ON THE "THREE SPACE PROBLEM"

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To Werner Fenchel on his 70th birthday.

1. Introduction.

Let $X$ be a Banach space and $Y$ a closed subspace of $X$. In this paper we will study the so-called three space problem: if one has information about two of the Banach spaces $X$, $Y$ and $X/Y$, what can be said about the third one. In the sequel we shall have information about $Y$ and $X/Y$ and draw conclusions about $X$. For other aspects of the problem cf. [7].

It is a classical result [1, II.4, p. 19–20] that if $Y$ and $X/Y$ are reflexive then $X$ is also reflexive. D. P. Giesy has shown [3, Th. II.9] that if $Y$ and $X/Y$ are $B$-convex then $X$ is also $B$-convex. We prove below that if $Y$ and $X/Y$ are super-reflexive then $X$ is also super-reflexive.

We also solve the following problem (apparently due to Palais): if each of the spaces $Y$ and $X/Y$ is isomorphic to a Hilbert space, is $X$ isomorphic to a Hilbert space? We prove that $X$ is in a certain sense close to being isomorphic to a Hilbert space, but that it need not be isomorphic to a Hilbert space.

2. Some inequalities.

In this section, we obtain information on the behavior of Rademacher series (resp. of martingales) with values in $X$ knowing the corresponding information for $Y$ and $X/Y$. We denote by $(\varepsilon_n)$ the Rademacher system on the interval $[0,1]$. Let $(\Omega, \mathcal{A}, P)$ be a probability space, a sequence of random variables $(X_n)_{n \geq 0}$ on $(\Omega, \mathcal{A}, P)$ with values in a Banach space is called a martingale if there exists an increasing sequence $(\mathcal{A}_n)_{n \geq 0}$ of sub-$\sigma$-algebras of $\mathcal{A}$ such that $\forall n \geq 0$, $X_n = E^{\mathcal{A}_n}(X_{n+1})$.

In this paper we shall say briefly ,,martingale” meaning ,,martingale defined on some probability space $(\Omega, \mathcal{A}, P)$”; moreover if $Z$ is a Banach space valued random variable on a probability space $(\Omega, \mathcal{A}, P)$ we shall write simply $\|Z\|_2$ for $(\int \|Z(\omega)\|^2 dP(\omega))^{\frac{1}{2}}$.

* Research was partially sponsored by NSF Grant GP 43083.
Sections 1-3 received November 25, 1974. Section 4 received February 3, 1975.

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When we wish to distinguish the norms of the Banach spaces involved, we will write \( \| \cdot \|_X, \| \cdot \|_Y, \ldots \) for the norms in the spaces \( X, Y, \ldots \). In particular recall the definition of the norm in \( X/Y \): let \( \pi \) denote the canonical projection from \( X \) onto \( X/Y \), by definition we have

\[
\forall x \in X, \| \pi(x) \|_{X/Y} = \inf \{ \| x + y \| \mid y \in Y \}.
\]

**Definition/Notation.** Let \( X \) be a Banach space and \( n \) an integer.

We define \( a_n^X \) as the smallest positive number \( a \) such that:

\[
\left( \int \| \sum_{i=1}^n \varepsilon_i(t)x_i \|^2 dt \right)^{1/2} \leq a \left( \sum_{i=1}^n \| x_i \|^2 \right)^{1/2}
\]

for all \( n \)-tuples \( (x_i)_{1 \leq i \leq n} \) of elements of \( X \).

We define \( \alpha_n^X \) as the smallest positive number \( \alpha \) such that:

\[
\| X_n \|_2 \leq \alpha \left( \sum_{i=1}^n \| X_i - X_{i-1} \|_2^2 \right)^{1/2}
\]

for all martingales \( (X_n)_{n \geq 0} \) with values in \( X \) and such that \( X_0 = 0 \). Obviously we have \( a_n^X \leq \alpha_n^X \leq \sqrt{n} \) for all integers \( n \). The following theorem motivates the preceding definitions.

**Theorem 1.** Let \( X \) be a Banach space and \( Y \) be a closed subspace of \( X \). The following inequalities hold for all integers \( n \) and \( k \):

1. \( a_{nk}^X \leq a_n^Y a_k^X + a_n^Y a_k^{X/Y} + a_n^X a_k^{X/Y} \)
2. \( \alpha_{nk}^X \leq \alpha_n^Y \alpha_k^X + 2 \alpha_n^Y \alpha_k^{X/Y} + 2 \alpha_n^X \alpha_k^{X/Y} \).

