ON THE INTEGRABILITY OF THE DERIVATIVE OF
A QUASIREGULAR MAPPING

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1. Introduction.

In [3] F. Gehring showed that the derivative of a quasiconformal mapping \( f: G \to \mathbb{R}^n, \quad n \geq 2, \) is integrable to a power \( \alpha = \alpha(K(f), n) > n \) over each compact subset of \( G. \) Here we prove using the terminology of [4]

1.1. Theorem. Let \( f: G \to \mathbb{R}^n, \quad n \geq 2, \) be quasiregular. Then over each compact subset \( C \) of \( G \) the derivative of \( f \) is \( L^2 - \text{integrable with} \)

\[
\beta = \beta(K(f), n, \mathcal{N}(f, C)) > n .
\]

Here \( \mathcal{N}(f, C) = \sup_{x \in C} i(x, f) . \)

The above theorem follows for \( n = 2 \) directly from Gehring's result, or originally from [1], since if \( f: G \to \mathbb{R}^2 \) is quasiregular then \( f = g \circ h \) where \( h \) is quasiconformal and \( g \) analytic.

The proof of Theorem 1.1 is based on Gehring's method, especially Lemma 3 of [3], and a new linear dilatation for quasiregular mappings.

N. Meyers has reported to the author that he has proved a corresponding result. However, his method is different and based on the theory of elliptic partial differential equations.

2. Capacity estimates for quasiregular mappings.

Suppose that \( f: G \to \mathbb{R}^n \) is a non-constant quasiregular mapping. We shall use the following capacity inequalities, see [4],

\[
(2.1) \quad \text{cap} f E \leq K_1(f) \text{cap} E
\]

where \( E \) is any condenser in \( G \) and

\[
(2.2) \quad \text{cap} E \leq K_0(f) N(f, A) \text{cap} f E
\]

if \( E = (A, C) \) is a normal condenser in \( G. \)

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Let $E_n(t)$ denote the Teichmüller condenser
\[(R^n \setminus I(-\infty, -t], I[0, 1])\,,\]
$t > 0$, in $R^n$.
\[I[a, b] = \{x \in R^n : x = te_1, a \leq t \leq b\} \,.
\]
It is well-known, see e.g. [2], that $\kappa_n(t) = \text{cap} E_n(t)$ is continuous, strictly decreasing, and
\[\lim_{t \to 0+} \kappa_n(t) = +\infty, \quad \lim_{t \to \infty} \kappa_n(t) = 0 \,.
\]
The following lemma contains the important symmetrization method in $R^n$ for obtaining significant lower bounds for the capacities of condensers, see [2], [5].

2.3. **Lemma.** Suppose that $E = (A, C)$ is a condenser in $R^n$ such that (1) $|A$ is connected and meets $S^{n-1}(x, r)$ for some $x$ and $r > 0$ and (2) $C$ is connected, $x \in C$, and $C$ meets $S^{n-1}(x, r')$, $r' > 0$. Then
\[\text{cap} E \geq \kappa_n(r/r') > 0 \,.
\]

3. **A new dilatation for quasiregular mappings.**

Suppose that $f : G \to R^n$ is non-constant and quasiregular. Let $x \in G$. For $r \in (0, d(x, \partial G))$ set
\[L(x, f, r) = \sup_{|x-y|=r} |f(y) - f(x)| \,,
\]
\[l(x, f, r) = \sup \{s > 0 : U(x, f, s) \subset B^n(x, r)\} \,.
\]
and for $r > 0$
\[L^*(x, f, r) = \sup_{y \in U(x, f, r)} |y - x| \,.
\]
We recall that $U(x, f, r)$ denotes the $x$-component of $f^{-1}B^n(f(x), r)$, see [4, p. 9]. Define $\tilde{H}(x, f, r) = L(x, f, r)/l(x, f, r) \,.
\]

3.1. **Remark.** In the theory of quasiconformal and quasiregular mappings the linear stretching
\[H(x, f) = \limsup_{r \to 0} H(x, f, r) \]
\[= \limsup_{r \to 0} \frac{L(x, f, r)}{\inf_{|y-x|=r} |f(y) - f(x)|} \]
is much used. It can be easily shown that
\[H(x, f) = \limsup_{r \to 0} \tilde{H}(x, f, r) \,.
\]
However, we are interested in the global properties of $\hat{H}(x,f,r)$ which are better than those of $H(x,f,r)$.

