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SOME OPERATOR INEQUALITIES FOR HERMITIAN BANACH $*$-ALGEBRAS

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Abstract

In this paper, we extend the Kubo-Ando theory from operator means on C*-algebras to a Hermitian Banach $*$-algebra $\mathscr{A}$ with a continuous involution. For this purpose, we show that if a and b are self-adjoint elements in $\mathscr{A}$ with spectra in an interval J such that $a \leq b$, then $f(a) \leq f(b)$ for every operator monotone function f on J, where $f(a)$ and $f(b)$ are defined by the Riesz-Dunford integral. Moreover, we show that some convexity properties of the usual operator convex functions are preserved in the setting of Hermitian Banach $*$-algebras. In particular, Jensen's operator inequality is presented in these cases.

1. Introduction

Let $\mathscr{A}$ be a Banach $*$-algebra. We say that $\mathscr{A}$ is a *Hermitian* Banach $*$-algebra if the spectrum of all self-adjoint elements of $\mathscr{A}$ comes from a subset of the real line. Shirali-Ford's theorem [19] states that a Hermitian Banach $*$-algebra is symmetric, that is, the spectrum of a^*a is non-negative for each $a \in A$. In Hermitian Banach $*$-algebras, we can define the positive elements. If $a \in \mathscr{A}$ is self-adjoint and its spectrum is a subset of $[0, \infty)$, then we say that a is *positive* or $a \geq 0$. Moreover, we say $a \leq b$ if $b - a \geq 0$ and $b > a$ if $b - a$ is positive and invertible. There are many studies on Hermitian Banach $*$-algebras; for instance, we refer readers to [3], [5], [6], [14], [7], [19].

Okayasu in [18] established Loewner-Heinz's inequality for positive elements in Hermitian Banach $*$-algebras and proved that if $\mathscr{A}$ is a unital Hermitian Banach $*$-algebra with continuous involution and $p \in (0, 1]$, then inequalities $0 < a \leq b$ and $0 < a < b$ imply that $a^p \leq b^p$ and $a^p < b^p$, respectively.

In continuation of [18], Tanahashi and Uchiyama in [20] obtained Furuta's inequality [10] for unital Hermitian Banach $*$-algebra with continuous involution and showed that

$$a^{(p+2r)/q} \leq (a^r b^p a^r)^{1/q}$$

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if $0 \le a \le b$ and the positive scalars p, q, and r satisfy $p + 2r \le (1 + 2r)q$ and $q \ge 1$.

In [8], the operator means $\sharp_\alpha$, $!_\alpha$, and $\triangledown_\alpha$ are defined for positive invertible elements in a unital Hermitian Banach $*$-algebra $\mathscr{A}$ with continuous involution. It is proved that

$$a \; !_\alpha \; b \le a \sharp_\alpha b \le a \triangledown_\alpha b \tag{1.1}$$

for positive elements a and b in $\mathscr{A}$.

Considering $\sharp$, $\triangledown$, and $!$ as examples of operator means on C^*-algebras motivated us to investigate an appropriate generalization of the notion of operator means in the context of Banach $*$-algebras. Analytic properties of operator monotone functions [4] and the results of Kubo and Ando [15] enable us to define the operator means on Banach $*$-algebras. Pursuing this investigation, we study the properties of operator means on Banach $*$-algebras in detail and extend the properties of operator means on $B(H)$ that remain valid in the Hermitian Banach $*$-algebras. In particular, by a different argument, we extend the inequality (1.1) and show that

$$a \; ! \; b \le a \, \sigma \, b \le \frac{a+b}{2}$$

for positive invertible elements a and b and for each symmetric operator mean σ.

In §2, we show that any operator monotone function preserves the order of elements in Banach $*$-algebras. Indeed, we introduce a method to extend some inequalities from C*-algebras to Banach $*$-algebras. In particular, by a different argument, we proved the Loewner-Heinz inequality for Hermitian Banach $*$-algebras that is proved in [18]. Moreover, we show that if $a, b \in \mathscr{A}$ are self-adjoint with spectra in (α, β) and $a - b \ge re > 0$ for some positive scalar r, then there exists a universal constant $\nu_r(a, b)$ such that

$$f(a) - f(b) \ge \nu_r(a, b) f'\left(\frac{\alpha+\beta}{2}\right) e > 0$$

for each non-constant operator monotone function f on (α, β). This is a new result, even if $\mathscr{A}$ is a C*-algebra.

Let f be an operator convex function on an open interval J (see Definition 2.1). We show that

$$f(\lambda a + (1-\lambda)b) \le \lambda f(a) + (1-\lambda) f(b)$$

for self-adjoint elements a and b in a Hermitian Banach $*$-algebra $\mathscr{A}$ with spectra in J and $0 \le \lambda \le 1$. We also introduce Jensen's inequality for operator convex functions in Hermitian Banach $*$-algebras. We prove that if g is an operator convex function on J and a is self-adjoint with spectra in J, then

$$g(c^*ac) \leq c^*g(a)c$$

for each element c in $\mathscr{A}$ such that $c^*c = e$.

In §3, we show that the family of all non-negative operator monotone functions f on $[0, \infty)$ such that $f(1) = 1$, is a normal family. As an application, we generalize the operator means in C*-algebras to the context of Hermitian Banach $*$-algebras and provide several properties and inequalities for operator means in these cases. In particular, we show that Ando-Hiai's inequality for positive elements in Hermitian Banach $*$-algebras.

Throughout the paper, we assume that a Hermitian Banach $*$-algebra $\mathscr{A}$ is unital with a continuous involution and take J to be an open interval of the real line.

2. Operator monotone and operator convex functions

We begin this section with recalling the definitions of operator monotone and operator convex functions as follows.

DEFINITION 2.1. Let A, B be self-adjoint operators acting on a Hilbert space with spectra in J. A continuous real valued function f defined on J is called:

- *operator monotone*, if $A \leq B$ implies $f(A) \leq f(B)$;
- *operator convex*, if $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$ for all $0 \leq \lambda \leq 1$.

