ON THE DIFFERENTIABILITY POINTS
OF A FUNCTION OF TWO REAL VARIABLES
ADMITTING PARTIAL DERIVATIVES

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1. Introduction.

Let \( f \) be a map from \( \mathbb{R}^2 \) into \( \mathbb{R} \). We define the partial derivatives, \( D_1(f, x) \) and \( D_2(f, x) \), in the usual way:

\[
D_j(f, x) = \lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h}
\]

where \( \{e_1, e_2\} \) is the unit vector basis in \( \mathbb{R}^2 \). We shall say that \( f \) is partially differentiable on \( A \subseteq \mathbb{R}^2 \), if \( D_1(f, x) \) and \( D_2(f, x) \) exists and are finite for all \( x \in A \).

We shall use the term "differentiable" in the sense of Stolz. That is, \( f \) is differentiable at \( x \) with differential \( D \in \mathbb{R}^2 \), if

\[
\lim_{h \to 0} \frac{|f(x + h) - f(x) - \langle h, D \rangle|}{\|h\|} = 0
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^2 \).

Stepanoff has shown in [3] that if \( f \) is continuous on \( \mathbb{R}^2 \) and partially differentiable on a continuum \( K \) (i.e. a compact connected subset of \( \mathbb{R}^2 \)) then the Lipshitzian \( L(f, x) \) is finite at every point \( x \in D \), for some dense subset \( D \) of \( K \). Here the Lipshitzian is defined by

\[
L(f, x) = \limsup_{h \to 0} \frac{|f(x + h) - f(x)|}{\|h\|} \quad \forall x \in \mathbb{R}^2.
\]

In this note we shall show that, if \( f \) is continuous and partially differentiable on a differentiable curve \( \Gamma \subseteq \mathbb{R}^2 \), then \( f \) is differentiable at \( x \) for all \( x \) in a dense \( G_\delta \)-subset of \( \Gamma \).

In [3] Stepanoff gives 3 important examples. The first example of Stepanoff is a continuous function \( f \), which is partially differentiable almost everywhere in \( \mathbb{R}^2 \), but nowhere differentiable. Stepanoff's second

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example is a continuous function $f$ which is partially differentiable on all of $\mathbb{R}^2$, but the set of differentiability points has Lebesgue measure smaller than any prescribed positive number $\varepsilon$. The last example of Stepanoff is a continuous function $f$ which is partially differentiable on all of $\mathbb{R}^2$, so that there exists a continuum $K$ with $\{x \in K \mid L(f, x) = \infty\}$ of second category in $K$.

2. Differentiability on a curve.

In this section we shall present a proof of the result announced in the introduction, but under essentially weaker conditions. In order to state the theorem we shall need the following definition: If $A \subseteq \mathbb{R}^2$, then $\theta(A)$ is defined to be the set of points $x \in A$, such that there exist $\beta(x) = \beta > 0$ and $\delta(x) = \delta > 0$ with the property

$$
(2.1) \quad \forall z \in b(x, \delta), \exists y \in A \text{ so that } ||y - z|| \leq \beta||z - x|| \text{ and either } p_1(y) = p_1(z) \text{ or } p_2(y) = p_2(z),
$$

where $b(x, \delta)$ denotes the closed ball with center at $x$ and radius $\delta$, and $p_j$ is the projection on $e_j$. Now we can state the main theorem:

**Theorem 2.1.** Let $f$ be a map: $\mathbb{R}^2 \to \mathbb{R}$ and $\Gamma$ a subset of $\mathbb{R}^2$ satisfying

1. $\Gamma$ is a $G_\delta$-set,
2. $f(\cdot + ae_j)|\Gamma$ is continuous for all $a \in \mathbb{R}$ and $j = 1, 2$,
3. $f$ is partially differentiable on $\Gamma$,
4. $\theta(\Gamma)$ contains a $G_\delta$-set $\Gamma_0$ which is dense in $\Gamma$.