**Proof.** We start with (1): let \( (x_j)_{j \leq nk} \) be a \( nk \)-tuple in \( X \). \( \forall \theta \in [0, 1] \), let \( X_i(\theta) \) denote \( \sum_{i=(i-1)k < j \leq ik} \varepsilon_j(\theta)x_j \). Let \( \pi \) denote the canonical projection from \( X \) onto \( X/Y \). Then \( \forall \theta \in [0, 1], \forall i = 1, 2, \ldots, n, \forall \gamma > 0 \) there exists \( Y_i(\theta) \) in \( Y \) such that: \( \| X_i(\theta) + Y_i(\theta) \|_X \leq \| \pi(X_i(\theta)) \|_{X/Y} + \gamma \). Let \( A(\theta) \) be the integral

\[
\left( \int \| \sum_{i=1}^n \varepsilon_i(t)X_i(\theta) \|^2 dt \right)^{1/2};
\]

by convexity of the norm, we have:

\[
A(\theta) \leq \left( \int \| \sum_{i=1}^n \varepsilon_i(t)Y_i(\theta) \|^2 dt \right)^{1/2} + \left( \int \| \sum_{i=1}^n \varepsilon_i(t)(X_i(\theta) + Y_i(\theta)) \|^2 dt \right)^{1/2},
\]

so that by the definitions of \( a_n^Y \) and \( a_n^X \):

\[
A(\theta) \leq a_n^Y \left( \sum_{i=1}^n \| Y_i(\theta) \|^2 \right)^{1/2} + a_n^X \left( \sum_{i=1}^n \| X_i(\theta) + Y_i(\theta) \|^2 \right)^{1/2}.
\]

But on one hand \( \| Y_i(\theta) \| \leq \| X_i(\theta) \| + \| Y_i(\theta) + X_i(\theta) \| \) and on the other hand \( \| X_i(\theta) + Y_i(\theta) \| \leq \| \pi(X_i(\theta)) \| + \gamma \), so that we have:

\[
A(\theta) \leq a_n^Y \left( \sum_{i=1}^n \| X_i(\theta) \|^2 \right)^{1/2} + (a_n^Y + a_n^X) \left[ \left( \sum_{i=1}^n \| \pi(X_i(\theta)) \|^2 \right)^{1/2} + \gamma \sqrt{n} \right].
\]
which gives after integration:

\[(\int A(\theta)^2 d\theta)^\frac{1}{2} \leq a_n^X(\sum_{i=1}^{\frac{\gamma n}{2}} ||X_i||_2)^\frac{1}{2} + (a_n^Y + a_n^X)[(\sum_{i=1}^{\frac{\gamma n}{2}} ||\tau(X_i)||_2)^\frac{1}{2} + \gamma \sqrt{n}] \]

\[\leq [a_n^X a_k^X + (a_n^Y + a_n^X)a_k^X/X](\sum_{i=1}^{\frac{\gamma n}{2}} ||x_j||_2)^\frac{1}{2} + (a_n^Y + a_n^X)\gamma \sqrt{n} \]

By an easy argument of symmetry one finds that

\[(\int A(\theta)^2 d\theta)^\frac{1}{2} = (\int ||\sum_{j=1}^{j=n_k} \varepsilon_j(t)x_j||^2 dt)^\frac{1}{2} \]

the result then follows since \(\gamma > 0\) is arbitrary.

Let us now prove (2): Let \((X_m)_{m \geq 0}\) be a martingale with values in \(X\) such that \(X_0 = 0\), adapted to a sequence of \(\sigma\)-algebras \((\mathcal{A}_m)_{m \geq 0}\). We write \(A_t = X_{tk} - X_{(t-1)k}\) for \(i = 1, 2, \ldots, n\), and set \(Z_0 = 0\) and \(\forall \lambda = 1, 2, \ldots, Z_\lambda = \sum_{i \leq t \leq \lambda} A_t\). Obviously \((Z_\lambda)_{\lambda \geq 0}\) is a martingale with respect to the sequence of \(\sigma\)-algebras \((\mathcal{A}_{tk})_{k \geq 0}\). Now, as easily seen, \(\forall \gamma > 0, \forall i = 1, 2, \ldots,\) there exists an \(\mathcal{A}_{tk}\)-measurable random variable \(Y_t\) with values in \(Y\) such that:

\[(3) \quad ||A_t + Y_t||_{L^2(X)} \leq ||\tau(A_t)||_{L^2(X/Y)} + \gamma \cdot \]

We define a martingale \((U_\lambda)_{\lambda \geq 0}\) with values in \(Y\) by setting \(U_0 = 0\) and \(\forall \lambda = 1, 2, \ldots,\)

\[U_\lambda = \sum_{1 \leq i \leq \lambda} E_{\mathcal{A}_{ti}}(Y_t) - E_{\mathcal{A}_{(t-1)i}}(Y_t) \]

\((U_\lambda)_{\lambda \geq 0}\) is a martingale adapted to the sequence of \(\sigma\)-algebras \((\mathcal{A}_{tk})_{k \geq 0}\).