The Loewner theorem states that a function f is operator monotone on J if and only if f has an analytic continuation to the upper half plan Π_+ such that f maps Π_+ into itself. Also, it is shown that a differentiable function f on an interval J is operator convex if and only if the function

$$g(t) = \begin{cases} \dfrac{f(t) - f(t_0)}{t - t_0}, & \text{if } t \neq t_0, \\ f'(t_0), & \text{if } t = t_0, \end{cases} \tag{2.1}$$

is operator monotone on J for each $t_0 \in J$; see [4]. So, operator convex functions are also analytic.

Let $a \in \mathscr{A}$ with $\mathrm{sp}(a) \subseteq J$ and f be an operator monotone or operator convex function on J. Since f is analytic on the $\Omega_J = \Pi_+ \cup \Pi_- \cup J$, we can define the element $f(a)$ by the integral representation

$$f(a) = \frac{1}{2\pi i}\int_{\gamma} f(z)(ze - a)^{-1}dz, \tag{2.2}$$

where γ is a closed rectifiable curve in Ω_J such that $\mathrm{sp}(a) \subset \mathrm{ins}(\gamma)$.

Note that $f(t)$ is an operator monotone (operator convex) function on (α, β) if and only if $f\big((2t - \alpha - \beta)/(\beta - \alpha)\big)$ is an operator monotone (operator convex) function on $(-1, 1)$; so we can study the family of operator monotone functions on $(-1, 1)$ and extend results to any finite open interval.

Let $\mathscr{K}$ denote the family of all operator monotone functions on $(-1, 1)$ such that $f(0) = 0$ and $f'(0) = 1$. Hansen and Pedersen in [12] showed that $\mathscr{K}$ is a compact convex subset of the space of all functions on $(-1, 1)$ with the topology of pointwise convergence and the extreme points of $\mathscr{K}$ are of the form $f_\lambda(t) = \frac{t}{1-\lambda t}$ with $|\lambda| < 1$. Let S be the convex hull of $\{f_\lambda : |\lambda < 1|\}$, where $f_\lambda(t) = \frac{t}{1-\lambda t}$, and let $A(\Omega)$ denote the set of all analytic functions on $\Omega = \Pi_+ \cup \Pi_- \cup (-1, 1)$. The author, Moslehian, and Uchiyama in [16] proved the following lemma.

LEMMA 2.2 ([16, Theorem 2.1]). *The family $\mathscr{K}$ is normal in $A(\Omega)$. In particular, for each function f in $\mathscr{K}$, there is a sequence $\{g_n\}$ in S such that it is uniformly compact convergence to f.*

In the next lemma, we collect some preliminary techniques for Hermitian Banach $*$-algebras that we used in this paper.

LEMMA 2.3. *Let $\mathscr{A}$ be a Hermitian Banach $*$-algebra. Let a, b, and c be self-adjoint elements in $\mathscr{A}$, and let $x \in \mathscr{A}$ be arbitrary. Then, the following statements are true.*

(i) *If $a \geq 0$ and $b \geq 0$, then $a + b \geq 0$ (see* [5]).

(ii) If $c > 0$ and $a \leq b$, then $cac \leq cbc$ (see [20]).

(iii) If $r(x)$ denotes the spectral radius of x, then $r(x) \leq \|x\|$ (see [5]).

(iv) If $0 < a \leq b$, then $0 < b^{-1} \leq a^{-1}$ and if $0 < a < b$, then $0 < b^{-1} < a^{-1}$ (see [18]).

(v) If $r(e - x) < 1$, then x is invertible (see [5]).

(vi) $r(ab) \leq r(a)r(b)$ (see [5]).

(vii) If D is an open subset of the complex plane containing the spectrum of x and $\{f_n\} \in A(D)$ uniformly converges to a function f on compact subsets of D, then $\{f_n(x)\}$ converges to $f(x)$ in the norm topology (see [13]).

We need the following lemmas to prove the main results in this section.

LEMMA 2.4. *Let $\mathscr{A}$ be a unital Hermitian Banach $*$-algebra with continuous involution, and let $\{a_n\}$ be a sequence of positive elements such that $a_n \to a$ in the norm topology. Then a is positive.*

PROOF. Since the involution is continuous, a is self-adjoint. It is sufficient to show that $\mathrm{sp}(a) \subseteq [0, \infty)$. Assume $\alpha > 0$ and $b_n = \frac{1}{\alpha}(a_n + \alpha e)$. Clearly $b_n \to b = \frac{1}{\alpha}(a + \alpha e)$ and $b_n \geq e$. So, by part (iv) of Lemma 2.3, b_n is invertible and $b_n^{-1} \leq e$. Moreover,

$$\begin{aligned} r(b_n^{-1}b - e) &= r(b_n^{-1}(b - b_n)) \\ &\leq r(b_n^{-1})r(b - b_n) \qquad \text{by part (vi) of Lemma 2.3} \\ &\leq \|b - b_n\| \to 0. \qquad \text{by part (iii) of Lemma 2.3} \end{aligned}$$

Hence by part (v) of Lemma 2.3, we have $b_n^{-1}b$ is invertible, so b or $a + \alpha e$ is invertible. Therefore, $-\alpha \notin \mathrm{sp}(a)$ and a is positive.

Let a, b, and c be positive elements in $\mathscr{A}$, and let $a \leq b$. Then, by part (ii) of Lemma 2.3, we have

$$\left(c + \frac{1}{n}e\right)a\left(c + \frac{1}{n}e\right) \leq \left(c + \frac{1}{n}e\right)b\left(c + \frac{1}{n}e\right)$$

for each $n \in \mathbb{N}$. Since $(c + \frac{1}{n}e)x(c + \frac{1}{n}e)$ converges to cxc for each $x \in \mathscr{A}$, by Lemma 2.4, we obtain $cac \leq cbc$.

Let $a \in \mathscr{A}$ and $\mathrm{sp}(a) \subset (-1, 1)$. We can see that $f_\lambda(a) = a(1 - \lambda a)^{-1}$ is self-adjoint and equals to the Riesz-Dunford integral defined in (2.2) for each $|\lambda| < 1$. Now, Lemma 2.2 and part (vii) of Lemma 2.3 imply that $f(a)$ is self-adjoint and can be defined by the Riesz-Dunford integral representation for any operator monotone function f in $\mathscr{K}$.

