PERFECT CODES IN
ANTIPODAL DISTANCE-TRANSITIVE GRAPHS

OLOF HEDEN

Let \( C \) be a perfect code in an antipodal distance-transitive graph. In this paper it is shown that if \( u \in C \) then any vertex at maximum distance from \( u \) also belongs to \( C \). This is a generalisation of a theorem for binary codes of Roos [1].

1.

A graph is a pair \((V(G), E(G))\) where \( V(G) \) is a finite and nonempty set of elements called vertices and \( E(G) \) is a set of unordered pairs of distinct elements of \( V(G) \) called edges.

\((v_0, v_1, \ldots, v_n)\) is a path from \( v_0 \) to \( v_n \) if \( v_i, i = 0, 1, \ldots, n \) are vertices and \( \{v_i, v_{i+1}\} \) are distinct edges. A graph is called connected if given any pair of vertices \( v, w \), there is a path from \( v \) to \( w \). In this paper we only consider connected graphs.

The number of edges in a path is the length of the path. Let \( d(u, v) \), the distance between the vertices \( u \) and \( v \), denote the length of the shortest path from \( u \) to \( v \). The function \( d(u, v) \) defines a metric on the set of vertices.

An automorphism \( \varphi \) of a graph is a permutation of \( V(G) \) such that for any given pair of vertices \( u \) and \( v \) it is true that \( d(\varphi(u), \varphi(v)) = d(u, v) \).

A graph is called distance-transitive if for any given two pairs of vertices \( u, v \) and \( w, z \) satisfying \( d(u, v) = d(w, z) \) there is an automorphism \( \varphi \) for which \( \varphi(u) = w \) and \( \varphi(v) = z \). All graphs in this paper are distance-transitive.

Let \( u \in V(G) \) and

\[
\Gamma_i(u) = \{v \in V(G) \mid d(u, v) = i\}.
\]

Let \( d \) be the maximum possible distance between any two vertices. \( d \) is called the diameter of \( G \). A graph is called antipodal if for all vertices \( v, w \in \Gamma_0(u) \cup \Gamma_d(u) \) either \( v = w \) or \( d(v, w) = d \).

Received June 20, 1974.
Example. Let $\mathbb{Z}_n$ be the integers modulo $n$. Let $\mathbb{Z}_n^r$ be the set of $r$-tuples of elements of $\mathbb{Z}_n$. Define the distance between $r$-tuples $\bar{s}=(s_1,\ldots,s_r)$ and $\bar{t}=(t_1,\ldots,t_r)$ to be
\[
d(\bar{s},\bar{t}) = |\{i \mid s_i \neq t_i\}|.
\]
$\mathbb{Z}_n^r$ is a distance-transitive graph where the $r$-tuples are vertices and $d(\bar{s},\bar{t})$ is the distance-function on the vertices. $\mathbb{Z}_2^r$ is an antipodal distance-transitive graph.

A subset $C$ of $V(G)$ is called a perfect $e$-error correcting code if for every vertex $v$ it is true that
\[
|\{u \in V(G) \mid d(v,u) \leq e\} \cap C| = 1.
\]
Let $u$ be a vertex. Define
\[
\gamma_i = |\Gamma_i(u) \cap C| \quad i=1,2,\ldots.
\]
Call the $d+1$-tuple $(\gamma_0,\gamma_1,\ldots,\gamma_d)$ the weight-enumerator of $C$. The weight enumerator is not independent of the choice of $u$. But we shall see in section 2 that it only depends on $d(u,C)$, the minimum possible distance between $u$ and any vertex of $C$. $d(u,C)$ is called the minimum weight of $C$.

