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FIBRATIONS AND COFIBRED PAIRS

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If A is a subspace of X such that the inclusion $A \to X$ is a cofibration then the pair (X,A) is called a *cofibred pair*. If A is a subspace of B and $q\colon Y \to B$ is a map then $Y|A$ will denote the subspace $q^{-1}(A)$ of Y. It is a result of Strøm [5, theorem 12] that: if (B,A) is a cofibred pair with A closed and $q\colon Y \to B$ is a fibration then $(Y, Y|A)$ is a cofibred pair. We give a new "convenient category" proof of this result, using the fibred mapping projection $(qr)\colon (YZ) \to B$ introduced in [1, p. 280]. We use the fact that the *exponential law of maps* bijection [2, theorem 1.1] is natural in the first variable (proposition 1) to show that: $(Y, Y|A)$ is a cofibred pair if and only if all extension-lifting problems of a certain type have solutions (theorem 2). The result (corollary 3) then follows.

A space X is said to be a $\mathfrak{k}$-space if it has the final topology with respect to all maps (= continuous functions) of compact Hausdorff spaces into X; we work in the convenient category of $\mathfrak{k}$-spaces [1, p. 276] (= the category $\mathscr{HG}$ of [6, p. 554]), and so all spaces and constructions should be considered in that sense (e.g. a *fibration* will mean a map between two $\mathfrak{k}$-spaces having the covering homotopy property with respect to all $\mathfrak{k}$-spaces). This category has certain technical advantages over it's well known convenient subcategory of compactly generated spaces (= Hausdorff k-spaces = Kelley spaces); it is better behaved with respect to quotients, but the important thing here is that it is closed with respect to the construction of fibred mapping spaces (YZ).

If X and Y are spaces then $M(X, Y)$ will denote the set of all maps $X \to Y$. If $p\colon X \to B$, $q\colon Y \to B$ and $f\colon X \to Y$ are maps such that $qf = p$ then we will write $f\colon p \to q$, and $M(p,q)$ will denote the set of all maps $p \to q$, $X \sqcap Y$ will denote the pullback space of X with Y, $p_q\colon X \sqcap Y \to Y$, $q_p\colon X \sqcap Y \to X$ will denote the projections and $p \sqcap q\colon X \sqcap Y \to B$ the map

$$(p \sqcap q)(x,y) = p(x) = q(y), \quad (x,y) \in X \sqcap Y .$$

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If $r: Z \to B$ is another map then Y_b and Z_b will denote the respective fibres of q and r over $b \in B$. We define the set

$$(YZ) = \bigcup_{b \in B} M(Y_b, Z_b)$$

and the function $(qr): (YZ) \to B$ as the obvious projection on B; we recall the definition [1, p. 280] of a topology on (YZ), with the following important properties:

(i) a bijection

$$\theta: M(p \sqcap q, r) \to M(p, (qr)),$$

is given by

$$\theta(f)(x)(y) = f(x,y), \quad f \in M(p \sqcap q, r),\ p(x) = q(y)$$

(the exponential law of maps = [2, theorem 1.1]);

(ii) if q and r are fibrations then so is (qr) [1, theorem 3.4].

If $f: p \to q$, and $f: X \to Y$ is a homeomorphism, then we will write $f: p \cong q$.

We list five more or less trivial facts about pullbacks and maps $q: Y \to B$:

(iii) if $s: B \times I \to B$ is the projection on the first factor then $q_s \cong q \times 1_I$;

(iv) if $f: W \to Z$ and $g: Z \to B$ are maps then $(q_g)_f \cong q_{(gf)}$;

(v) if $i: A \to B$ denotes the inclusion of subspace A in B and

$$q \mid A = q \mid (Y \mid A): (Y \mid A) \to A$$

then $q_i \cong q \mid A$;

(vi) if (v) is used to identify $Y \sqcap A$ with $Y \mid A$ then the projection $i_q: Y \sqcap A \to Y$ is essentially just the inclusion $Y \mid A \to Y$; and

(vii) if W is a space and $r: B \times W \to B$ is the projection on the first factor then there is a bijection $\varphi: M(Y, W) \to M(q, r)$

$$\varphi(f)(y) = (q(y), f(y)), \quad f \in M(Y, W),\ y \in Y$$

(the inverse of φ is the function $\psi: M(q,r) \to M(Y, W)$,

$$\psi(g) = r'g,$$

where $r': B \times W \to W$ is the projection on the second factor).