LEMMA 2.5. *Let a and b be positive invertible elements of a unital Hermitian Banach $*$-algebra $\mathscr{A}$ such that $a - b \geq re > 0$ for some positive scalar r. Then*

$$b^{-1} - a^{-1} \geq \frac{r}{(\|a\| - r)\|a\|}e.$$

PROOF. By part (iv) of Lemma 2.3, we obtain $b^{-1} \geq (a - re)^{-1}$. We can see that the function $h(t) = t^2 - rt$ is increasing on the interval $[r, \|a\|]$. So $t^2 - rt \leq \|a\|^2 - r\|a\|$ for each $t \in \mathrm{sp}(a) \subseteq [r, \|a\|]$. By the spectral mapping theorem, we have

$$\mathrm{sp}\big(\|a\|^2 e - r\|a\|e - a^2 + ra\big) = \big\{\|a\|^2 - r\|a\| - t^2 + rt : t \in \mathrm{sp}(a)\big\} \subseteq [0, \infty).$$

Hence, $a^2 - ra \le (\|a\|^2 - r\|a\|)e$. Therefore

$$ra - r^2e = ra^{-1/2}(a^2 - ra)a^{-1/2} \le r\big(\|a\|^2 - r\|a\|\big)a^{-1}.$$

By part (iv) of Lemma 2.3, we obtain

$$\left(a^{-1} + \frac{re}{\|a\|^2 - r\|a\|}\right)(a - re) = \frac{1}{\|a\|^2 - r\|a\|}(ra - r^2e) - ra^{-1} + e \le e.$$

Again, part (iv) of Lemma 2.3 is used to get $(a - r)^{-1} \ge a^{-1} + \frac{re}{(\|a\|-r)\|a\|}$.

THEOREM 2.6. *Let $\mathscr{A}$ be a unital Hermitian Banach $*$-algebra with continuous involution, and let f be an operator monotone function on an open interval J. Let a and b be self-adjoint elements of $\mathscr{A}$ with spectra in J and with $a \ge b$. Then $f(a) \ge f(b)$. Moreover, if f is non-constant and $a > b$, then $f(a) > f(b)$.*

PROOF. By translations, we can assume that $J = (-1, 1)$ and that $f \in \mathscr{K}$. Note that $f(a)$ and $f(b)$ are self-adjoint. First, We prove the theorem for $f_\lambda(t) = \frac{t}{1-\lambda t}$, which $-1 < \lambda < 1$. For $\lambda = 0$, the claim is trivial; so assume that λ is non-zero. We can see that

$$\mathrm{sp}(e - \lambda a) = \{1 - \lambda t : t \in \mathrm{sp}(a)\} \subseteq \{1 - \lambda t : t \in (-1, 1)\} \subseteq (0, 2).$$

Similarly $\mathrm{sp}(e - \lambda b) \subseteq (0, 2)$. As $\lambda(e - \lambda a) \le \lambda(e - \lambda b)$, by part (iv) of Lemma 2.3, we obtain $\frac{1}{\lambda}(e - \lambda a)^{-1} \ge \frac{1}{\lambda}(e - \lambda b)^{-1}$. Hence,

$$\begin{aligned} f(a) = a(e - \lambda a)^{-1} &= \frac{1}{\lambda}\big(-e + (e - \lambda a)^{-1}\big) \\ &\ge \frac{1}{\lambda}\big(-e + (e - \lambda b)^{-1}\big) = f(b). \end{aligned}$$

By part (i) of Lemma 2.3, the claim is true for the function $f = \sum_{i=1}^n c_i f_{\lambda_i}$, where $c_i > 0$ and $\sum_{i=1}^n c_i = 1$. Now, assume that $f \in \mathscr{K}$ is arbitrary. By Lemma 2.2, there exists a sequence $\{f_n\}$ in $\mathscr{S}$ such that f_n converges to f in uniform compact convergence topology on Ω. Now, part (vii) of Lemma 2.3 implies that $f_n(a)$ converges to $f(a)$ and $f_n(b)$ converges to $f(b)$. By Lemma 2.4, we have $f(a) \ge f(b)$.

If $a > b$, since $\mathrm{sp}(a - b)$ is a compact subset of $(0, \infty)$, we can find a scalar $r > 0$ such that $\mathrm{sp}(a - b) \subseteq (r, \infty)$ or $a - b \ge re$. We shall show that $f_\lambda(a) - f_\lambda(b)$ is bounded below, and so it is invertible for each $-1 < \lambda < 1$. It is clear that the claim is true for $\lambda = 0$. If $0 < \lambda < 1$, then

$(e-\lambda b)-(e-\lambda a)=\lambda(a-b)\geq \lambda re>0$. Note that $e-\lambda b$ and $e-\lambda a$ are positive and invertible. By Lemma 2.5, we have

$$\begin{aligned} f_\lambda(a)-f_\lambda(b) &= \frac{1}{\lambda}\big((e-\lambda a)^{-1}-(e-\lambda b)^{-1}\big) \\ &\geq \frac{1}{\lambda}\left(\frac{\lambda r}{(\|e-\lambda b\|-\lambda r)\|e-\lambda b\|}\right)e \qquad \text{(by (3.2))} \\ &= \frac{r}{(\|e-\lambda b\|-\lambda r)\|e-\lambda b\|}e>0. \end{aligned}$$

A similar argument shows that

$$f_\lambda(a)-f_\lambda(b)\geq \frac{r}{(\|e-\lambda a\|+\lambda r)\|e-\lambda a\|}e>0$$

for each $-1<\lambda<0$. Consider $h_1\colon[-1,0]\to\mathbb{R}$ and $h_2\colon[0,1]\to\mathbb{R}$ as

$$\begin{aligned} h_1(\lambda) &= \frac{r}{\big(\|e-\lambda a\|+\lambda r\big)\|e-\lambda a\|}, \\ h_2(\lambda) &= \frac{r}{\big(\|e-\lambda b\|-\lambda r\big)\|e-\lambda b\|}. \end{aligned}$$

Clearly h_1, h_2 are continuous, and since $[-1,0]$ and $[0,1]$ are compact, we can define $\nu_1=\min_{-1\leq\lambda\leq 0}\{h_1(\lambda)\}$ and $\nu_2=\min_{0\leq\lambda\leq 1}\{h_2(\lambda)\}$. Now, put $\nu_r(a,b)=\min\{\nu_1,\nu_2\}$. If $f=\sum_{i=1}^n c_i f_{\lambda_i}$ such that $c_i>0$, $-1<\lambda_i<1$, and $\sum_{i=1}^n c_i=1$, then

$$f(a)-f(b)=\sum_{i=1}^n c_i(f_{\lambda_i}(a)-f_{\lambda_i}(b))\geq \sum_{i=1}^n c_i\nu_r(a,b)e=\nu_r(a,b)e.$$

For an arbitrary non-constant operator monotone function f, there exists a sequence $\{f_n\}$ in S such that $f_n(t)$ converges to $f(t)$ in a uniform compact convergence topology. Again, use the part (vii) of Lemma 2.3 and Lemma 2.4 to get $f(a)-f(b)\geq \nu_r(a,b)e>0$.