The following lemma is a slightly modified version of [4, Lemma 4.8].

3.2. Lemma. Suppose that $U(x,f,r)$, $0 < r < r_0$, is a normal domain of $f$. Then the mapping $r \mapsto L^*(x,f,r)$ is strictly increasing and continuous from the left for $0 < r \leq r_0$.

3.3. Lemma. If $\bar{B}^n(x,r) \subset G$, then $U(x,f,\bar{l}(x,f,r)) \subset B^n(x,r)$.

Proof. Let $\bar{l} = l(x,f,r)$ and $l > \varepsilon > 0$. Clearly $U(x,f,\bar{l} - \varepsilon) \subset B^n(x,r)$, hence by [4, Lemma 2.5], $U(x,f,\bar{l} - \varepsilon)$ is a normal domain of $f$ and by Lemma 3.2,

$L^*(x,f,\bar{l}) = \lim_{\varepsilon \to 0} L^*(x,f,\bar{l} - \varepsilon) \leq r$.

3.4. In the following discussion we fix $x_0 \in G$ and pick $r_0 > 0$ so that for $r \in (0,4r_0]$

(a) $U(x_0,f,r)$ is a normal neighborhood of $x_0$,
(b) $\partial U(x_0,f,r)$ is connected.

This is possible by [4, pp. 9–11]. Let $U_0 = U(x_0,f,r_0)$ and $d_0 = d(\partial U(x_0,f,2r_0), U_0) > 0$.

3.5. Lemma. $\hat{H}(x,f,r) \leq C$ for all $x \in U_0$ and $r \in (0,d_0]$. Here $C$ depends only on $K(f)$, $n$, and $i(x_0,f)$.

Proof. Fix $x \in U_0$ and $r \in (0,d_0]$. Let $L = L(x,f,r)$ and $l = l(x,f,r)$. Denote $U = U(x,f,L)$. Then $U \subset U(x_0,f,4r_0)$. Suppose that $L > l$ and let $0 < \varepsilon < L - l$. Now $U(x,f,l + \varepsilon)$ meets $S^{n-1}(x,r)$. The condenser

$E = (U, \bar{U}(x,f,l + \varepsilon)) = (U, C)$

is a normal condenser. On the other hand $\partial U$ is connected, for if there exists a bounded component $F$ of $\partial U$, then by (b), $F \subset U(x_0,f,4r_0)$. Now $f \circ F \subset f \circ U = f \circ U = S^{n-1}(f(x), L)$, hence there exists $z \in F$ such that $f(z) \in S^{n-1}(f(x), L)$. Pick a line $T$ passing through $f(x_0)$ and $f(z)$ and let $T'$ be the $f(z)$-component of $T \cap (\bar{B}^n(f(x_0), 4r_0) \setminus B^n(f(x), L))$.

Now $f^{-1}T' \subset \partial U$, thus the $z$-component of $f^{-1}T'$, say $T_1$, is contained in $F$. Because $U(x_0,f,4r_0)$ is a normal domain, $T_1$ meets $\partial U(x_0,f,4r_0)$ which is impossible since $F \subset U(x_0,f,4r_0)$. Since $C$ is connected and both
Lemma 2.3 yields \( \cap E \geq \theta_n > 0 \). The inequality (2.2) implies
\[
\theta_n \leq \text{cap } E \leq K_0(f)N(f, U) \text{cap } E
\]
\[
\leq K(f)i(x_0,f)\omega_{n-1} \ln(L/(\tilde{l}+\varepsilon))^{1-n}.
\]
Thus
\[
L/(\tilde{l}+\varepsilon) \leq C = C(n, K(f), i(x_0,f)).
\]
Letting \( \varepsilon \to 0 \) we deduce the result.