We notice that:\(\forall i = 1, 2, \ldots, n\)

\[U_t - U_{t-1} = E_{\mathcal{A}_{tk}}(Y_t) - E_{\mathcal{A}_{(t-1)k}}(Y_t) \]

and (since \(E_{\mathcal{A}_{(t-1)k}}(A_t) = 0\)) that:

\[A_t + U_t - U_{t-1} = E_{\mathcal{A}_{tk}}(A_t + Y_t) - E_{\mathcal{A}_{(t-1)k}}(A_t + Y_t) \]

by the triangle inequality:

\[||U_t - U_{t-1}||_2 \leq ||A_t||_2 + ||A_t + U_t - U_{t-1}||_2 \]

so that, by the continuity of the conditional expectations on \(L^2(X)\), we have:

\[(4) \quad ||A_t + U_t - U_{t-1}||_2 \leq 2||A_t + Y_t||_2 \]

hence:

\[(5) \quad ||U_t - U_{t-1}||_2 \leq ||A_t||_2 + 2||A_t + Y_t||_2 \]

Now, using the definition of \(\alpha_n^X\), we get:

\[(6) \quad ||X_{nk}||_2 = ||Z_n||_2 \leq ||U_n||_2 + ||Z_n + U_n||_2 \]

\[\leq \alpha_n^X(\sum_{i=1}^{n_k} ||U_t - U_{t-1}||_2^2)^\frac{1}{2} + \alpha_n^X(\sum_{i=1}^{n_k} ||A_t + U_t - U_{t-1}||_2^2)^\frac{1}{2} \]

\[\leq \alpha_n^X(\sum_{i=1}^{n_k} ||A_t||_2^2)^\frac{1}{2} + 2(\alpha_n^Y + \alpha_n^X)(\sum_{i=1}^{n_k} ||A_t + Y_t||_2^2)^\frac{1}{2} \]

the last inequality being deduced from (4) and (5).
Now fix \( i \) between 1 and \( n \), and set \( V_0^i = 0 \) and \( \forall \lambda = 1, 2, \ldots, V_{\lambda}^i = \sum_{(\lambda - 1)k < j \leq (\lambda - 1)k + k} X_j - X_{j-1} \), so that \( (V_\lambda^i)_{\lambda \geq 0} \) is a martingale adapted to the sequence of \( \sigma \)-algebras \( (\mathcal{A}_{(\lambda - 1)k})_{\lambda \geq 0} \); we have therefore:

\[
\|A_\lambda^i\|_2 = \|V_k^i\|_2 \leq \alpha_k^X \left( \sum_{(\lambda - 1)k < j \leq \lambda k} \|X_j - X_{j-1}\|_2 \right)^\lambda
\]

and:

\[
\|\pi(A_\lambda^i)\|_2 = \|\pi(V_k^i)\|_2 \leq \alpha_k^X \left( \sum_{(\lambda - 1)k < j \leq \lambda k} \|\pi(X_j) - \pi(X_{j-1})\|_2 \right)^\lambda
\]

\[
\leq \alpha_k^X \left( \sum_{(\lambda - 1)k < j \leq \lambda k} \|X_j - X_{j-1}\|_2 \right)^\lambda.
\]

With (3), (6) and the inequalities above, we finally obtain:

\[
\|X_n^k\|_2 \leq \left( \alpha_n^Y \alpha_k^X + 2 \alpha_n^Y \alpha_k^X + 2 \alpha_n^X \alpha_k^X \right) \left( \sum_{j=1}^{k} \|X_j - X_{j-1}\|_2 \right)^k + 2\gamma \sqrt{n} \left( \alpha_n^Y + \alpha_n^X \right),
\]

and this concludes the proof of (2) since \( \gamma > 0 \) is arbitrary.

3. Applications.

We first recall some definitions:

A Banach space \( X \) is called \( B \)-convex if there exist an integer \( n \) and \( \epsilon > 0 \) such that

\[
\inf \|\sum_{i=1}^n \epsilon_i x_i\| \leq n(1 - \epsilon)
\]

for all \( n \)-tuples \((x_1, x_2, \ldots, x_n)\) in the unit ball of \( X \) and the infimum is over all choices of \( n \)-signs \((\epsilon_1, \ldots, \epsilon_n)\) in \((-1, 1)^n\).

Following James, we say that a Banach space \( Z \) is finitely representable in a Banach space \( X \) if for all \( \epsilon > 0 \) and any finite dimensional subspace \( M \) of \( Z \) there exist a subspace \( N \) of \( X \) and an isomorphism \( T \) from \( M \) onto \( N \) such that

\[
\|T\| \|T^{-1}\| \leq 1 + \epsilon.
\]

A Banach space \( X \) is called super-reflexive if all the Banach spaces \( Z \) which are finitely representable in \( X \) are reflexive.