Let $u$ and $v$ be two vertices such that $d(u,v)=j$. The numbers
\[
\begin{align*}
k_i &= |\Gamma_i(u)| \quad i=0,1,\ldots,d \\
 a_j &= |\Gamma_1(v) \cap \Gamma_j(u)| \\
b_j &= |\Gamma_1(v) \cap \Gamma_{j+1}(u)| \quad \text{(defined for } j \leq d-1) \\
c_j &= |\Gamma_1(v) \cap \Gamma_{j-1}(u)| \quad \text{(defined for } j \geq 1)
\end{align*}
\]
are independent of the choices of $u$ and $v$. They satisfy the following relations
\begin{align}
(1) \quad a_j + b_j + c_j &= k_1, \quad j=0,1,\ldots,d, \\
 k_i b_i &= k_{i+1} c_{i+1}, \quad i=0,1,\ldots,d-1,
\end{align}
(2) $k_1 = b_0 > b_1 \geq \ldots \geq b_{d-1} \geq 1, \quad 1 = c_1 \leq c_2 \leq \ldots \leq c_d$.

For a proof of this see [4]. Let
\[
\Gamma(G) = \begin{pmatrix}
0 & c_1 & 0 \\
b_0 & a_1 & c_2 \\
0 & b_1 & a_2 \\
& b_2 & \ldots & c_{d-1} & 0 \\
& & \ldots & a_{d-1} & c_d \\
0 & & & b_{d-1} & a_d
\end{pmatrix}
\]
$\Gamma(G)$ is called the intersection matrix of $G$. If $[1,v_1(\lambda),\ldots,v_d(\lambda)]^t$ is an right eigenvector of $\Gamma(G)$ belonging to the eigenvalue $\lambda$, then it must satisfy the relations

\begin{equation}
\begin{aligned}
    v_i(\lambda) &= \lambda, \\
    c_{i+1}v_{i+1}(\lambda) + (a_i - \lambda)v_i(\lambda) + b_{i-1}v_{i-1}(\lambda) &= 0 \quad (v_0(\lambda) = 1) \\
    i &= 1, 2, \ldots, d - 1.
\end{aligned}
\end{equation}

\begin{equation}
    b_{d-1}v_{d-1}(\lambda) + (a_d - \lambda)v_d(\lambda) = 0.
\end{equation}

The functions $v_i(\lambda)$, $i = 1, \ldots, d$, are polynomials in $\lambda$ of degree $i$.

Biggs has shown [2] and [3] that the $d + 1$ eigenvalues of $\Gamma(G)$ are distinct and that they are zeros of the polynomial

\begin{equation}
    (\lambda - k_1)(1 + v_1(\lambda) + \ldots + v_d(\lambda)).
\end{equation}

2.

In [3] Biggs shows that if a perfect $e$-error correcting code exists in the distance-transitive graph $G$ then the polynomial $1 + v_1(\lambda) + \ldots + v_e(\lambda)$ divides the polynomial $1 + v_1(\lambda) + \ldots + v_d(\lambda)$. It is natural to ask which polynomial $f(\lambda)$ satisfy

\begin{equation}
    (1 + v_1(\lambda) + \ldots + v_e(\lambda))f(\lambda) = 1 + v_1(\lambda) + \ldots + v_d(\lambda).
\end{equation}

We shall prove a lemma saying that if $(\gamma_0, \gamma_1, \ldots, \gamma_d)$ is the weight-enumerator of the code then $1 + v_1(\lambda) + \ldots + v_d(\lambda)$ divides

\begin{equation}
    (1 + v_1(\lambda) + \ldots + v_e(\lambda))(\gamma_0 + \gamma_1v_1(\lambda)/k_1 + \ldots + \gamma_dv_d(\lambda)/k_d).
\end{equation}

Consequently at least $d - e$ eigenvalues of the intersection matrix must be zeros of the polynomial $\gamma_0 + \gamma_1v_1(\lambda)/k_1 + \ldots + \gamma_dv_d(\lambda)/k_d$. The solution of a system of $n$ such linear equations will only depend on $\gamma_0, \gamma_1, \ldots, \gamma_{d-n}$ as we shall see in lemma 2. Knowing this it will be easy to prove the theorem of Biggs and to prove that the weight-enumerator of the code only depends on the minimum weight for the code.