The following result describes the naturality of the exponential law (i) with respect to the first variable p:

Proposition 1. *Given maps* $p\colon X \to B$, $p'\colon X' \to B$, $q\colon Y \to B$, $r\colon Z \to B$, $k\colon p \to p'$, $h\colon p \sqcap q \to r$, $h'\colon p' \sqcap q \to r$. *If* $k \times 1\colon p \sqcap q \to p' \sqcap q$ *is the map* $(k \times 1)(x,y) = (k(x),y)$, *where* $p(x) = q(y)$, *then diagram* (i) *commutes if and only if diagram* (ii) *commutes*:

(i)
$$\begin{tikzcd} p \sqcap q \arrow[d, "k \times 1"] \arrow[r, "h"] & r \\ p' \sqcap q \arrow[ur, "h'"'] & \end{tikzcd}$$

(ii)
$$\begin{tikzcd} p \arrow[d, "k"] \arrow[r, "\theta(h)"] & (qr) \\ p' \arrow[ur, "\theta(h')"'] & \end{tikzcd}$$

Proof. $h(x,y) = h'(k(x),y)$ for all $(x,y) \in X \sqcap Y$ if and only if $\theta(h)(x)(y) = \theta(h')(k(x))(y)$ for all such (x,y). This is simply the condition that $\theta(h)(x) = \theta(h')(k(x))$ for all $x \in X$, i.e. that $\theta(h) = \theta(h')k$.

Let $s\colon B \times I \to B$ denote the projection on the first factor.

Theorem 2. *If* $q\colon Y \to B$ *is a map and* A *is a subspace of* B *then the following conditions are equivalent*:

(a) $(Y, Y|A)$ *is a cofibred pair*;

(b) *for every space* W (*with associated projection* $r\colon B \times W \to B$) *and map* $f\colon (B \times \{0\}) \cup (A \times I) \to (Y\ B \times W)$ *such that the following square commutes*

$$\begin{tikzcd} (B \times \{0\}) \cup (A \times I) \arrow[r, "f"] \arrow[d, "\cap"'] & (Y\ B \times W) \arrow[d, "(qr)"] \\ B \times I \arrow[r, "s"] & B \end{tikzcd}$$

there exists a map $f'\colon B \times I \to (Y\ B \times W)$ *filling in the diagram, i.e. such that the triangles formed are commutative.*

Proof. This is a direct application of proposition 1, to the case where p and p' are the respective projections of $(B \times \{0\}) \cup (A \times I)$ and $B \times I$ on B, $k\colon (B \times \{0\}) \cup (A \times I) \to B \times I$ is the inclusion and $r\colon B \times W \to B$ is the usual projection; with (iii)–(vii) above being used to simplify some of the statements involved. Let us consider the following conditions:

(c) for every space W and every map $g\colon X \sqcap Y \to W$ there exists a map $g'\colon X' \sqcap Y \to W$ with $g'(k \times 1) = g$;

(d) for every space W and every map $h\colon p \sqcap q \to r$ there exists a map $h'\colon p' \sqcap q \to r$ with $h'(k \times 1) = h$; and

(e) for every space W and every map $\theta(h)\colon p \to (qr)$ there is a map $\theta(h')\colon p' \to (qr)$ such that $\theta(h')k = \theta(h)$.

It follows from remark (iii) that $X' \sqcap Y$ is essentially $Y \times I$; from (iv) that $q_p = q_{(p'k)} \cong (q_{p'})_k$, from (iii) that

$$(q_{p'})_k \cong (q \times 1_I)k ,$$

and hence from (v) that $X \sqcap Y$ is essentially $(Y \times \{0\}) \cup ((Y|A) \times I)$; and from (vi) that

$$k \times 1 : X \sqcap Y \to X' \sqcap Y$$

is essentially the inclusion $(Y \times \{0\}) \cup ((Y|A) \times I) \to Y \times I$. Hence (a) is equivalent to condition (c), now (c) is equivalent to (d) by (vii), and (d) to (e) by proposition 1. (e) is simply a restatement of (b) above, and so the result is proved.

To apply this theorem we notice that if q is a fibration then so is (qr) (see (ii) above). If (B,A) is a relative CW-complex then it follows by the $\mathfrak{k}$-space version of [3, p. 416, theorem 9] that $(Y, Y|A)$ is a cofibred pair. If we use the $\mathfrak{k}$-space version of the stronger theorem [4, theorem 4] then we immediately obtain the following result.

Corollary 3. *If (B,A) is a cofibred pair with A closed and $q\colon Y \to B$ is a fibration then $(Y, Y|A)$ is a cofibred pair.*

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