R. C. James has recently produced [4] an example of a \( B \)-convex Banach space which is not super-reflexive. It is proved in [2] that a Banach space is super-reflexive if and only if there is an equivalent norm on \( X \) for which the space is uniformly convex, i.e. \( \forall \epsilon \in (0, 2) \)

\[
\delta(\epsilon) = \inf \{1 - \|\frac{1}{2}(x+y)\| \mid \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\} > 0.
\]

Moreover (cf. [9]) it is possible to choose a renorming for which the modulus of convexity \( \delta(\epsilon) \) is greater than \( K\epsilon^q \) for some constant \( K > 0 \) and some \( q < \infty \).
D. P. Giesy proved [3, th. II.9] that if $Y$ and $X/Y$ are $B$-convex then $X$ is also $B$-convex; actually this also follows from (1) since it is known that a Banach space $X$ is $B$-convex iff $a_n^X < \sqrt{n}$ for some integer $n$ or iff $a_n^{X/\sqrt{n}}$ tends to 0 when $n$ tends to infinity (cf. [8, exp. VII. p. 12–13]). The situation is quite similar in the case of super-reflexivity; the following proposition is used and discussed in [10].

**Proposition 1:** Let $X$ be a Banach space; the following conditions are equivalent:

(i) $X$ is super-reflexive.

(ii) $\alpha_n^X < \sqrt{n}$ for some integer $n$.

(iii) $\alpha_n^{X/\sqrt{n}} \rightarrow 0$ when $n \rightarrow \infty$.

(iv) There exists a real number $p > 2$ such that

$$\alpha_n^{X/n^{1/p}} \rightarrow 0 \quad \text{when } n \rightarrow \infty.$$ 

**Remark 1.** The equivalence of (ii), (iii) and (iv) can be easily deduced from (2) which becomes, when taking $Y = X$, $\alpha_{nk}^X \leq \alpha_n^X \alpha_k^X$ (since $\alpha_n^{(0)} = 0$ for all $n \in \mathbb{N}$).

**Theorem 2.** If a Banach space $X$ has a closed subspace $Y$ such that both $Y$ and $X/Y$ are super-reflexive then $X$ is super-reflexive.

**Proof.** From (2) we deduce: (since obviously $\alpha_n^X \leq \sqrt{n}$ for all $n \in \mathbb{N}$)

$$\forall n \in \mathbb{N}, \quad \alpha_n^{x_n^X} \leq n[\alpha_n^Y / \sqrt{n} + 2\alpha_n^X / \sqrt{n} \cdot \alpha_n^{X/Y}/\sqrt{n} + 2\alpha_n^{X/Y}/\sqrt{n}] .$$

If $Y$ and $X/Y$ are super-reflexive, then (proposition 1) $\alpha_n^Y / \sqrt{n} \rightarrow 0$ and $\alpha_n^{X/Y}/\sqrt{n} \rightarrow 0$ when $n \rightarrow \infty$; hence when $n$ is sufficiently large we must have $\alpha_n^{x_n^X} < n$ which implies (proposition 1) that $X$ itself is super-reflexive.

We will now focus our attention on the case where both $Y$ and $X/Y$ are isomorphic to a Hilbert space. The sequences $(a_n^X)_{n \geq 1}$ and $(\alpha_n^X)_{n \geq 1}$ give information on the isomorphic structure of the Banach space $X$. For instance, S. Kwapien has proved in [5] that $\sup_{n \geq 1} a_n^X a_n^{x_n^X}$ is finite if and only if the Banach space $X$ is isomorphic to a Hilbert space. Also, it is proved in [10] (see also [9]) that $\sup_{n \geq 1} \alpha_n^X$ is finite if and only if the Banach space $X$ has an equivalent norm $\| \cdot \|$ for which the modulus of smoothness

$$q(t) = \sup \left\{ \frac{1}{2}(|x+ty| + |x-ty|) - 1 \mid x, y \in X, |x| = |y| = 1 \right\}$$

satisfies $q(t) \leq Kt^2$ for all $t > 0$, for some constant $K$. 

The Banach spaces \( X \) for which \( \sup_{n \geq 1} a_n^X \) is finite are usually referred to as spaces of type 2.

**Theorem 3.** Let \( X \) be a Banach space and \( Y \) a closed subspace of \( X \).

(a) If both spaces \( Y \) and \( X/Y \) are of type 2 (i.e. both \( \sup_{n \geq 1} a_n^Y \) and \( \sup_{n \geq 1} a_n^{X/Y} \) are finite) then there exist constants \( c \) and \( \alpha \) such that:

\[
\forall n \geq 2, \quad a_n^X \leq c (\log n)^\alpha.
\]

(b) If both \( \sup_{n \geq 1} \alpha_n^X \) and \( \sup_{n \geq 1} \alpha_n^{X/Y} \) are finite then there exist constants \( c \) and \( \alpha \) such that:

\[
\forall n \geq 2, \quad \alpha_n^X \leq c (\log n)^\alpha.
\]

**Proof.** Let \( c_1 = \sup_{n \geq 1} a_n^Y, \ c_2 = \sup_{n \geq 1} a_n^{X/Y} \); the inequality (1) yields:

\[
\forall n, \ k \in \mathbb{N}, \quad a_{nk}^X \leq c_1 a_k^X + c_1 c_2 + a_n^X c_2;
\]

since obviously (unless \( X = \{0\} \)), \( 1 \leq a_n^X \) for all integers \( n \), we obtain in particular:

