(N_x) < M(D) < 0 .$$

According to Strebel [12, 13] the degenerate differential $J_c d\zeta^2$ on X has the following extremal property with respect to any D_1, D_2 as constructed above:

$$2\pi(M(D_1) + M(D_2)) \leqq \log r_1(X) + \log r_2(X) ,$$

since $M(X_i') = (\log r_i)/2\pi$. Combining this with (1) on choosing D_i to contain $N_{g(x_i)}$ we obtain

LEMMA 4.1. *There exists $R_1 > 0$ such that for all $(X,g) \in T(S_c)$, $r_i(X) \geqq R_1$, $i = 1,2$.*

4.4. Now we can prove the required uniformity property. For $R < \exp(-2\pi)$ and ζ the canonical coordinate in N_x set $N_x(R) = \{p \in N_x : |\zeta(p)| < R\}$.

LEMMA 4.2. *There exists $R^* > 0$ such that for each $(X,g) \in T(S_c)$ and for each puncture $x = g(x_i)$ of X, it is true that $N_x(R^*) \subset X_i'$, $i = 1,2$.*

PROOF. Using Lemma 4.1 consider the composed map $F(z) = P \circ \pi^{-1} \circ p_i^{-1}(z)$ of $D(R_1)$ into X_i' and then into $D(1)$. $F(z)$ is conformal with $F(0) = 0$ and $F'(0) = 1$. Consequently by the Koebe $\frac{1}{4}$ theorem, the F-image of $D(R_1)$ contains the punctured disk $D(R_2)$ with $R_2 = R_1/4$. Now set $R^* = \min(R_2, \exp(-2\pi))$.

LEMMA 4.3. *Given $r \leqq R^*$ set $X(r) = X - N_{g(x_1)}(r) - N_{g(x_2)}(r)$. There exists $M = M(r) < \infty$ such that for all $(X,g) \in T(S_c)$,*

$$\iint_{X(r)} |J_c| < M .$$

PROOF. It suffices to prove this for each component X_i' of X' with $X_i'(r) = X_i' \cap X(r)$, $i = 1,2$. Given r apply the Koebe $\frac{1}{4}$ theorem to the normalized map $p_i(\zeta)$ of $N_{g(x_i)}(r)$ into $D(r_i)$. Its image in $D(r_i)$ contains the punctured disk $D(\varrho)$ for $\varrho = r/4$. Hence comparing moduli of annuli we have

$$\iint_{X_i'(r)} |J_c| \leqq 2\pi \log(r_i/\varrho) < -2\pi \log \varrho .$$

COROLLARY 4.4. *$J_c d\zeta^2$ varies continuously on $T(S_c)$.*

PROOF. Suppose $(X_n, g_n) \to (X, g) \in T(S_c)$ and assume for simplicity that X has two components. In particular $(Y_n, g_n) \to (Y, g)$ where Y_n is one of the components of X_n and Y of X. We can choose fuchsian equivalents Γ_n of the Y_n as to converge to the fuchsian equivalent Γ of Y. Let $\mathscr{J}_n$ denote the automorphic form on U resulting from lifting the differential $J_c d\zeta^2$ on Y_n. Using Lemma 4.3 and passing to a subsequence if necessary we see that $\mathscr{J}_n$ converges uniformly on compact subsets to a non-zero automorphic form $\mathscr{J}$ with respect to Γ. $\mathscr{J}$ projects to a quadratic differential $Hd\zeta^2$ on Y. Examining (cf. [13]) the convergence of $\mathscr{J}_n$ in U we see each non-critical trajectory of $Hd\zeta^2$ is closed on Y, has length 2π in the metric $|H|^{\frac{1}{2}}|d\zeta|$ (this is why $\mathscr{J} \neq 0$) and is retractable to either $g(x_1)$ or $g(x_2)$. These properties are true because each compact segment σ of length $\geqq 2\pi + \varepsilon$ of a non-critical (horizontal) trajectory of $\mathscr{J}$ in U is the limit of a segment σ_n of a non-critical trajectory of $\mathscr{J}_n$ with $L(\sigma_n) \cap \sigma_n \neq \emptyset$ (L is the unit translation in Γ_n). Thus $L(\sigma) \cap \sigma \neq \emptyset$ and the projection $\pi(\sigma) \subset Y$ is closed and retractable to the puncture which determines $L \in \Gamma$. This trajectory structure in turn implies that $Hd\zeta^2$ has a double pole at whichever of $g(x_1), g(x_2)$ is associated with Y, necessarily with expansion $(-1/\zeta^2 + \ldots)d\zeta^2$. But these properties uniquely characterize $J_c d\zeta^2$.