The proof of Theorem 2.6 shows that if f is a non-constant operator monotone function on $(-1,1)$ and $a-b\geq re>0$, then

$$f(a)-f(b)\geq \nu_r(a,b)f'(0)e>0,$$

where $\nu_r(a,b)$ is independent of f.

If we replace $f(t)$ with $f\big((2t-\alpha-\beta)/(\beta-\alpha)\big)$, then we get the following corollary.

COROLLARY 2.7. *Let $\mathscr{A}$ be a unital Hermitian Banach $*$-algebra with continuous involution. Let a and b be self-adjoint elements of $\mathscr{A}$ with spectra in (α, β) and $a - b \geq re > 0$. Then, there exists a universal constant $\nu_r(a, b)$ such that*

$$f(a) - f(b) \geq \nu_r(a, b) f'\big(\tfrac{1}{2}(\alpha + \beta)\big)e > 0$$

for each non-constant operator monotone function f on (α, β).

Let $\mathscr{G}$ denote the family of all operator convex functions g on $(-1, 1)$ such that $g(0) = g'(0) = 0$ and $g''(0) = 1$. Also, let F be the convex hull of $g_\lambda(t) = t^2/(1 - \lambda t)$, where $|\lambda| < 1$. In the following lemma we prove that $\mathscr{G}$ is a normal family.

LEMMA 2.8 ([16, Corollary 2.2]). *The family $\mathscr{G}$ is bounded in $A(\Omega)$. In particular, for each function f in $\mathscr{G}$, there is a sequence $\{g_n\}$ in F that has uniformly compact convergence to f.*

LEMMA 2.9. *Let a and b be self-adjoint elements with spectra in $(-1, 1)$. Then*

$$g_\lambda\big(\beta a + (1 - \beta)b\big) \leq \beta g_\lambda(a) + (1 - \beta)g_\lambda(b)$$

for each $|\lambda| < 1$ and $0 \leq \beta \leq 1$.

PROOF. Assume that $0 \leq \beta \leq 1$ and $\lambda = 0$. Then

$$\begin{aligned} &\beta g_0(a) + (1 - \beta)g_0(b) - g_0(\beta a + (1 - \beta)b) \\ &\quad = \beta a^2 + (1 - \beta)b^2 - (\beta a + (1 - \beta)b)^2 \\ &\quad = \beta(1 - \beta)(a^2 + b^2 - ab - ba) \\ &\quad = \beta(1 - \beta)(a - b)^2 \geq 0. \end{aligned}$$

As $g_\lambda(t) = t^2/(1 - \lambda t) = -\frac{1}{\lambda}t - \frac{1}{\lambda^2} + \frac{1}{\lambda^2(1-\lambda t)}$ for each $0 < |\lambda| < 1$, it is sufficient to prove the theorem for function $1/(1 - \lambda t)$. But by the arithmetic-geometric inequality [8] or part (iv) of Theorem 3.4, we have

$$\begin{aligned} \big(e - \lambda(\beta a + (1 - \beta)b)\big)^{-1} &= \big(\beta(e - \lambda a) + (1 - \beta)(e - \lambda b)\big)^{-1} \\ &\leq \beta(e - \lambda a)^{-1} + (1 - \beta)(e - \lambda b)^{-1}. \end{aligned}$$

By a proof similar to that of Theorem 2.6, we obtain the following theorem.

THEOREM 2.10. *Let g be an operator convex function on an open interval J. Let a and b be self-adjoint elements with spectra in J. Then*

$$g(\beta a + (1 - \beta)b) \leq \beta g(a) + (1 - \beta)g(b)$$

for each $0 \leq \beta \leq 1$.

The set $M_n(\mathscr{A})$ of all $n \times n$ matrices with elements from $\mathscr{A}$ can be made into a Banach $*$-algebra with the norm

$$\|X\| = \max_{i=1,2,\dots,n} \sum_{j=1}^{n} \|a_{ij}\| \tag{2.3}$$

for each $X = (a_{ij}) \in M_n(\mathscr{A})$. It is easy to check that if the involution on $\mathscr{A}$ is continuous, then the involution on $M_n(\mathscr{A})$ is also continuous. Moreover, we can construct the following lemma.

LEMMA 2.11. *Let $\mathscr{A}$ be a Hermitian Banach $*$-algebra. Then $M_2(\mathscr{A})$ with the norm defined by (2.3) is Hermitian.*

PROOF. Without loss of generality, we only show that $X = \begin{pmatrix} a+ie & x \\ x^* & b+ie \end{pmatrix}$ is invertible for each self-adjoint operator $\begin{pmatrix} a & x \\ x^* & b \end{pmatrix}$ in $M_2(\mathscr{A})$. First note that, since $\mathscr{A}$ is Hermitian, $a + ie$ and $b + ie$ are invertible. Now, we can conclude that X is equal to

$$\begin{pmatrix} e & x(b+ie)^{-1} \\ 0 & e \end{pmatrix} \cdot \begin{pmatrix} a+ie-x(b+ie)^{-1}x^* & 0 \\ 0 & b+ie \end{pmatrix} \cdot \begin{pmatrix} e & 0 \\ (b+ie)^{-1}x^* & e \end{pmatrix}. \tag{2.4}$$

The first and last factors are invertible with respective inverses

$$\begin{pmatrix} e & x(b+ie)^{-1} \\ 0 & e \end{pmatrix}^{-1} = \begin{pmatrix} e & -x(b+ie)^{-1} \\ 0 & e \end{pmatrix}$$

and

$$\begin{pmatrix} e & 0 \\ (b+ie)^{-1}x^* & e \end{pmatrix}^{-1} = \begin{pmatrix} e & 0 \\ -(b+ie)^{-1}x^* & e \end{pmatrix}.$$