(7)

\[
\forall n \in \mathbb{N}, \quad a_{n^2}^X \leq (c_1 + c_1 c_2 + c_2) a_n^X.
\]

Let \( \alpha \) be such that \( 2^\alpha = c_1 + c_1 c_2 + c_2 \) and set \( b_n = a_n^X/(\log n)^\alpha \) for all \( n = 2, 3, \ldots \); then (7) becomes:

(8)

\[
\forall n \in \mathbb{N}, \quad b_{n^2} \leq b_n.
\]

Let \( n \) be an integer, \( n \geq 2 \); there exist \( k \geq 0 \) such that:

\[
N_k = 2^{2^k} \leq n < 2^{2^{k+1}} = N_{k+1}^2.
\]

Since \( (a_n^X)_{n \geq 1} \) is clearly increasing, we can write:

\[
b_n = a_n^X/(\log n)^\alpha \leq a_{N_k^2}^X/(\log N_k^2)^\alpha \leq 2^\alpha a_{N_k^2}^X/(\log N_k^2)^\alpha = 2^\alpha b_{N_k^2};
\]

from (8) it follows that \( \forall k \geq 0 b_{N_k} \leq b_{N_0} = b_2 \), hence \( b_n \leq 2^\alpha b_2 \) for all integers \( n \geq 2 \); this completes the proof of (a). It is clear that the proof of (b) is entirely similar.

**Corollary 1.** In the situation of theorem 2, if each of \( Y \) and \( X/Y \) is isomorphic to a Hilbert space, then for all \( p < 2 \) there exists a constant \( c_p > 0 \) such that:

\[
c_p^{-1} (\sum \|x_n\|p')^{1/p'} \leq \|\sum \varepsilon_n x_n\|_2 \leq c_p (\sum \|x_n\|p)^{1/p}
\]

for all finite sequences \( (x_n) \) in \( X \).
Proof. The assumptions imply that $Y$ and $X/Y$ are of type 2, and also that $X^*/Y^\perp$ and $Y^\perp$ are of type 2. By theorem 3.a we have:

$$\forall n \in \mathbb{N}: a_n^X \leq c(\log n)^\alpha, \ a_n^{X^*} \leq c(\log n)^\alpha,$$

for some constants $c$ and $\alpha$. By a known argument (see [8, exp. 7, p. 5]) one can prove that for all $p < 2$ there exists a constant $c_p$ such that:

$$\left\| \sum \varepsilon_n x_n \right\|_2 \leq c_p(\sum \left\| x_n \right\|^p)^{1/p}$$

for all finite sequences $(x_n)$ in $X$ and

$$\left\| \sum \varepsilon_n x_n^* \right\|_2 \leq c_p(\sum \left\| x_n^* \right\|^p)^{1/p}$$

for all finite sequences $(x_n^*)$ in $X^*$. The conclusion follows then from an argument of duality.

Corollary 2. If each of the spaces $Y$ and $X/Y$ is isomorphic to a Hilbert space, then for all $p < 2$ there exists an equivalent renorming of $X$ for which the modulus of smoothness $\varrho$ satisfies $\forall t > 0: \varrho(t) \leq K_p t^p$, for some constant $K_p$; moreover, for all $q > 2$ there exists an equivalent renorming of $X$ for which the modulus of convexity $\delta$ satisfies $\forall \varepsilon \leq 2: \delta(\varepsilon) \geq K_q \varepsilon^q$, for some constant $K_q > 0$.

Proof. The assumptions imply, using theorem 3.b, that there exist constants $c$ and $\alpha$ such that:

$$\forall n \in \mathbb{N}: a_n^X \leq c(\log n)^\alpha \quad \text{and} \quad a_n^{X^*} \leq c(\log n)^\alpha.$$

As proved in [10], [9], this is sufficient to imply the conclusions of corollary 2.

Remark 2. It is proved in [6] that if a Banach space $X$ has an equivalent norm for which the modulus of smoothness $\varrho$ satisfies $\forall t > 0: \varrho(t) \leq K t^2$ and an equivalent norm for which the modulus of convexity $\delta$ satisfies $\forall \varepsilon \in (0,2): \delta(\varepsilon) \geq L \varepsilon^2$, for some constants $K$ and $L > 0$, then $X$ is isomorphic to a Hilbert space.

Remark 3. A Banach space is called of type $p$ if there exists a constant $c$ such that:

$$\left( \int \left\| \sum \varepsilon_n(t) x_n \right\|^p dt \right)^{1/p} \leq c(\sum \left\| x_n \right\|^p)^{1/p}$$

for all finite sequences $(x_n)$ in $X$; let us call briefly $p$-smooth a Banach space for which there is an equivalent norm such that the modulus of smoothness $\varrho$ satisfies $\forall t > 0: \varrho(t) \leq K t^p$, for some constant $K$. 
If in the definitions of $a_n^X$ and $\alpha_n^X$ we replace 2 by a number $p$ in $(1, 2)$, then clearly Theorem 1 is still valid. This can be used to prove in an entirely similar way as the preceding lines: If $X$ has a closed subspace $Y$ such that both $Y$ and $X/Y$ are of type $p$ (respectively are $p$-smooth) then $X$ is of type $q$ (respectively is $q$-smooth) for all $q < p$.