5. A foliation by Teichmüller disks.

5.1. Fixing a point $(S, f) \in T_g$ and a non-trivial simple loop γ on S_0 we constructed the Jenkins differential $Jd\zeta^2$ on S in section 2.2. Denoting by S' the annular region resulting from the removal of its critical trajectories from S we saw that $Jd\zeta^2$ determines a conformal map of S' onto the annulus $A = \{1 < |z| < R\}$. In section 3.4 we constructed the cusp ray in the Teichmüller disk $D[J]$ from the origin (S, f) and found a pinched surface S_c corresponding to its end point and consequently to the end point of the cusp ray from any point of $D[J]$. Actually S_c was represented as the two punctured disks $D(1) = \{0 < |z| < 1\}$ and $D(1)' = \{1 < |z| < \infty\}$ with certain identifications involving the unit circles as dictated by $Jd\zeta^2$. Furthermore the differential $-dz^2/z^2$ on $D(1) \cup D(1)'$ lifts to a differential $\varphi d\zeta^2$ on S_c. This lifted differential is in fact the degenerate $J_c d\zeta^2$ for S_c because $\varphi d\zeta^2$ has the right expansion $(-1/\zeta^2 \ldots)$ $d\zeta^2$ at the punctures and the right trajectory structure which together characterize $J_c d\zeta^2$.

An "opening up" process can be described by reversing these steps, starting at S_c. Take the image of the annulus $\{s^{-\frac{1}{2}} < |z| < 1\} \subset D(1)$, $\infty > s \geqq R$, under the map $z \to sz$ and attach it to the annulus $\{1 < |z| <$

$s^{\frac{1}{2}}\} \subset D(1)'$ with the two points at $z=\sqrt{s}$ corresponding. The identifications involving the boundary components of the resulting annulus are as dictated by $J_c d\zeta^2$ in the manner suggested in section 3.4. Thus we get a one parameter family of compact surfaces of genus g, which when $s=R$ is the original surface S.

However this opening process as a mapping into T_g is not well defined. The problem is that the map $f_c f$: $S_0-\gamma \to S_c$ determines f only up to Dehn twists about γ (cf. section 4.1) so that a presentation of the fundamental group $\pi_1(S_c)$ does not uniquely determine one of $\pi_1(S)$. Before embarking on the opening process a decision must be made fixing a geometric prescription in $D(1)\cup D(1)'$ on how a presentation of the fundamental group of each opened surface is to be obtained. Once this is done the opening process described above flows from S_c along the cusp ray of some point Q of the orbit of (S,f) under $T[\gamma]$, terminating at Q.

5.2. The result S_c' of deleting the critical trajectories of $J_c d\zeta^2$ from S_c has two components which we label so that S'_{c1} is obtained from $D(1)$ and S'_{c2} from $D(1)'$. Let x_i denote the puncture of S'_{ci} and Γ_i the fuchsian equivalent of x_i and the component of S_c associated with it.

Fix a point $\xi_i \in U$ so that

a) $\xi(x_i)\equiv\pi(\xi_i)$ lies in the universal canonical neighborhood $N_{x_i}(R^*)$ $\subset S'_{ci}$ of x_i,

(here π is the projection $U \to U/\Gamma_i$), and

b) under the map $S'_{ci} \to D(1)$ or $D(1)'$, the image of $\xi(x_i)$ lies on the positive real axis, $i=1,2$.

With $\xi_1,\xi_2 \in U$ and R now fixed and determined from (S,f) we are ready to move through points $(X,g)\in T(S_c)$. By Corollary 4.4 the differentials $J_c d\zeta^2$ and the associated mapping radii $r_i(X)$ vary continuously on $T(S_c)$. The same is true of the points $\xi(g(x_i)) \in N_{g(x_i)}(R^*)$ determined by projecting ξ_i using the fuchsian equivalent corresponding to $g(x_i)$.

Label the two regions resulting from removing the critical trajectories of $J_c d\zeta^2$ from X so that $g(x_i)$ is the puncture of X_i'. In section 4.2 we introduced $p_i(\xi)$ which maps X_i' onto $D(r_i)$. Now define

$$q_1(\zeta) = e^{i\theta_1}p_1(\zeta)/r_1, \quad q_2(\zeta) = r_2 e^{i\theta_2}p_2(\zeta)^{-1}$$

where θ_i is determined by the condition that $q_i\big(\xi(g(X_i))\big)>0$, $i=1,2$. $q_1(\zeta)$ maps X_1' to $D(1)$ and $q_2(\zeta)$ maps X_2' to $D(1)'$.

5.3. We have fixed R as determined from (S,f) by the Jenkins differential J on S. Let T denote the parabolic transformation in D resulting from pulling back $T[\gamma]$ from $D[J]$ (Lemma 3.2). We will construct a homeomorphism (in terms of the quotient metrics)

$$M_*: T(S_c)\times D/\{T\} \to T_g/\{T[\gamma]\}$$

where $\{T[\gamma]\}$ denotes the infinite cyclic and fixed point free subgroup of the Teichmüller modular group generated by $T[\gamma]$.