So, it is sufficient to show that

$$\begin{pmatrix} a+ie-x(b+ie)^{-1}x^* & 0 \\ 0 & b+ie \end{pmatrix}$$

or that $a + ie - x(b+ie)^{-1}x^*$ is invertible. Note that $(b+ie)^{-1} = b(b^2+e)^{-1} - i(b^2+e)^{-1}$. So

$$a + ie - x(b+ie)^{-1}x^* = a - xb(b^2+e)^{-1}x^* + i(e + x(b^2+e)^{-1}x^*).$$

Put $c = e + x(b^2 + e)^{-1}x^*$. As c is positive and invertible, and as $c^{-1/2}(a - xb(b^2 + e)^{-1}x^*)c^{-1/2}$ is self-adjoint, we have

$$a + ie - x(b + ie)^{-1}x^* = c^{1/2}\big(c^{-1/2}(a - xb(b^2 + e)^{-1}x^*)c^{-1/2} + ie\big)c^{1/2}$$

is invertible.

Note that, if $\left(\begin{smallmatrix} a & x \\ x^* & b \end{smallmatrix}\right)$ in $M_2(\mathscr{A})$ is positive, then

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & x \\ x^* & b \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}.$$

This equality implies that a is positive. By using the decomposition of equation (2.4), we can construct the following corollary for Hermitian Banach ∗-algebras. It is well known for C*-algebras.

COROLLARY 2.12. *Let a and b be positive invertible elements of a Hermitian Banach ∗-algebra $\mathscr{A}$. If $x \in \mathscr{A}$, then*

$$\begin{pmatrix} a & x \\ x^* & b \end{pmatrix}$$

in $M_2(\mathscr{A})$ is positive if and only if $a \geq xb^{-1}x^*$.

By a standard calculus we have the following lemma.

LEMMA 2.13. *Let f be an analytic function on an open set D. Let $\mathscr{A}$ be a Hermitian Banach ∗-algebra, and let $x, y \in \mathscr{A}$ be elements such that* $\mathrm{sp}(x)$ *and* $\mathrm{sp}(y)$ *are included in D. Then, we have the following statements:*

(i) *if $u \in \mathscr{A}$ is unitary, then $f(u^*xu) = u^*f(x)u$;*

(ii) *if $X = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$, then*

$$f(X) = \begin{pmatrix} f(x) & 0 \\ 0 & f(y) \end{pmatrix}.$$

In the next theorem, using the same strategy as in [12], we introduce Jensen's operator inequality for Hermitian Banach ∗-algebras.

THEOREM 2.14. *Let $\mathscr{A}$ be a unital Hermitian Banach ∗-algebra, and let g be an operator convex function on an open interval J. Then*

$$g(c^*ac) \leq c^*g(a)c$$

for each self-adjoint element a with spectra in J and for each $c \in \mathscr{A}$ such that $c^*c = e$.

PROOF. First note that, for a self-adjoint element a of $\mathscr{A}$, we have $\mathrm{sp}(a) \subseteq (\alpha, \beta)$ if and only if $\alpha e < a < \beta e$. Hence $\mathrm{sp}(c^*ac) \subseteq J$. Also, since $\mathrm{sp}(cc^*) \subseteq \mathrm{sp}(cc^*) \cup \{0\} = \{0, 1\}$, we have $cc^* \leq e$. Consider $X = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $d = (e - cc^*)^{1/2}$, $V = \begin{pmatrix} c & -d \\ 0 & c^* \end{pmatrix}$ and $U = \begin{pmatrix} c & d \\ 0 & -c^* \end{pmatrix}$. It can be concluded that U and V are unitary in $M_2(\mathscr{A})$. Thus

$$\begin{aligned}
\begin{pmatrix} g(c^*ac) & 0 \\ 0 & g(dad + cac^*) \end{pmatrix} &= g \begin{pmatrix} c^*ac & 0 \\ 0 & dad + cac^* \end{pmatrix} \\
&= g\big(\tfrac{1}{2} U^*XU + V^*XV\big) \\
&\leq \tfrac{1}{2}\big(g(U^*XU) + g(V^*XV)\big) \\
&= \tfrac{1}{2}\big(U^*g(X)U + V^*g(X)V\big) \\
&= \begin{pmatrix} c^*g(a)c & 0 \\ 0 & dg(a)d + cg(a)c^* \end{pmatrix}.
\end{aligned}$$

Hence $g(c^*ac) \leq c^*g(a)c$.

3. Operator means

A binary operation σ on the class of positive operators is called an *operator connection*, if for positive operators A, B, and C in $B(H)$, the following requirements are fulfilled:

(1) $A \leq C$ and $B \leq D$ imply $A \,\sigma\, B \leq C \,\sigma\, D$;
(2) $C(A \,\sigma\, B)C \leq (CAC) \,\sigma\, (CBC)$;
(3) $A_n \searrow A$ and $B_n \searrow B$ imply $A_n \,\sigma\, B_n \longrightarrow A \,\sigma\, B$, in the strong norm topology.

If $I \,\sigma\, I = I$, the connection σ is called the *operator mean*. A mean σ is called *symmetric*, if $A \,\sigma\, B = B \,\sigma\, A$ for any pair of positive operators.

Kubo and Ando in [15] showed that for each operator mean σ, there exists a unique non-negative operator monotone function f on $[0, \infty)$ with $f(1) = 1$ such that $A \,\sigma\, B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}$ for positive invertible operators A and B in $B(H)$. In particular, arithmetic, geometric, and harmonic means correspond to the functions $(t+1)/2$, $\sqrt{t}$, and $2t/(t+1)$, respectively. Operator means and operator monotone functions are studied by several researchers. For more details, see [2], [17], [21].

Based on [15], for any non-negative operator monotone function f on $[0, \infty)$, we define

$$a \,\sigma_f\, b = a^{1/2} f(a^{-1/2} b a^{-1/2}) a^{1/2}$$

for positive invertible elements a and b of a Hermitian Banach $*$-algebra $\mathscr{A}$.