Remark 4. C. Stegall proved that if both $[X/Y]^*$ and $Y^*$ have the Radon-Nikodym property then $X^*$ also has the Radon-Nikodym property [11, corollary 6]. We mention this result because (cf. [9]) super-reflexivity happens to be equivalent to the super-Radon-Nikodym property.

4. The counterexample to Palais’ problem.

We turn now to a construction of an example which shows that if $Y$ and $X/Y$ are both Hilbert spaces $X$ itself need not be a Hilbert space.

We start by mentioning an elementary numerical inequality which we shall need in the sequel. Let $t$ and $s$ be real numbers and consider the complex numbers $u = 1 + is$, $v = 1 + it$. Then

\begin{equation}
|t(1 + t^2)^{-1} - s(1 + s^2)^{-1}|^2 = (\text{Imag}(u/|u| - v/|v|))^2
\end{equation}

\begin{equation}
\leq |u/|u| - v/|v||^2 = 2 - 2 \text{ Rea } u\overline{v}/|u| \cdot |v|
\end{equation}

\begin{equation}
= 2((1 + t^2)(1 + s^2) - (1 + ts))/(|u| \cdot |v|) \leq 2((1 + t^2)(1 + s^2) - (1 + ts))
\end{equation}

We define now a class $B_n$ of functions from the $n$ dimensional real Hilbert space $l^2_n$ into the infinite-dimensional Hilbert space $l^2$. These functions are defined so as to resemble linear operators. The main point in the construction below is to show that if $n$ is large there are however functions in $B_n$ whose distance (in a natural definition of such a notion) from the set of linear operators is large.

Definition. Let $n$ be an integer. A function $f: l^2_n \to l^2$ is said to belong to the class $B_n$ if

\begin{equation}
f(\lambda x) = \lambda f(x), \quad x \in l^2_n, \lambda \text{ real}
\end{equation}

and

\begin{equation}
\|\sum_{i=1}^k f(x_i)\| \leq \sum_{i=1}^k \|x_i\|
\end{equation}

whenever $\{x_i\}_{i=1}^k \subseteq l^2_n$ are such that $\sum_{i=1}^k x_i = 0$.

Clearly every linear operator belongs to $B_n$. The next lemma enables an inductive construction of members of $B_n$ whose non-linearity increases with $n$. 
Lemma 1. Let \( n \) be a positive integer and let \( f \in B_n \). Then the map \( g : \ell^2_{2n} \to \ell^2 \) defined by
\[
(12) \quad g(x,y) = (f(x), f(y), x \cdot y/\|(x\|_2^2 + \|y\|_2^2)^{1/2}), \quad x,y \in \ell^2_n
\]
belongs to \( B_{2n} \).

In (12) the pair \((x,y)\) denotes an element in \( \ell^2_{2n} = \ell^2_n \oplus \ell^2_n \) (the direct sum in the Hilbert sense). Similarly the element in the right hand side of (12) determines in an obvious way an element in \( \ell^2 \). The third component in the right hand side of (12) is taken as 0 if \( x = y = 0 \).