Fix $(X,g)\in T(S_c)$. Given $z\in D$ set

$$w = 2z/(1-z) = u+iv \in \{\mathrm{Re}\, w > -1\}.$$

Note that T corresponds to the translation $w\mapsto w+2\pi i/\log R$. Take the image of the annulus

$$\{(R^{1+u})^{-\frac{1}{2}} < |\zeta| < 1\} \subset D(1)$$

under the map

$$H_z: \zeta \to e^{i\varphi}R^{1+u}\zeta, \qquad \varphi = v\log R$$

and glue it to the outer contour of the annulus

$$\{1 < |\zeta| < (R^{1+u})^{\frac{1}{2}}\} \subset D(1)'$$

without further rotation obtaining the annulus

$$A_z = \{1 < |\zeta| < R^{1+u}\}.$$

By means of the conformal maps $\zeta\mapsto\zeta$, $\zeta\mapsto H_z^{-1}(\zeta)$ in thin neighborhoods of the components of ∂A_z and the prescriptions concerning identifications on $\partial D(1)$, $\partial D(1)'$ as given by q_1, q_2 (cf. section 3.4) we obtain by making identifications on ∂A_z a compact surface $S_z[X,g]$.

In particular for $z=0$ we obtain a surface $S[X,g]$ and a natural homeomorphism given by the construction $g_c: X\to S[X,g]-\gamma'$ where γ' is an appropriate simple loop on $S[X,g]$. The homeomorphism $g_c g f_c f$: $S_0-\gamma \to S[X,g]-\gamma'$ determines a presentation of $\pi_1(S[X,g])$ up to Dehn twists about γ'. Correspondingly this is true for each $S_z[X,g]$, $z\in D$.

The differential $J_c d\zeta^2$ on X determines a Jenkins differential on each $S_z[X,g]$ by lifting $-d\zeta^2/\zeta^2$ from A_z. For $z=0$ this differential J corresponds to γ' and the annular region determined by it on $S[X,g]$ has modulus $(\log R)/2\pi$.

Comparing the calculations here with those of section 3.5 we conclude that for fixed (X,g) the map M_* determined by the above construction

$$M_*: D/\{T\} \to D[J]/\{T[\gamma]\},$$

with $z=0$ mapped to the orbit derermined by $S[X,g]$, is a holomorphic homeomorphism. It is, in fact, the projection of a map F described in section 3.2.

Now let (X,g) vary in $T(S_c)$. Because the modulus $(\log R)/2\pi$ of the annulus determined by J on $S[X,g]$ is independent of (X,g), the parabolic transformation T in D is independent of (X,g) as well. For fixed z the map M_* is continuous because $J_c d\zeta^2$ and hence q_1, q_2 vary continuously on $T(S_c)$. M_* sends (X,g) to a point on some slice $D[J]/\{T[\gamma]\}$ where $D[J]$ is uniquely determined by the condition that its cusp rays terminate at (X,g). That is, M_* in injective. Since every point of T_g lies on some disk $D[J]$, M_* is also surjective and therefore a homeomorphism.

Because $T(S_c)$ and T_g are simply connected, M_* can be lifted to a homeomorphism

$$M: T(S_c) \times D \to T_g$$

which is uniquely determined by the requirement that M map $(S_c, \text{id.}) \times \{0\}$ to (S,f).

The proof of the Theorem is now complete.

5.4. Except for briefly describing the basic situation for finitely punctured compact surfaces in section 2.4 we have dealt only with T_g and its boundary space $\partial_\gamma T_g$ obtained by pinching γ. However everything we have done goes through without change if we start with $T(g,n)$, the Teichmüller space of a compact surface of genus g with n punctures S_0, and refer this to its boundary space $\partial_\gamma T(g,n)$, where γ is now a non-trivial simple loop in S_0 not retractable to any puncture.

In the case of $T(1,1)$ or $T(0,4)$ our procedure represents the Teichmüller space itself as a Teichmüller disk. The classical modular group is the Teichmüller modular group of $T(0,4)$ but for $T(1,1)$ is a subgroup of index two.

In general, starting with S_0, an n-times punctured surface of genus g, choose $3g+n-3$ mutually disjoint simple loops that together cut S_0 into a union of triply connected regions. Start by decomposing $T(g,n)$ into $\partial_{\gamma_1} T(g,n) \times D$. Then write the (possibly product) Teichmüller space $\partial_{\gamma_1} T(g,n)$ as the product that results on pinching γ_2, etc. Alternatively, $T(g,n)$ can be built up from $T(1,1)$ and $T(0,4)$ (or $T(0,3)$) as a product D^N. In either way we obtain a proof of the Corollary.

In the germinal paper [3], Fenchel made use of the fact that T_g is a cell to show that every element of finite order in the Teichmüller modular group (mapping class group) can be represented as a conformal automorphism of some Riemann surface of genus g.

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