4. Proof of Theorem 1.1.

We may assume that \( f \) is non-constant. By [3, Lemma 3] it suffices to show that each \( x_0 \in G \) has a neighborhood \( U \) such that
\[
m_n(Q)^{-1} \int_Q |f'|^n \, dm_n \leq b(m_n(Q)^{-1} \int_Q |f'| \, dm_n)^n
\]
for each cube \( Q \subset U \) parallel to the coordinate axis with \( b = b(n, K(f), i(x_0,f)) \). Fix \( x_0 \in G \) and let \( r_0 > 0 \), \( U_0 \), and \( d_0 \) be as in 3.4. Set \( U = U_0 \cap B^n(x_0, d_0) \).

Let \( Q \subset U \). We may assume that
\[
Q = \{ x \in \mathbb{R}^n : |x_i| < s, 1 \leq i \leq n \}
\]
and \( f(0) = 0 \). Now \( s/n \leq d_0 \). At first we shall show that
\[
\tilde{l}(0,f, s/n) \leq C'\tilde{l}(0,f, s)
\]
where \( C' = C'(n, K(f), i(x_0,f)) \).

Let \( L = L(0,f, s/n) \) and \( U' = U(0,f, L) \). The condenser \( E = (U', \overline{B}^n(0,s)) \) is a normal condenser in \( U(x_0,f, 4r_0) \). Clearly \( \cap U' \) meets \( S^{n-1}(s/n) \). As in the proof of Lemma 3.5 it can be shown that \( \cap U' \) is connected. Lemma 2.3 and (2.2) applied to the condenser \( E \) give
\[
\kappa_n(s/n) = \kappa(s/n) \leq \text{cap } E \leq K_0(f)N(f, U') \text{cap } E
\]
\[
\leq K(f)i(x_0,f)\omega_{n-1} \left( \ln \frac{L(0,f, s/n)}{L(0,f, s)} \right)^{1-n}.
\]
This implies \( L(0,f, s/n) \leq C''L(0,f, s) \) which, by Lemma 3.5, yields
\[
L(0,f, s/n) \leq C''C\tilde{l}(0,f, s) = C'\tilde{l}(0,f, s).
\]
Since \( \tilde{l}(0,f, s/n) \leq L(0,f, s/n) \), (4.2) follows.

Let \( r \in (0, s/n) \) and
\[
Q' = \{ x \in \mathbb{R}^n : |x_i| \leq r, 1 \leq i \leq n \}.
\]
Then $E=(Q,Q')$ is a condenser in $G$ and

\begin{equation}
(4.3) \quad \cap E \leq \omega_{n-1} \left( \ln \frac{s}{r/\sqrt{n}} \right)^{1-n}.
\end{equation}

Define $r'=\sup_{x \in Q'} |f(x)|$, $L=L(0,f,s/\sqrt{n})$, and \( \tilde{l} = \tilde{l}(0,f,s) \).

\begin{equation}
(4.4) \quad \cap fE = \cap (fQ,fQ') \geq \cap (fB^n(s/\sqrt{n}),fQ') \geq \cap (B^n(L),fQ') \geq \kappa_n(L/r')
\end{equation}

where Lemma 2.3 is used in the last step. Combining (4.3), (4.4), and (2.1) we get

\begin{equation}
(4.5) \quad \kappa_n(L/r') \leq \cap fE \leq K(f) \cap E \leq K(f) \omega_{n-1} \left( \ln \frac{s}{r/\sqrt{n}} \right)^{1-n}.
\end{equation}

Let $\alpha=\alpha(n,K(f),i(x_0,f))=\max(C,C')$ where $C$ is given by Lemma 3.5 and $C'$ by (4.2). Choose $r$ so that the right hand side of (4.5) is $\kappa_n(2x^2)$. Then $s=\gamma r$ where $\gamma=\gamma(n,K(f),i(x_0,f))$. Since $\kappa_n$ is decreasing, (4.5) gives $L/\alpha \geq 2r' \alpha$ and so

\[ \tilde{l}(0,f,s/\sqrt{n}) \geq 2r' \alpha \geq 2r'C'. \]