Proof. It is trivial that \( g \) satisfies (10) and thus we have only to check (11). Let \( \{x_i\}_{i=1}^k \) and \( \{y_i\}_{i=1}^k \) be elements in \( \ell^2_n \) such that
\[
(13) \quad \sum_{i=1}^k x_i = \sum_{i=1}^k y_i = 0.
\]
Put
\[
(14) \quad \alpha_i = \|y_i\|/(\|x_i\|_2^2 + \|y_i\|_2^2)^{1/2}, \quad i = 1, \ldots, k
\]
(we assume as we clearly can that \( \|x_i\|_2^2 + \|y_i\|_2^2 > 0 \)). By (11) (for the given \( f \)) and (13) we get that for any choice of the scalar \( c \)
\[
(15) \quad \|\sum_i g(x_i, y_i)\|_2^2 = \|\sum_i f(x_i), \sum_i f(y_i), \sum_i \alpha_i x_i - c \sum_i x_i\|_2^2
\]
\[
= \|\sum_i f(x_i)\|_2^2 + \|\sum_i f(y_i)\|_2^2 + \|\sum_i (\alpha_i - c)x_i\|_2^2
\]
\[
\leq \left( \sum_i \|x_i\|_2^2 + \sum_i \|y_i\|_2^2 + \sum_i \alpha_i - c \|x_i\|_2^2 \right).
\]
Put now \( c = \sum_i \alpha_i \|x_i\|_2 / \sum_i \|x_i\|_2 \). Then
\[
(16) \quad \left( \sum_i |\alpha_i - c| \|x_i\|_2 \right)^2 \leq \sum_i \|x_i\|_2 \sum_i \|x_i\|_2 (\alpha_i - c)^2
\]
\[
= \sum_i \|x_i\|_2 \sum_i \|x_i\|_2 (\alpha_i^2 + c^2 - 2\alpha_i c)
\]
\[
= \sum_i \|x_i\|_2 \sum_i \|x_i\|_2 (\alpha_i^2 - (\sum_i \|x_i\|_2)^2 c^2
\]
\[
= \frac{1}{2} \sum_i \sum_j \|x_i\|_2 \|x_j\|_2 (\alpha_i - \alpha_j)^2.
\]
By (9), (14), (15) and (16) we get that
\[
\|\sum_i g(x_i, y_i)\|_2^2 \leq \left( \sum_i \|x_i\|_2 \right)^2 + \left( \sum_i \|y_i\|_2 \right)^2 +
\]
\[
+ \sum_i \sum_j \left[ \left( \|x_i\|_2^2 + \|y_i\|_2^2 \right)(\|x_j\|_2^2 + \|y_j\|_2^2)^{1/2} - (\|x_i\|_2 \|x_j\|_2 + \|y_i\|_2 \|y_j\|_2) \right]
\]
\[
= \sum_i \|x_i\|_2^2 + \sum_i \|y_i\|_2^2 + \sum_i \sum_{j, i \neq j} \left( \|x_i\|_2^2 + \|y_i\|_2^2 \right)(\|y_j\|_2^2 + \|y_j\|_2^2)^{1/2}
\]
\[
= \left( \sum_i (\|x_i\|_2^2 + \|y_i\|_2^2)^{1/2} \right)^2 = \left( \sum_i ||(x_i, y_i)||_2 \right)^2
\]
and this concludes the proof of the lemma.
We introduce next a natural notion of the distance of a function \( f : l^2_n \to l^2 \) from the set of linear operators.

**Definition.** Let \( f : l^2_n \to l^2 \) be a bounded function. Put

\[
D_n(f) = \inf_T \sup_{|x| = 1} \| f(x) - Tx \|,
\]

where the infimum is taken over all linear operators \( T : l^2_n \to l^2 \). Put also

\[
D_n = \sup \{ D_n(f) \mid f \in B_n \}.
\]

(Observe that every \( f \in B_n \) is automatically bounded).

From Lemma 1 it is easy to get an estimate from below on the growth of \( D_n \).

**Lemma 2.** For every integer \( n \) we have

\[
D_{2n}^2 \geq D_n^2 + 1/16.
\]

**Proof.** Let \( \varepsilon > 0 \) and let \( f \in B_n \) be such that \( D_n(f) > D_n - \varepsilon \). Let \( g \in B_{2n} \) be the function given by (12). Let \( T \) be a linear operator from \( l^2_{2n} \) into \( l^2 \). In accordance with the decomposition of \( l^2_{2n} \) and \( l^2 \) into direct summands appearing in (12) we define six linear operators from \( l^2_n \) into \( l^2 \) by the relations

\[
T(x,0) = (U_1x, U_2x, U_3x) \quad T(0,y) = (V_1y, V_2y, V_3y).
\]

By the definition of \( D_n(f) \) there are \( x_0 \) and \( y_0 \) in \( l^2_n \), both of norm 1, so that

\[
\| U_1x_0 - f(x_0) \| > M_n - \varepsilon, \quad \| V_2y_0 - f(y_0) \| > M_n - \varepsilon.
\]

By considering the point \( (x_0, 0) \) in \( l^2_{2n} \) we get that

\[
D_{2n}^2 \geq D_{2n}^2(g) \geq \| U_1x_0 - f(x_0) \|^2 + \| U_3x_0 \|^2 \geq (M_n - \varepsilon)^2 + \| U_3x_0 \|^2.
\]

Consider next the points \( (x_0, \pm y_0)\sqrt{2} \) in \( l^2_{2n} \). We have by (10) and (12) that

\[
g(x_0/\sqrt{2}, \pm y_0/\sqrt{2}) - T(x_0/\sqrt{2}, \pm y_0/\sqrt{2})
\]

\[= (f(x_0) - U_1x_0, -U_2x_0, x_0/\sqrt{2} - U_3x_0)/\sqrt{2} \mp (V_1y_0, V_3y_0 - f(y_0), V_3y_0)/\sqrt{2}.
\]
Since for every two vectors \( z \) and \( w \) in \( l_2 \) there is a sign \( \theta \) such that
\[
||z + \theta w||^2 \geq ||z||^2 + ||w||^2
\]
we get that
\[
(19) \quad D_{2n}^2 \geq D_{2n}^2(g)
\]
\[
\geq \frac{1}{2}(\|f(x_0) - U_1x_0\|^2 + \|U_2x_0\|^2 + \|x_0/\sqrt{2} - U_3x_0\|^2 + \|V_2y_0\|^2 + \|V_3y_0\|^2)
\]
\[
\geq (M_n - \varepsilon)^2 + \|\frac{1}{2}x_0 - U_3x_0/\sqrt{2}\|^2.
\]
One of the numbers \( \|U_3x_0\| \) and \( \|\frac{1}{2}x_0 - U_3x_0/\sqrt{2}\| \) must be larger than \( \frac{1}{4} \).
Since \( \varepsilon \) was arbitrary the lemma follows by comparing (18) with (19).