**Lemma 1.** If $C$ is a perfect code that corrects $e$ errors and $(\gamma_0, \gamma_1, \ldots, \gamma_d)$ is the weight enumerator of $C$ then the polynomial $1 + v_1(\lambda) + \ldots + v_d(\lambda)$ divides the polynomial

\begin{equation}
    (1 + v_1(\lambda) + \ldots + v_e(\lambda))(\gamma_0 + \gamma_1v_1(\lambda)/k_1 + \ldots + \gamma_dv_d(\lambda)/k_d).
\end{equation}

**Proof.** Let $\mu$ be an eigenvalue of the intersection matrix, and $u$ a vertex of $G$. To every vertex $v$ of $G$ associate the following number

\begin{equation}
    v_{d(u,v)}(\mu)/k_{d(u,v)} = f(\mu, v).
\end{equation}
Using induction over \( i \) and the relations (1), (3) and (4) it is straightforward to prove that
\[
v_i(\mu)f(\mu, v) = \sum_{w, d(\sigma, w) = i} f(\mu, w) \quad \text{for} \quad i = 0, 1, \ldots, d.
\]
Consequently if \( C \) is a perfect \( e \)-error correcting code
\[
(\sum_{v \in C} f(\mu, v))(1 + v_1(\mu) + \ldots + v_e(\mu)) = \sum_{v \in V(G)} f(\mu, v),
\]
that is,
\[
(\gamma_0 + \gamma_1 v_1(\mu)/k_1 + \ldots + \gamma_d v_d(\mu)/k_d)(1 + v_1(\mu) + \ldots + v_e(\mu)) = 1 + v_1(\mu) + \ldots + v_d(\mu).
\]
Since the zeros of \( 1 + v_1(\lambda) + \ldots + v_d(\lambda) \) are eigenvalues of the intersection matrix, it is necessary that the zeros of \( 1 + v_1(\lambda) + \ldots + v_d(\lambda) \) are zeros of
\[
(\gamma_0 + \gamma_1 v_1(\lambda)/k_1 + \ldots + \gamma_d v_d(\lambda)/k_d)(1 + v_1(\lambda) + \ldots + v_e(\lambda)).
\]
Consequently the lemma 1 is true.

**Lemma 2.** If \( \lambda_1, \ldots, \lambda_j \) are distinct eigenvalues of the intersection matrix of \( G \) then

\[
\begin{vmatrix}
\frac{v_{d-j+1}(\lambda_1)}{k_{d-j+1}} & \ldots & \frac{v_d(\lambda_1)}{k_d} \\
\vdots & \ddots & \vdots \\
\frac{v_{d-j+1}(\lambda_j)}{k_{d-j+1}} & \ldots & \frac{v_d(\lambda_j)}{k_d}
\end{vmatrix} \neq 0
\]

**Proof.** Suppose \( \mu \) is an eigenvalue of \( I(G) \) and \( v_d(\mu) = 0 \). Then we get by recursion using (3) and (4) that \( v_0(\mu) = 0 \). This is impossible since \( v_0(\mu) = 1 \). We conclude that \( v_d(\mu) \neq 0 \). So by dividing by the nonzero number \( v_d(\mu) \) we get an eigenvector
\[
(1/v_d(\mu), \ldots, v_{d-1}(\mu)/v_d(\mu), v_d(\mu)/v_d(\mu))^t
\]
\[
= (v'_0(\mu), \ldots, v'_{d-1}(\mu), 1)^t
\]
of \( I(G) \) belonging to the eigenvalue \( \mu \). Now \( v'_i(\mu) \), \( i = 0, 1, \ldots, d - 1 \) must satisfy the relations
\[
b_{d-i} v'_{d-i}(\mu) = \mu - a_d,
\]
\[
c_{i+1} v'_{i+1}(\mu) + (a_i - \mu)v'_i(\mu) + b_{i-1} v'_{i-1}(\mu) = 0 \quad i = 1, 2, \ldots, d - 1.
\]
Using recursion we see that \( v'_i(\mu) \) is a polynomial in \( \mu \) of degree \( d - i \).
So by elementary determinant calculus

