ON FOURIER STIELTJES TRANSFORMS OF DISCRETE MEASURES

RON C. BLEI

It is well-known that if a set of integers $E$ is sufficiently lacunary then every restriction of a Fourier transform of an $L^1$ function to $E$ can be interpolated by a transform of a discrete measure (see [5], for example). In this note we show that a diophantine condition used in our previous work (e.g. [2]) is sufficient to imply the above property (Corollary 1). As usual, we deduce the abundance of non-Sidon sets with this property, and, as another application, we obtain a short proof of the existence of sets $E_1$ and $E_2$ so that

$$c_0(E_1) \hat{\otimes} c_0(E_2) \approx A(E_1 + E_2)$$

(see Theorem 3.1 of [7]). We conclude with an open question.

Let $G$ be any countable discrete abelian group, $G'$ its dual, and let the Bohr compactification of $G' (= (G_d)^\wedge)$ be denoted by $\hat{G}$. We refer to [3] and [6] for basic notation and facts.

Let $D$ be a dense countable subgroup of $G$, and $\varphi_D \equiv \varphi: G' \to \hat{D}$ be the injective canonical map: $(\varphi(\gamma), d) = (\gamma, d)$. We set

$$A(E, G') = L^1(G')^\wedge/[f \in L^1(G')^\wedge : f = 0 \text{ on } E],$$

and

$$A((\varphi(D(E))^\wedge, \hat{D}) = L^1(D)^\wedge/[f \in L^1(D)^\wedge : f = 0 \text{ on } \varphi(D(E))],$$

where $(\varphi(D(E))^\wedge$ denotes the closure of $\varphi(D(E))$ in $\hat{D}$. To simplify notation, we shall refer to $A(E)$ and $A((\varphi(E))^\wedge)$, respectively. Let $E \subseteq G'$ satisfy the following condition:

(*) $(\varphi(E))^\wedge$ is a countable set so that $\partial(\varphi(E))^\wedge \cap \varphi(E) = \emptyset$

$\partial(\varphi(E))^\wedge$ denotes the derived set of $(\varphi(E))^\wedge$.

We set

$$A_0((\varphi(E))^\wedge) = \{f \in A((\varphi(E))^\wedge) : f = 0 \text{ on } \partial(\varphi(E))^\wedge\}.$$

We need the following lemma, a similar version of which appears in [2] (Lemma 2.2); for the sake of completeness we prove it here:

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LEMMA. Let $E \subset \Gamma$ satisfy (*). Then, $A(E)$ and $A_0((\varphi(E))^{-})$ are isometric in the natural way:

$$A(E) \ni \lambda \leftrightarrow \lambda \varphi^{-1} \in A_0((\varphi(E))^{-}).$$

PROOF. We first note that finitely supported functions are norm dense in $A(E)$. Similarly, since $\partial(\varphi(E))^{-}$ obeys synthesis in $\hat{D}$ (7.2.4 of [6]), and since for any open set $O$ containing $\partial(\varphi(E))^{-}$, $\varphi(E) \setminus (O \cap \partial(\varphi(E))^{-})$ is finite, it follows that finitely supported functions are norm dense in $A_0((\varphi(E))^{-})$ as well. Therefore, it suffices to prove that

$$\|h\|_{A(E)} = \|h \circ \varphi^{-1}\|_{A((\varphi(E))^{-})},$$

where $h$ is a finitely supported function on $E$. But, by the way that the above norms are computed, it suffices to check that if $\{a_i\}_{i=1}^N$ is any finite set of complex numbers, and $\{\gamma_i\}_{i=1}^N$ is any finite subset of $E$, then

$$\sup_{\gamma \in G} |\sum_{i=1}^N a_i(\gamma_i, g)| = \sup_{\gamma \in \hat{G}} |\sum_{i=1}^N a_i(\varphi(\gamma_i), y)|.$$

The above equality follows from the density of $D$ in both $\hat{D}$ and $G$.

We now let

$$B_d(E) = \mathcal{V}(G)^{\sim}/\{h \in \mathcal{V}(G)^{\sim} : h = 0 \text{ on } E\}.$$

That is, $B_d(E)$ is the algebra of restrictions to $E$ of Fourier Stieltjes transforms of discrete measures on $G$. It is clear from the definition of $\varphi$ that $A((\varphi(E))^{-})$ can be canonically identified with a closed subalgebra of $B_d(E)$: If $h$ is the restriction of a function in $\mathcal{V}(D)^{\sim}$ to $\varphi(E)$, then $h \circ \varphi^{-1}$ is the restriction of the same function in $\mathcal{V}(D)^{\sim} \subset \mathcal{V}(G)^{\sim}$ to $E$. We therefore obtain

COROLLARY 1. Let $E \subset \Gamma$ satisfy (*) with respect to some $D \subset G$; then, $A(E)$ is a closed subalgebra of $B_d(E)$.