Then the set

$$
\Lambda = \{x \in \Gamma \mid f \text{ is differentiable at } x\}
$$

contains a $G_\delta$-set which is dense in $\Gamma$.

**Proof.** Let $\Gamma_j^+(\varepsilon)$ be the set of $x \in \Gamma$ so that there exists a neighborhood $U$ of $x$, relatively in $\Gamma$, and a $\delta > 0$ so that

$$
|(f(y + te_j) - f(y)|/|t - D_j(y)| \leq \varepsilon \quad \forall y \in U, \forall 0 < t \leq \delta.
$$

Let $\Gamma_j^-(\varepsilon)$ be the set of $x \in \Gamma$ so that there exists a neighborhood $U$ of $x$, relatively to $\Gamma$, and a $\delta > 0$ so that

$$
|(f(y + te_j) - f(y)|/|t - D_j(y)| \leq \varepsilon \quad \forall y \in U, \forall -\delta \leq t < 0,
$$

where $D_1$ and $D_2$ are the partial derivatives of $f$ in the directions $e_1$ and $e_2$. Then we have:
(2.2) \( \Gamma_1^+(\varepsilon), \Gamma_1^-(\varepsilon), \Gamma_2^+(\varepsilon) \) and \( \Gamma_2^-(\varepsilon) \) are open and dense relatively in \( \Gamma \) for all \( \varepsilon > 0 \).

Let us consider \( \Gamma_1^+(\varepsilon) \). It is obvious that \( \Gamma_1^+(\varepsilon) \) is open relatively in \( \Gamma \).
Since \( \Gamma \) is a \( G_\delta \)-set in \( \mathbb{R}^2 \) we can find a complete metric \( \varrho(x,y) \) on \( \Gamma \) which generates the topology of \( \Gamma \). Now suppose that \( \Gamma_1^+(\varepsilon) \) is not dense in \( \Gamma \).
Then we can find \( x_0 \in \Gamma \) and \( 0 < r_0 \leq 1 \) so that

\[
B(x_0, r_0) \cap \Gamma_1^+(\varepsilon) = \emptyset
\]

where we define

\[
B(x, r) = \{ y \in \Gamma \mid \varrho(x,y) \leq r \}, \\
B^0(x, r) = \{ y \in \Gamma \mid \varrho(x,y) < r \}
\]

for \( x \in \Gamma \) and \( r > 0 \). Now \( x_0 \notin \Gamma_1^+(\varepsilon) \) and \( B^0(x_0, r_0) \) is a neighborhood of \( x_0 \), so there exist \( 0 < t_1 < \frac{1}{2} \) and \( x_1 \in B^0(x_0, r_0) \) with

\[
|(f(x_1 + t_1e_1) - f(x_1))|t_0 - D_1(x_1)| > \varepsilon.
\]

Then we can find \( 0 < s_1 < \frac{1}{2} \) with

\[
\left| \frac{f(x_1 + t_1e_1) - f(x_1)}{t_1} - \frac{f(x_1 + s_1e_1) - f(x_1)}{s_1} \right| > \varepsilon,
\]

since \( f \) is partially differentiable at \( x_1 \) by (2.1.3). Now by (2.1.2) we can find \( 0 < r_1 \leq \frac{1}{2} \) so that \( B(x_1, r_1) \subseteq B(x_0, r_0) \) and

\[
\left| \frac{f(x + t_1e_1) - f(x)}{t_1} - \frac{f(x + s_1e_1) - f(x)}{s_1} \right| > \varepsilon
\]

for all \( x \in B(x_1, r_1) \). Continuing in this way we can inductively define \( x_n \in \Gamma, t_n, s_n \) and \( r_n \) in \((0, 2^{-n}]\) so that

(i) \( B(x_{n+1}, r_{n+1}) \subseteq B^0(x_n, r_n) \) \( \forall n \geq 0 \),

(ii) \( \left| \frac{f(x + t_ne_1) - f(x)}{t_n} - \frac{f(x + s_ne_1) - f(x)}{s_n} \right| > \varepsilon \)

for all \( x \in B(x_n, r_n) \) and all \( n \geq 1 \).