\[
\det \begin{bmatrix}
v_{d-j+1}(\lambda_1) & \cdots & v_d(\lambda_1) \\
k_{d-j+1} & \cdots & k_d \\
\vdots & & \vdots \\
v_{d-j+1}(\lambda_j) & \cdots & v_d(\lambda_j) \\
k_{d-j+1} & \cdots & k_d
\end{bmatrix} = \frac{\prod_{i=1}^{j} v_d(\lambda_i)}{\prod_{i=1}^{j} k_{d-i+1}} \det \begin{bmatrix}
v'_{d-j+1}(\lambda_1) & \cdots & 1 \\
\vdots & & \vdots \\
v'_{d-j+1}(\lambda_j) & \cdots & 1
\end{bmatrix}
\]

\[
= r \det \begin{bmatrix}
\lambda_1^{i-1} & \cdots & \lambda_1 & 1 \\
\vdots & & \vdots & \vdots \\
\lambda_j^{i-1} & \cdots & \lambda_j & 1
\end{bmatrix} \quad \text{for some } r \neq 0.
\]

Since the \(\lambda_i\)'s, \(i = 1, 2, \ldots, j\) are distinct the last determinant is nonzero and the lemma is proved.

**Theorem 1** (Biggs). *If there exists a perfect e-error correcting code \(C\) in the distance-transitive graph \(G\) then the polynomial \(1 + v_1(\lambda) + \cdots + v_e(\lambda)\) divides the polynomial \(1 + v_1(\lambda) + \cdots + v_d(\lambda)\).*

**Proof.** For every perfect code \(C\) with minimum weight less than \(e\) there exists an automorphism \(\varphi\) of \(G\) such that \(\varphi(C) = C'\) is a perfect code with minimum weight equal to \(e\). Suppose that the polynomial \(1 + v_1(\lambda) + \cdots + v_e(\lambda)\) has less than \(e\) zeros among the eigenvalues of \(\Gamma(G)\). If \(\gamma_0 = \ldots = \gamma_{e-1} = 0\) there exists a perfect code with such a weight-enumerator, as we saw above. Then by lemma 2 the solutions of the linear system of equations

\[
\gamma_0 + \gamma_1 v_1(\lambda_i)/k_1 + \cdots + \gamma_d v_d(\lambda_i)/k_d = 0, \quad \lambda_i \text{ eigenvalue of } \Gamma(G) \text{ and } i = 1, 2, \ldots, d - e + 1
\]

should be \(\gamma_j = 0, j = e, e + 1, \ldots, d\). This is impossible.

**Theorem 2.** The weight-enumerator of a perfect code in a distance-transitive graph only depends on the minimum-weight of the code.

**Proof.** Let \((\gamma_0, \gamma_1, \ldots, \gamma_d)\) be the weight enumerator of the perfect \(e\)-error correcting code \(C\). From lemma 1 we know that there exist \(d - e\) eigenvalues \(\lambda_s, s = 1, 2, \ldots, d - e\) of \(\Gamma(G)\) such that

\[
\gamma_0 + \gamma_1 v_1(\lambda_s)/k_1 + \cdots + \gamma_d v_d(\lambda_s)/k_d = 0.
\]

Suppose that the minimum weight of \(C\) is equal to \(i\), that is, \(\gamma_0 = \ldots = \gamma_{i-1} = \gamma_{i+1} = \ldots = \gamma_e = 0, \gamma_i = 1\). We then get that

\[
(*) \quad \gamma_{e+1} v_{e+1}(\lambda_s)/k_{e+1} + \cdots + \gamma_d v_d(\lambda_s)/k_d = v_i(\lambda_s)/k_i \quad s = 1, 2, \ldots, d - e
\]

\[\text{Math. Scand. 35 — 3}\]
Since
\[
\det \begin{pmatrix}
v_{e+1}(\lambda_1) & \cdots & v_d(\lambda_1) \\
\frac{k_{e+1}}{} & \cdots & \frac{k_d}{k_d} \\
\vdots & & \vdots \\
v_{e+1}(\lambda_{d-e}) & \cdots & v_d(\lambda_{d-e}) \\
\frac{k_{e+1}}{} & \cdots & \frac{k_d}{k_d}
\end{pmatrix} + 0
\]
we get that the solutions of the system of linear equations (*) are unique.

3.