Now (4.2) yields

\begin{equation}
(4.6) \quad \tilde{l} \geq 2r'.
\end{equation}

Let $P: \mathbb{R}^n \to \mathbb{R}^n$ be the projection $P(x)=x-x_ne_n$. For $y \in PQ'$ let $J_y$ be the closed upper segment of $(Q \setminus Q') \cap P^{-1}(y)$. By Fubini’s theorem

\[ \int_Q |f'| \, dm_n \geq \int_{PQ} dm_{n-1}(y) \int_{J_y} |f'| \, dm_1, \]

hence there exists $y \in PQ'$ such that

\begin{equation}
(4.7) \quad \int_{J_y} |f'| \, dm_1 \leq m_{n-1}(PQ')^{-1} \int_Q |f'| \, dm_n = (2r)^{-(n-1)} \int_Q |f'| \, dm_n
\end{equation}

and $f$ is absolutely continuous on $J_y$. Let $l(fJ_y)$ denote the length of the path $f|J_y$. Observe that $f|J_y$ need not be injective. We claim that

\begin{equation}
(4.8) \quad \tilde{l} - r' \leq l(fJ_y).
\end{equation}

If not, then $l(fJ_y) < \tilde{l} - r'$ and so $fJ_y \subset B^n(\tilde{l})$. By [4, Lemma 2.6] the components of $f^{-1}fJ_y \cap \bar{U}(0,f,\tilde{l})$ are in $U(0,f,\tilde{l})$ and each of them is mapped onto $fJ_y$. Now $J_y$ is a part of such a component since $U(0,f,\tilde{l}) \supset Q'$ by (4.6). On the other hand, by Lemma 3.3, $U(0,f,\tilde{l}) \subset B^n(s)$. This implies $J_y \subset B^n(s)$, a contradiction.

The inequalities (4.6) and (4.8) give

\[ l = 2l - \tilde{l} \leq 2(\tilde{l} - r') \leq 2l(fJ_y) \leq 2 \int_{J_y} |f'| \, dm_1 \leq 2(2r)^{-(n-1)} \int_Q |f'| \, dm_n, \]

where Lemma 2.3 is used in the last step. Combining (4.3), (4.4), and (2.1) we get

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where \((4.7)\) is used in the last step. Lemma 3.5, (4.2), and the relation \(s = yr\) now yield
\[
m_n(fQ) \leq \Omega_n L^n \leq \Omega_n Cl(0, f, s/n)^n \leq \Omega_n C^n l(0, f, s)^n
\[
\leq \Omega_n \alpha^{n+1} l^n \leq \Omega_n \alpha^{n+1} (2(2r)^{-(n-1)}) \int_Q |f'| \, dm_n
\[
= \frac{q}{K(f) i(x_0, f)} m_n(Q) \left( \frac{1}{m_n(Q)} \int_Q |f'| \, dm_n \right)^n.
\]
Here \(q = q(n, K(f), i(x_0, f))\). The integration formula \([4, 2.15]\) and the fact \(N(y, f, Q) \leq i(x_0, f)\) for \(y \in fQ\) with the above inequality imply
\[
\frac{1}{m_n(Q)} \int_Q |f'| ^n \, dm_n \leq \frac{K(f)}{m_n(Q)} \int_Q J(x, f) \, dm_n(x) \leq \frac{K(f)}{m_n(Q)} \int_Q N(y, f, Q) \, dm_n(y)
\]
\[
\leq \frac{K(f) i(x_0, f)}{m_n(Q)} m_n(fQ) \leq q \left( \frac{1}{m_n(Q)} \int_Q |f'| \, dm_n \right)^n.
\]
This is \((4.1)\). The theorem follows.

REFERENCES