From (i) it follows that we can find \( \hat{x} \) with \( \hat{x} \in B(x_n, r_n) \) for all \( n \geq 1 \), since the metric \( \varrho \) is complete. Hence by (ii) we have

\[
\left| \frac{f(\hat{x} + t_ne_1) - f(\hat{x})}{t_n} - \frac{f(\hat{x} + s_ne_1) - f(\hat{x})}{s_n} \right| > \varepsilon
\]
for all \( n \geq 1 \). Now \( f \) is partially differentiable at \( \hat{x} \) and \( \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n = 0 \), so for \( n \to \infty \) we find

\[
|D_1(\hat{x}) - D_1(\hat{x})| \geq \varepsilon ,
\]

which is impossible. Hence \( \Gamma^+_1(\varepsilon) \) is dense in \( \Gamma \). Similarly one may prove that \( \Gamma^+_2(\varepsilon) \), \( \Gamma^-_2(\varepsilon) \) and \( \Gamma^+_2(\varepsilon) \) are open and dense relatively in \( \Gamma \), and so (2.2) is proved.

Now let

\[
\Gamma(\varepsilon) = \Gamma^+_1(\varepsilon) \cap \Gamma^-_1(\varepsilon) \cap \Gamma^+_2(\varepsilon) \cap \Gamma^-_2(\varepsilon) .
\]

Then we obviously have:

(2.3) \( x \in \Gamma(\varepsilon) \) if and only if there exists a \( \delta(x) = \delta > 0 \) so that for all \( y \in \Gamma \cap b(x, \delta) \), all \( 0 < |t| \leq \delta \) and \( j = 1 \) or \( 2 \),

\[
|f(y + te_j) - f(y)| / |t - D_j(y)| \leq \varepsilon .
\]

Since \( \Gamma \) is a Baire space it follows from (2.2) that

(2.4) \( \Gamma(\varepsilon) \) is open and dense relatively in \( \Gamma \) for all \( \varepsilon > 0 \).

Now we shall prove:

(2.5) \( \forall x \in \Gamma(\varepsilon), \exists \delta > 0 \) such that

\[
|D_j(x) - D_j(y)| \leq 3\varepsilon \quad \forall y \in \Gamma \cap b(x, \delta), \quad \forall j = 1, 2 .
\]

First we choose \( \delta_0 > 0 \) so that the inequalities in (2.4) are satisfied. Then we choose \( 0 < \delta \leq \delta_0 \) so that

\[
|f(x) - f(y)| \leq \frac{1}{2} \epsilon \delta_0 ,
\]

\[
|f(x + \delta_0 e_j) - f(y + \delta_0 e_j)| \leq \frac{1}{2} \epsilon \delta_0
\]

for all \( y \in \Gamma \cap b(x, \delta) \) and for \( j = 1, 2 \), which is possible by (2.1.2). Then we have for \( y \in \Gamma \cap b(x, \delta) \) and for \( j = 1 \) or \( 2 \):

\[
|D_j(x) - D_j(y)| \leq \left| D_j(x) - \frac{f(x + \delta_0 e_j) - f(x)}{\delta_0} \right| + \left| \frac{f(x + \delta_0 e_j) - f(y)}{\delta_0} - D_j(y) \right| +
\]

\[
+ \delta_0^{-1} |f(x) - f(y)| + \delta_0^{-1} |f(x + \delta_0 e_j) - f(y + \delta_0 e_j)|
\]

\[
\leq 3\varepsilon ,
\]

and so (2.5) is proved.