Let $P(0,\infty)$ denote the set of all positive operator monotone functions on $(0,\infty)$, and consider the convex set

$$\mathscr{P} = \{f \in P(0,\infty) : f(1) = 1\}.$$

Hansen in [11] showed that $\mathscr{P}$ is compact in the topology of pointwise convergence and extreme points in $\mathscr{P}$ are necessarily of the form

$$f_\alpha(t) = \frac{t}{\alpha + (1-\alpha)t},$$

where $0 \le \alpha \le 1$. By the Krein-Milman theorem, $\mathscr{P}$ is generated in this topology by convex hull of its extreme points. In the next theorem, we show that the family $\mathscr{P}$ is generated in uniformly compact topology by convex hull of its extreme points.

THEOREM 3.1. *Let $\Omega = \Pi_+ \cup \Pi_- \cup (0,\infty)$. The family $\mathscr{P}$ is bounded in $A(\Omega)$, and therefore it is a normal family.*

PROOF. Let P_0 denote the convex hull of $f_\alpha(t) = t/(\alpha + (1-\alpha)t)$, where $|\alpha| \le 1$. Since $\mathscr{P}$ is convex and compact in the topology of pointwise convergence, by the Krein-Milman theorem, $\mathscr{P}$ is the closed convex hull of its extreme points in this topology. Fix $K \subseteq \Pi_+ \cup \Pi_- \cup (0,\infty)$ as a compact set. Then $h(\alpha, z) = |\alpha + (1-\alpha)z|$ is continuous on $[-1,1] \times K$ and so takes its minimum value. Note that the minimum value m_K of h on $[-1,1] \times K$ is non-zero. Put $M_K := \sup\{|z| : z \in K\}$. Then

$$|f_\alpha(z)| = \frac{|z|}{|\alpha + (1-\alpha)z|} \le \frac{M_K}{m_K}.$$

If $f = \sum_{i=1}^n c_i f_{\alpha_i} \in \mathscr{P}$, then

$$|f(z)| = \left|\sum_{i=1}^n c_i f_{\alpha_i}(z)\right| \le \sum_{i=1}^n c_i |f_{\alpha_i}(z)| \le \sum_{i=1}^n c_i \frac{M_K}{m_K} = \frac{M_K}{m_K}.$$

Therefore $\|f\|_K \le \frac{M_K}{m_K}$. Now assume that $f \in \mathscr{P}$ is arbitrary. There exists $\{f_n\}$ in P_0 such that $f_n(t) \to f(t)$ for each $t \in (0,\infty)$. Since P_0 is bounded, the sequence $\{f_n\}$ is uniformly bounded on compact subsets of Ω. By applying Montel's theorem, there exists a subsequence $\{f_{n_j}\}$ converging to a function h in uniform compact convergence topology on Ω. Since h and f are analytic and $f = h$ on $(0,\infty)$, we have $f(z) = h(z)$ for all $z \in \Omega$. Hence

$$|f(z)| = |h(z)| = \lim_{n_j \to \infty} |f_{n_j}(z)| \le \frac{M_K}{m_K},$$

for each $z \in K$. Therefore $\mathscr{P}$ is a normal family.

Theorem 3.1 states that for any non-negative operator monotone function f on $(0, \infty)$, where $f(1) = 1$, there exists a sequence $\{f_n\}$ in P such that it is uniformly compact convergence to f, and for each n, we have

$$f_n = \sum_{i=1}^{k_n} \gamma_i f_{\alpha_i}, \tag{3.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{k_n}$ and $\gamma_1, \gamma_2, \ldots, \gamma_{k_n}$ are positive scalars with $\sum_{i=1}^{k_n} \gamma_i = 1$. So, we have the following corollary.

Corollary 3.2. *Let σ_f be an operator mean. Then, there exists a sequence $\{\sigma_{f_n}\}$, where f_n satisfies the equation (3.1) such that $a \, \sigma_{f_n} \, b$ converges to $a \, \sigma_f \, b$ for each positive invertible elements a and b in $\mathscr{A}$.*

Lemma 3.3. *Let a, b, and c be positive elements of a unital Hermitian Banach $*$-algebra $\mathscr{A}$. If*

$$H(a, b, c) = \begin{pmatrix} 2a - c & 2a \\ 2a & 2a + 2b \end{pmatrix} \tag{3.2}$$

is positive, then $c \leq a \,!\, b$. Moreover, $H(a, b, a \,!\, b)$ is positive.

Proof. Suppose that $c = a \,!\, b$. We can see that

$$a \,!\, b = 2a - 2a(a + b)^{-1}a. \tag{3.3}$$

By Corollary 2.12, $H(a, b, a \,!\, b)$ is positive. Now, assume that $H(a, b, c)$ is positive. Again use the Corollary 2.12 to get

$$2a - c \geq 2a(a + b)^{-1}a.$$

Equation (3.3) implies that $c \leq a \,!\, b$.

Let $0 \leq \alpha \leq 1$, and let $a, b \in \mathscr{A}$ be positive invertible elements. We define the operator means

- $a \,!_\alpha\, b = ((1 - \alpha)a^{-1} + \alpha b^{-1})^{-1}$,
- $a \,\sharp_\alpha\, b = a^{1/2}(a^{-1/2}ba^{-1/2})^\alpha a^{1/2}$,
- $a \,\triangledown_\alpha\, b = (1 - \alpha)a + \alpha b$.

In the next theorem we obtain some properties of operator means in Hermitian Banach $*$-algebras.

THEOREM 3.4. *Let $\mathscr{A}$ be a unital Hermitian Banach $*$-algebra with a continuous involution. Assume that a, b, c, d are positive invertible elements of $\mathscr{A}$ and that σ is an operator mean. Then, the following statements hold:*

(i) $c(a\,\sigma\,b)c = (cac)\,\sigma\,(cbc)$;

(ii) $(a\,\sigma\,b) + (c\,\sigma\,d) \le (a+c)\,\sigma\,(b+d)$;

(iii) *if* $a_1 \le a_2$ *and* $b_1 \le b_2$, *then* $a_1\,\sigma\,b_1 \le a_2\,\sigma\,b_2$;

(iv) $a\,!\,b \le (a\,\sigma\,b) \triangledown (b\,\sigma\,a) \le a \triangledown b$; *in particular, for any symmetric operator mean σ, we have*

$$a\,!\,b \le a\,\sigma\,b \le a \triangledown b;$$

(v) $a\,!_\alpha\,b \le a\,\sharp_\alpha\,b \le a \triangledown_\alpha b$.