As usual, the following is a consequence of Lemma 2.3 in [1] (see also Lemma 1.1 in [2]):

COROLLARY 2. Let $E \subset \Gamma$ be a non-Sidon set. Then, there exists a non-Sidon set $F \subset E$ such that $A(F)$ is a closed subalgebra of $B_d(F)$.

AN APPLICATION TO TENSOR PRODUCTS. We recall that if $E_1$ and $E_2$ are countable sets, then

$$c_0(E_1) \hat{\otimes} c_0(E_2) = \{\varphi \in c_0(E_1 \times E_2) : \varphi = \sum f_j g_j, \text{ where } f_j \in c_0(E_1), g_j \in c_0(E_2), \text{ and } \sum \|f_j\|_{\infty} \|g_j\|_{\infty} < \infty\}.$$
We set
\[ \|\varphi\|_\hat{\otimes} = \inf \left\{ \sum \|f_j\|_\infty \|g_j\|_\infty : \varphi = \sum f_j g_j \right\}. \]

The reader is referred to [7] for a detailed study of tensor algebras in the context of discrete abelian groups. We give here a short proof of the existence of sets \( E_1 \) and \( E_2 \) in \( \Gamma \) so that \( c_0(E_1) \hat{\otimes} c_0(E_2) \cong A(E_1 + E_2) \) (see Theorem 3.1 of [7]). Fix a \( D \), a dense countable subgroup of \( G \), and let \( E_1 \) and \( E_2 \) in \( \Gamma \) be any two sets so that

1. \( E_1 \cap E_2 = \emptyset \), \( 0 \notin E_1 \cup E_2 \), and \( E_1 \cup E_2 \) is \( \mathbb{Z}_3 \)-independent, i.e., the relation \( \sum_{j=1}^{N} \omega_j \lambda_j = 0 \), where \( \omega_j = -1, 1, \) or \( 0 \), and \( \{\lambda_j\}_{j=1}^{N} \subset E_1 \cup E_2 \), can hold only if \( \omega_j = 0 \) for all \( j \).
2. \( \partial(\varphi(E_1 \cup E_2))^{-1} = \{x_0\} \subset \hat{D} \), and without loss of generality we assume that \( x_0 = 0 \).

It is clear from the independence assumption that functions on \( E_1 \times E_2 \) can be freely identified with functions on \( E_1 + E_2 \). In fact, if \( f \in c_0(E_1) \) and \( g \in c_0(E_2) \), we think of \( f \cdot g \) as a function \( h \) on \( E_1 + E_2 \):
\[ h(\lambda + \nu) = f(\lambda) \cdot g(\nu), \quad \text{where } \lambda \in E_1, \nu \in E_2. \]

Also, it follows from (2) that \( \partial(\varphi(E_1 + E_2))^{-1} = E_1 \cup E_2 \cup \{0\} \). Therefore, via Corollary 1, we conclude that \( A(E_1 + E_2) \) is a closed subalgebra of \( B_d(E_1 + E_2) \), and in particular
\[ A(E_1 + E_2) \cong c_0(E_1) \hat{\otimes} c_0(E_2). \]

To prove the reverse inclusion, it suffices to check that if \( f \in c_0(E_1) \) and \( g \in c_0(E_2) \), where \( \|f\|_\infty \) and \( \|g\|_\infty \leq 1 \), then \( f \cdot g \) (as a function on \( E_1 + E_2 \)) can be interpolated by \( \hat{\mu} \in M(G) \), where \( \|\mu\|_M \leq 1 \). This follows easily by considering a Riesz product whose transform equals \( f \) on \( E_1 \), and \( g \) on \( E_2 \).

Another consequence of our lemma is that if a Sidon set \( E \subset \Gamma \) satisfies \( (*) \) then \( \overline{E} \) (closure in \( \Gamma \)) is a Helson set in \( \Gamma \).

**Open Question.** Is a Sidon set in \( \Gamma \) a finite union of (Sidon) sets that satisfy \( (*) \) \( ((*) \) may be satisfied with respect to different \( D \)'s)?

Recalling that a Sidon set in \( \bigoplus \mathbb{Z}_p \) (\( p \) a prime) is a finite union of independent sets (see [4]), we easily answer affirmatively the above question in this setting: Let \( E = \{\chi_j\} \) be an (infinite) independent set in \( \bigoplus \mathbb{Z}_p \).
We think of $D = GP(\chi_I)$ as sitting in $\bigoplus \mathbb{Z}_p$, (note that $D$ is isomorphic to $\bigoplus \mathbb{Z}_p$) and map, as before, $\bigoplus \mathbb{Z}_p$ into $GP(\chi_I)^\wedge$. It follows that the closure of the image of $E$ in $GP(\chi_I)^\wedge$ is a countable set with 0 in $GP(\chi_I)^\wedge$ as its only accumulation point.

REFERENCES

THE UNIVERSITY OF CONNECTICUT
STORRS, CONNECTICUT 06268