Now let \( \Gamma_0 \) be the dense \( G_\delta \)-set from (2.1.4), and put

\[
\Gamma_1 = \Gamma_0 \cap \bigcap_{n=1}^{\infty} \Gamma(1/n) .
\]
Then $\Gamma_1$ is a $G_\delta$-set which is dense in $\Gamma$, since $\Gamma$ is a Baire space and (2.4) holds. We shall now prove that $f$ is differentiable at all points of $\Gamma_1$. So let $x \in \Gamma_1$ and let $\varepsilon > 0$ be given. Since $x \in \theta(\Gamma)$ we can find $\beta > 0$ and $\delta > 0$, so that (2.1) holds.

Now we choose $k \geq 1$ so large that $k \geq \varepsilon^{-1}(2\beta + 5)$. Since $x \in \Gamma(1/k)$, we can find $0 < \delta_1 \leq \delta$ so that

(iii) \[ |(f(y + te_j) - f(y))t - D_j(y)| < 1/k, \]

(iv) \[ |D_j(x) - D_j(y)| < 3/k \]

for all $y \in \Gamma \cap B(x, \delta_1)$, all $0 < |t| \leq \delta_1$, and $j = 1, 2$. Now let $\delta_2 = (\beta + 1)^{-1}\delta_1$. Then we shall show that

(2.6) \[ |f(z) - f(x) - \langle x - z, D(x) \rangle| \leq \varepsilon ||z - x|| \quad \forall z \in B(x, \delta_2). \]

So let $z \in B(x, \delta_2)$, and put $r = ||z - x||$. Now $||z - x|| = r \leq \delta_2 \leq \delta$. Hence by (2.1) we can find $y \in \Gamma$ so that $||y - z|| \leq \beta r$ and either $p_1(y) = p_1(z)$ or $p_2(y) = p_2(z)$. Let us assume that the first case occurs. Then we put $x' = (p_1(z), p_2(x))$, and we have:

\[
\begin{align*}
  z &= y + te_2 \quad \text{with} \quad |t| = ||y - z|| \leq \beta r \leq \delta_1, \\
  x' &= y + se_2 \quad \text{with} \quad |s| = ||x' - y|| \leq (\beta + 1)r \leq \delta_1, \\
  x' &= x + ue_1 \quad \text{with} \quad |u| = ||x' - x|| \leq r \leq \delta_1.
\end{align*}
\]

So by (iii) and (iv),

\[
\begin{align*}
  |f(z) - f(x) - \langle x - z, D(x) \rangle| \\
  \leq |f(z) - f(y) - \langle z - y, D(y) \rangle| + |f(y) - f(x') - \langle y - x', D(y) \rangle| + \\
  + |f(x') - f(x) - \langle x' - x, D(x) \rangle| + |\langle z - x', D(y) - D(x) \rangle| \\
  \leq k^{-1}|t| + k^{-1}|s| + k^{-1}|u| + ||z - x'|| ||D_2(y) - D_2(x)|| \\
  \leq k^{-1}(\beta r + (\beta + 1) r + r + 3r) = k^{-1}(2\beta + 5)r \\
  \leq r \varepsilon.
\end{align*}
\]

Hence (2.6) is proved, and so $f$ is differentiable at all points of the $G_\delta$-set $\Gamma_1$, and $\Gamma_1$ is dense in $\Gamma$.

**Proposition 2.2.** Let $\gamma$ be a differentiable non-constant map from $[0, 1]$ into $\mathbb{R}^2$ satisfying:

(2.2.1) **There exist an $F_\sigma$-set $T_0 \subseteq [0, 1]$ so that $S_0 \subseteq T_0$ and $T_0 \setminus S_0$ is a Lebesgue-nullset,**

where $S_0 = \{ t \mid \gamma'(t) = 0 \}$. Then the curve $\Gamma = \gamma([0, 1])$ satisfies (2.1.1) and (2.1.4) in Theorem 2.1.
Remark. If $\gamma'$ only has finitely many discontinuities, then it is easily checked that (2.2.1) holds for $T_0=S_0$. If $\gamma'$ only has countably many zeros, then obviously (2.2.1) holds with $T_0=S_0$.