PROOF. If $f_n = \sum_{i=1}^{k_n} \gamma_i f_{\alpha_i}$, then $\sigma_{f_n} = \sum_{i=1}^{k_n} \gamma_i \sigma_{f_{\alpha_i}}$. Applying Corollary 3.2 and Lemma 2.4, it is sufficient to prove the theorem for

$$f_\alpha(t) = \frac{t}{\alpha + (1-\alpha)t} = (\alpha t^{-1} + (1-\alpha))^{-1},$$

where $0 < \alpha < 1$. Note that $\sigma_{f_\alpha} = !_\alpha$.

(i) Consider $z = (cac)^{1/2}$. We have

$$\begin{aligned}(cac)\,!_\alpha\,(cbc) &= z\big(\alpha(z^{-1}cbcz^{-1})^{-1} + (1-\alpha)e\big)^{-1}z\\ &= \big(\alpha z^{-1}(z^{-1}cbcz^{-1})^{-1}z^{-1} + (1-\alpha)z^{-2}\big)^{-1}\\ &= \big(\alpha(cbc)^{-1} + (1-\alpha)z^{-2}\big)^{-1}\\ &= \big(\alpha c^{-1}b^{-1}c^{-1} + (1-\alpha)c^{-1}a^{-1}c^{-1}\big)^{-1}\\ &= c\big(\alpha b^{-1} + (1-\alpha)a^{-1}\big)^{-1}c\\ &= ca^{1/2}\big(\alpha(a^{-1/2}ba^{-1/2})^{-1} + (1-\alpha)e\big)^{-1}a^{1/2}c\\ &= c(a\,!_\alpha\,b)c.\end{aligned}$$

(ii) Note that $a\,!_\alpha\,b = \big(\frac{1}{2}(1-\alpha)^{-1}a\big)\,!\,\big(\frac{1}{2}\alpha^{-1}b\big)$ for each $0 < \alpha < 1$. According to Lemma 3.3, $H\big(\frac{1}{2}(1-\alpha)^{-1}a, \frac{1}{2}\alpha^{-1}b, a\,!_\alpha\,b\big)$ and $H\big(\frac{1}{2}(1-\alpha)^{-1}c, \frac{1}{2}\alpha^{-1}d, c\,!_\alpha\,d\big)$ are both positive. Hence

$$\begin{aligned}&H\big(\tfrac{1}{2}(1-\alpha)^{-1}(a+c), \tfrac{1}{2}\alpha^{-1}(b+d), a\,!_\alpha\,b + c\,!_\alpha\,d\big)\\ &\quad = H\big(\tfrac{1}{2}(1-\alpha)^{-1}a, \tfrac{1}{2}\alpha^{-1}b, a\,!_\alpha\,b\big) + H\big(\tfrac{1}{2}(1-\alpha)^{-1}c, \tfrac{1}{2}\alpha^{-1}d, c\,!_\alpha\,d\big)\end{aligned}$$

is positive. Again use Lemma 3.3 to get

$$(a \,!_\alpha\, b) + (c \,!_\alpha\, d) \le \left(\tfrac{1}{2}(1-\alpha)^{-1}(a+c)\right) \,!\, \left(\tfrac{1}{2}\alpha^{-1}(b+d)\right)$$
$$= (a+c) \,!_\alpha\, (b+d).$$

(iii) Fix a, and assume that $b_1 \le b_2$. So $a^{-1/2}b_1a^{-1/2} \le a^{-1/2}b_2a^{-1/2}$, and since f_α is an operator monotone function for each $0 < \alpha < 1$, by Theorem 2.6, we obtain $f_\alpha(a^{-1/2}b_1a^{-1/2}) \le f_\alpha(a^{-1/2}b_2a^{-1/2})$. Hence

$$a \,!_\alpha\, b_1 = a^{1/2} f_\alpha(a^{-1/2}b_1a^{-1/2})a^{1/2} \le a^{1/2} f_\alpha(a^{-1/2}b_2a^{-1/2})a^{1/2} \le a \,!_\alpha\, b_2.$$

If $a_1 \le a_2$, then for any positive invertible element b, we have

$$a_1 \,!_\alpha\, b = b \,!_{1-\alpha}\, a_1 \le b \,!_{1-\alpha}\, a_2 = a_2 \,!_\alpha\, b.$$

Therefore, if $a_1 \le a_2$ and $b_1 \le b_2$, then

$$a_1 \,!_\alpha\, b_1 \le a_2 \,!_\alpha\, b_1 \le a_2 \,!_\alpha\, b_2.$$

(iv) Note that

$$4(1+t^{-1})^{-1} \le f_\alpha(t) + f_{1-\alpha}(t) \le t+1, \tag{3.4}$$

for any $t > 0$. Since any parts of inequality (3.4) are operator monotone functions, $(e+(a^{-1}ba^{-1})^{-1})^{-1}$, $f_\alpha(a^{-1}ba^{-1})+f_{1-\alpha}(a^{-1}ba^{-1})$, and $(a^{-1}ba^{-1})+e$ are self-adjoint. By the spectral mapping theorem, we obtain

$$4(e+(a^{-1}ba^{-1})^{-1})^{-1} \le f_\alpha(a^{-1}ba^{-1}) + f_{1-\alpha}(a^{-1}ba^{-1})$$
$$\le (a^{-1}ba^{-1}) + e.$$

Therefore,

$$a \,!\, b = 2a^{1/2}(e+(a^{-1}ba^{-1})^{-1})^{-1}a^{1/2}$$
$$\le 1/2\left(a^{1/2} f_\alpha(a^{-1/2}ba^{-1/2})a^{1/2} + a^{1/2} f_{1-\alpha}(a^{-1/2}ba^{-1/2})a^{1/2}\right)$$
$$\le a^{1/2}\tfrac{1}{2}\left((a^{-1/2}ba^{-1/2}) + e\right)a^{1/2}.$$

Since

$$a^{1/2} f_\alpha(a^{-1/2}ba^{-1/2})a^{1/2} + a^{1/2} f_{1-\alpha}(a^{-1/2}ba^{-1/2})a^{1/2} = a \,!_\alpha\, b + a \,!_{1-\alpha}\, b$$
$$= a \,!_\alpha\, b + b \,!_\alpha\, a,$$

the proof is completed.