**Corollary.** There is a constant \( C > 0 \) so that \( D_n \geq C(\log n)^4 \).

For the construction below it is convenient and also of interest to note that the fact that the range of the functions in \( B_n \) was allowed to be the infinite-dimensional Hilbert space \( l^2 \) was not really used. We could just as well have defined \( B_n \) by considering maps from \( l^2_n \) into \( l^2_{n^2} \).

Let \( n \) be an integer, let \( f: l^2_n \to l^2_{n^2} \) be an element of \( B_n \) and let \( \| \| \) denote the usual inner product norm in \( l^2_n \) and \( l^2_{n^2} \). In the direct sum \( Z_n = l^2_n \oplus l^2_{n^2} \) we introduce a norm \( \| \| \) by taking as its unit ball the closed convex hull of all the points of the form \( (0, y) \) with \( \|y\| \leq 1 \) and all the points of the form \( (x, f(x)) \) with \( \|x\| \leq 1 \). The subspace of \( Z_n \) of all the points of the form \( (0, y) \) is denoted by \( Y_n \). With these notations we have

**Proposition 2.** The space \( Y_n \) is isometric to \( l^2_{n^2} \). The space \( Z_n/ Y_n \) is isometric to \( l^2_n \). Any linear projection of \( Z_n \) onto \( Y_n \) has norm \( \geq D_n(f) \).

**Proof.** Whenever \( \|y\| \leq 1 \) the point \( (0, y) \) is in the unit ball of \( Z_n \) and hence \( \|((0, y))\| \leq 1 \). Assume conversely that \( \|((0, y))\| < 1 \). Then there is a \( y_0 \in l^2_n \) and \( \{x_i\}_{i=1}^n \in l^2_n \) so that
\[
\sum_i x_i = 0, \quad y_0 + \sum_i f(x_i) = y, \quad \|y_0\| + \sum_i \|x_i\| \leq 1.
\]
Hence, by (11)
\[
\|y\| \leq \|y_0\| + \sum_i \|f(x_i)\| \leq \|y_0\| + \sum_i \|x_i\| \leq 1.
\]
This proves the first statement of the proposition.

Consider now \( Z_n/ Y_n \). For every \( x \in l^2_n \) we have
\[
\inf_{y \in Y_n} \|((x, 0) + y)\| \leq \|((x, f(x))\| \leq \|x\|.
\]
Also assume that \( \|((x, 0) + Y_n)\| \leq 1 \). Since the first (i.e. the \( l^2_n \)) coordinate of the points in the unit ball of \( Z_n \) has \( \| \cdot \| \) less or equal to 1 we get that \( \|x\| \leq 1 \). This proves the second statement in the proposition.
Finally let $P$ be a bounded linear projection from $Z_n$ onto $Y_n$. Then
\[ P(x,0) = (0, Tx) \] for some linear operator $T$ from $l^2_n$ to $l^2_{n_2}$. Hence
\[ P(x,f(x)) = P(x,0) + P(0,f(x)) = (0, Tx) + (0, f(x)) \]
and thus
\[ |||P||| \geq \sup_{||x||=1} |||P(x, f(x))||| = \sup_{||x||=1} ||Tx + f(x)|| \geq D_n f. \]

**Theorem 4.** There exists a Banach space $Z$ and a subspace $Y$ of $Z$ so that $Y$ and $Z/Y$ are both isometric to $l_2$ but $Z$ is not isomorphic to $l_2$.

**Proof.** By the Corollary to Lemma 2 we may choose for every integer $n$ a map from $l^2_n$ to $l^2_{n_2}$ so that if $Z_n \supseteq Y_n$ are the spaces constructed above any projection from $Z_n$ onto $Y_n$ will have norm $\geq C(\log n)^t$. The spaces
\[ Z = (\sum_n \oplus Z_n)_2 \supseteq Y = (\sum_n \oplus Y_n)_2 \]
have the properties required in the statement of the theorem.

**Remark.** If $1 < p < \infty$ the spaces $Z_p = (\sum_n \oplus Z_n)_p$ and $Y_p = (\sum_n \oplus Y_n)_p$ are examples of spaces such that $Y_p$ and $Z_p/Y_p$ are both isomorphic to $l_p$ but $Z_p$ is not an $\mathcal{L}_p$ space.

**REFERENCES**


**STANFORD UNIVERSITY, STANFORD, CALIFORNIA, U.S.A.**
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