Proof. Let

$$S_+ = \{ t \mid 0 < t < 1 \text{ and } \gamma'(t) \neq 0 \},$$

$$\Gamma_+ = \gamma(S_+).$$

We shall then show that

$$\Gamma_+ \subseteq \theta(\Gamma). \tag{2.7}$$

So let $x_0 = \gamma(t_0)$ for some $t_0 \in S_+$. Then one of the following four cases must occur: (i) $\gamma'_1(t_0) > 0$, (ii) $\gamma'_2(t_0) < 0$, (iii) $\gamma'_2(t_0) > 0$, or (iv) $\gamma'_2(t_0) < 0$. If the first case occurs we can find $r_0 > 0$ so that $[t_0-r_0, t_0+r_0] \subseteq [0,1]$ and

$$\gamma_1(t_0+r) - \gamma_1(t_0) \geq ar \quad \forall 0 \leq r \leq r_0,$$

$$\gamma_1(t_0+r) - \gamma_1(t_0) \leq ar \quad \forall -r_0 \leq r \leq 0,$$

$$|\gamma_2(t_0+r) - \gamma_2(t_0)| \leq A|r| \quad \forall -r_0 \leq r \leq r_0,$$

where $a = \frac{1}{2}\gamma'_1(t_0)$ and $A = 1 + |\gamma'_2(t_0)|$. Let $\beta = a^{-1}A + 1$ and $\delta = ar_0$. If $z \in B(x_0, \delta)$ we have

$$\gamma_1(t_0-r_0) \leq p_1(x_0) - ar_0 \leq p_1(z) \leq p_1(x_0) + ar_0 \leq \gamma_1(t_0+r_0).$$

So there exist $r_1$ with $|r_1| \leq r_0$ and $p_1(z) = \gamma_1(t_0+r_1)$. Let $y = \gamma(t_0+r_1)$. Then $y \in \Gamma$ and $p_1(y) = p_1(z)$. Moreover,

$$||y-z|| = |p_2(y) - p_2(z)| \leq |p_2(y) - p_2(x_0)| + |p_2(x_0) - p_2(z)|$$

$$\leq ||x_0 - z|| + |\gamma_2(t_0+r) - \gamma_2(t_0)|$$

$$\leq ||x_0 - z|| + A|r_1|$$

$$\leq ||x_0 - z|| + a^{-1}A|\gamma_1(t_0+r_1) - \gamma_1(t_0)|$$

$$\leq \beta||x_0 - z||.$$

This shows that $x_0 \in \theta(\Gamma)$, and since the three remaining cases may be proved similarly, we have proved (2.7).

We may of course assume that $0 \in T_0$ and $1 \in T_0$. Then $T_0 \cup S_+ = [0,1]$, and so $\Gamma_0 \cup \Gamma_+ = \Gamma$ where $\Gamma_0 = \gamma(T_0)$. Moreover, $\Gamma_0$ is an $F_\sigma$-set, since $T_0$ is $\sigma$-compact. By Theorem 3.2.3 in [1] we have

$$\int_{T_0} ||\gamma'(t)|| \, dt = \int_{\mathbb{R}^2} \# \{ \gamma^{-1}(y) \cap T_0 \} H^1(dy),$$

where $H^1$ is the 1-dimensional Hausdorff measure in $\mathbb{R}^2$. Now the left hand side is 0 and

$$\# \{ \gamma^{-1}(y) \cap T_0 \} \geq 1 \quad \forall y \in \Gamma_0.$$
So we find that $H^1(\Gamma_0) = 0$. Moreover, $\Gamma$ is connected and locally connected and contains at least 2 points, since $\gamma$ is continuous and non-constant. Hence, if $U$ is open relatively in $\Gamma$ and $U \neq \emptyset$, then $U$ contains a connected set with at least 2 points. So by Corollary 2.10.12 in [1] we have

$$H^1(U) > 0$$

for all non-empty sets $U$ which are relatively open in $\Gamma$. This implies that the interior of $\Gamma_0$ relatively in $\Gamma$ is empty. Hence $\Gamma_1 = \Gamma \setminus \Gamma_0$ is a $G_\delta$-set which is dense in $\Gamma$ (note that $\Gamma$ is compact and so a fortiori a $G_\delta$-set), and $\Gamma_1 \subseteq \Gamma_+ \subseteq \theta(\Gamma)$. So (2.1.1) and (2.1.4) holds.