(v) Note that $!_\alpha$, $\sharp_\alpha$, and ∇_α correspond to the operator monotone functions $(\alpha t^{-1}+(1-\alpha))^{-1}$, t^α, and $\alpha t+(1-\alpha)$, respectively. Since $(\alpha t^{-1}+(1-\alpha))^{-1} \leq t^\alpha \leq \alpha t + (1-\alpha)$ for each positive scalar t, by a similar argument of part (iv), we have

$$a \,!_\alpha\, b \leq a \,\sharp_\alpha\, b \leq a \,\nabla_\alpha\, b.$$

By Furuta's inequality for Banach $*$-algebras [20], we can see that if $a \geq b > 0$, then

$$a^{-r} \,\sharp_{\frac{1+r}{p+r}}\, b^p \leq a$$

for any $p \geq 1$ and $r \geq 0$. Now, based on the proof of [9, Theorem 2], we can state the Ando-Hiai theorem [1] for Banach $*$-algebras.

THEOREM 3.5. *Let a and b be positive invertible elements of a unital Hermitian Banach $*$-algebra $\mathscr{A}$ and $0 \leq \alpha \leq 1$. If $a \,\sharp_\alpha\, b \leq e$, then $a^r \,\sharp_\alpha\, b^r \leq e$ for each $r \geq 1$.*

PROOF. It is sufficient to prove the theorem for any $1 \leq r \leq 2$. Consider $c = a^{-1/2}ba^{-1/2}$. Since $a \,\sharp_\alpha\, b \leq e$, we have $c^\alpha \leq a^{-1}$. Apply Furuta's inequality for c^α, a^{-1}, $r_0 = r-1$, and $p = \frac{1+r_0(1-\alpha)}{\alpha}$ to get

$$a^{r_0} \,\sharp_\alpha\, c^{1+r_0(1-\alpha)} \leq a^{-1}.$$

By part (i) of Theorem 3.4, we have

$$\begin{aligned}
a^r \,\sharp_\alpha\, b^r &= a^{1/2}\big(a^{r_0} \,\sharp_\alpha\, (a^{-1/2}b^r a^{-1/2})\big)a^{1/2} \\
&= a^{1/2}\big(a^{r_0} \,\sharp_\alpha\, (ca^{1/2}(a^{-1/2}c^{-1}a^{-1/2})^{2-r}a^{1/2}c)\big)a^{1/2} \\
&= a^{1/2}\big(a^{r_0} \,\sharp_\alpha\, (c(a \,\sharp_{2-r}\, c^{-1})c)\big)a^{1/2} \\
&\leq a^{1/2}\big(a^{r_0} \,\sharp_\alpha\, (c(c^{-\alpha} \,\sharp_{2-r}\, c^{-1})c)\big)a^{1/2} \\
&= a^{1/2}\big(a^{r_0} \,\sharp_\alpha\, c^{1+r_0(1-\alpha)}\big)a^{1/2} \\
&\leq e.
\end{aligned}$$

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REFERENCES

1. Ando, T., and Hiai, F., *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl. 197/198 (1994), 113–131.
2. Ando, T., and Hiai, F., *Operator log-convex functions and operator means*, Math. Ann. 350 (2011), no. 3, 611–630.
3. Bailey, D. W., *On symmetry in certain group algebras*, Pacific J. Math. 24 (1968), 413–419.
4. Bendat, J., and Sherman, S., *Monotone and convex operator functions*, Trans. Amer. Math. Soc. 79 (1955), 58–71.
5. Bonsall, F. F., and Duncan, J., *Complete normed algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 80, Springer-Verlag, New York-Heidelberg, 1973.
6. Civin, P., and Yood, B., *Involutions on Banach algebras*, Pacific J. Math. 9 (1959), 415–436.
7. El Kinani, A., *Holomorphic functions operating in Hermitian Banach algebras*, Proc. Amer. Math. Soc. 111 (1991), no. 4, 931–939.
8. Feng, B. Q., *The geometric means in Banach $*$-algebras*, J. Operator Theory 57 (2007), no. 2, 243–250.
9. Fujii, M., and Kamei, E., *Ando-Hiai inequality and Furuta inequality*, Linear Algebra Appl. 416 (2006), no. 2-3, 541–545.
10. Furuta, T., *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$*, Proc. Amer. Math. Soc. 101 (1987), no. 1, 85–88.
11. Hansen, F., *The fast track to Löwner's theorem*, Linear Algebra Appl. 438 (2013), no. 11, 4557–4571.
12. Hansen, F., and Pedersen, G. K., *Jensen's inequality for operators and Löwner's theorem*, Math. Ann. 258 (1981/82), no. 3, 229–241.
13. Kadison, R. V., and Ringrose, J. R., *Fundamentals of the theory of operator algebras. Vol. I*, Pure and Applied Mathematics, vol. 100, Academic Press, Inc., New York, 1983.
14. Kaplansky, I., *Symmetry of Banach algebras*, Proc. Amer. Math. Soc. 3 (1952), 396–399.
15. Kubo, F., and Ando, T., *Means of positive linear operators*, Math. Ann. 246 (1979/80), no. 3, 205–224.
16. Moslehian, M. S., Najafi, H., and Uchiyama, M., *A normal family of operator monotone functions*, Hokkaido Math. J. 42 (2013), no. 3, 417–423.
17. Najafi, H., *Operator means and positivity of block operators*, Math. Z. 289 (2018), no. 1-2, 445–454.
18. Okayasu, T., *The Löwner-Heinz inequality in Banach $*$-algebras*, Glasg. Math. J. 42 (2000), no. 2, 243–246.
19. Shirali, S., and Ford, J. W. M., *Symmetry in complex involutory Banach algebras. II*, Duke Math. J. 37 (1970), 275–280.
20. Tanahashi, K., and Uchiyama, A., *The Furuta inequality in Banach $*$-algebras*, Proc. Amer. Math. Soc. 128 (2000), no. 6, 1691–1695.
21. Uchiyama, M., *Operator monotone functions, positive definite kernel and majorization*, Proc. Amer. Math. Soc. 138 (2010), no. 11, 3985–3996.

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