3. Differentiability along a curve.

In this section we shall prove a result supplementary to the result in section 2. The result is based on the following simple lemma:

**Lemma 3.1.** Let $f$ be a map from $\mathbb{R}^2$ in $\mathbb{R}$ whose partial derivatives $D_1(f,x_0)$ and $D_2(f,x_0)$ exist at the point $x_0$. If one of the partial derivatives exists and is bounded in the neighborhood of $x_0$, then the Lipshitzian $L(f,x_0)$ of $f$ is finite at $x_0$.

**Proof.** Suppose that $|D_1(f,x)| \leq M$ for all $x \in B(x_0, \delta)$, and suppose that $\delta > 0$ is chosen so small that

$$|f(x_0 + h_2e_2) - f(x_0)| \leq K|h_2| \quad \forall |h_2| \leq \delta,$$

where $K = |D_2(f,x_0)| + 1$. Then we have for all $h = (h_1, h_2) \in B(x_0, \delta)$, by the Mean Value Theorem:

$$|f(x_0 + h) - f(x_0)| \leq |f(x_0 + h_1e_1 + h_2e_2) - f(x_0 + h_2e_2)| + |f(x_0 + h_2e_2) - f(x_0)|$$

$$\leq |h_1| |D_1(f,x_0 + \theta h_1e_1 + h_2e_2)| + K|h_2|$$

where $0 < \theta < 1$. So we find $L(f,x_0) \leq K + M$.

**Theorem 3.2.** Let $\gamma$ be a map from $[0,1]$ into $\mathbb{R}^2$ which is differentiable at almost all points in $[0,1]$. Suppose that $f$ maps $\mathbb{R}^2$ into $\mathbb{R}$ so that:

(3.2.1) For almost all points $t \in [0,1]$ the partial derivatives, $D_1(f, \gamma(t))$ and $D_2(f, \gamma(t))$, exists at the point $\gamma(t)$.

(3.2.2) For almost all points $t \in [0,1]$, the one of the partial derivatives of $f$ exists and is bounded in a neighborhood of $\gamma(t)$.

Then $f \circ \gamma$ is differentiable almost everywhere.
Proof. From Lemma 3.1 it follows that there exists a nullset 
\( N \subseteq [0,1] \) so that

\[
L(f; \gamma(t)) < \infty \quad \forall t \notin N,
\]
\[
L(\gamma, t) < \infty \quad \forall t \notin N.
\]

Let \( g = f \circ \gamma \) and let \( t \in [0,1] \setminus N \). Then there exists a \( \delta > 0 \) so that

\[
|f(\gamma(t) + h) - f(\gamma(t))| \leq K\|h\| \quad \forall \|h\| \leq \delta,
\]

where \( K = L(f, \gamma(t)) + 1 \). Then we choose \( r > 0 \) so that

\[
\|\gamma(t+s) - \gamma(t)\| \leq M|s| \quad \forall |s| \leq r,
\]

where \( M = L(\gamma, t) + 1 \). We may of course assume that \( Mr \leq \delta \), and so for \( |s| \leq r \),

\[
|g(t+s) - g(t)| \leq K\|\gamma(t+s) - \gamma(s)\| \leq KM|s|.
\]

Hence \( L(g, t) < \infty \) for almost all \( t \), and so \( g \) is differentiable for almost all \( t \) by Denjoy's theorem (see Theorem (4.2) p. 270 in [2]).

References


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