The following relations are easy but useful consequences of the definition of antipodal distance-transitive graph of diameter \(d\).

(5) If \(d(u, v) < d\) then \(\Gamma_d(u) \cap \Gamma_d(v) = \emptyset\).

(6) If \(d(u, v) = d\) and \(d(v, w) = i < d/2\) then \(d(u, w) = d - i\).

(7) If \(d(u, v) = d = 2n + 1\) then \(\Gamma_n(v) \subseteq \Gamma_{n+1}(u)\).

(8) If \(d(u, v) = d = 2n\) then \(\Gamma_n^{-1}(v) \subseteq \Gamma_{n+1}(u)\).

(9) If \(d = 2n + 1\) then \(\Gamma_{n+1}(u) = \bigcup_{v \in \Gamma_d(u)} \Gamma_n(v)\).

(10) If \(d = 2n\) then \(\Gamma_{n+1}(u) = \bigcup_{v \in \Gamma_d(u)} \Gamma_{n-1}(v)\).

We need two lemmas for the proof of theorem 3.

**Lemma 3.** If \(G\) is an antipodal distance-transitive graph with diameter \(d\) then \(1 \leq k_1 \leq k_2 \leq \ldots \leq k_j > k_{j+1} > \ldots > k_d\) for some

\[j \geq \begin{cases} n+1 & \text{if } d = 2n+1 \\ n & \text{if } d = 2n. \end{cases}\]

**Proof.** Suppose that \(k_j > k_{j+1}\). Then from relation (1) we get that \(c_{j+1} > b_j\). So by using relation (2) we see that \(c_{s+1} > b_s\) if \(s > j\) and consequently \(k_s > k_{s+1}\) if \(s > j\). By (7) and (8) is \(k_n \leq k_{n+1}\) when \(d = 2n + 1\) and \(k_{n-1} \leq k_{n+1}\) when \(d = 2n\). It follows that \(j \geq n + 1\) if \(d = 2n + 1\) and \(j \geq n\) if \(d = 2n\).

**Lemma 4.** If \(G\) is an antipodal distance-transitive graph with diameter \(d\) then

\[k_d = \begin{cases} b_n/c_{n+1} & \text{if } d = 2n+1 \\ b_n/c_n & \text{if } d = 2n. \end{cases}\]
Proof. First assume that \( d = 2n + 1 \). Let \( z \in \Gamma_n(u) \), that is, \( d(u, z) = n \). By (9) we have

\[
|\Gamma_{n+1}(u) \cap \Gamma_1(z)| = \sum_{v \in \Gamma_d(u)} |\Gamma_n(v) \cap \Gamma_1(z)| ,
\]

that is, \( b_n = c_{n+1} \), since \( d(v, z) = n + 1 \). When \( d = 2n = d(u, v) \) choose \( z \) such that \( d(u, z) = d(v, z) \), and use (10) similarly.

Theorem 3. If \( C \) is a perfect code in an antipodal distance-transitive graph with diameter \( d \) then for any vertex \( u \) it is that either \( \Gamma_0(u) \cup \Gamma_d(u) \subseteq C \) or \( (\Gamma_0(u) \cup \Gamma_d(u)) \cap C = \emptyset \).

Proof. Suppose that \( u \in C \) and that there exists a vertex \( v \in \Gamma_d(u) \setminus C \). Since \( C \) is perfect and corrects \( e \) errors there must be a vertex \( v' \) for which \( d(v, v') = i \leq e \).

Let \( w \in \Gamma_t(u) \) and \( d(w, v') = d \). It is easy to see that such a vertex must exist. Let \( \varphi \) be an automorphism that satisfy \( \varphi(w) = u \) and \( \varphi(u) = w \).

If \( (\gamma_0, \gamma_1, \ldots, \gamma_d) \) is the weight enumerator of \( \varphi(C) \) then \( \gamma'_t = 1 \) and \( \gamma'_d \geq 1 \).

But we get from lemma 3 that \( |\Gamma_t(u)| \geq k_1 \) (in the nontrivial cases \( e \leq d/2 \)) and from lemma 4, since \( b_n < k_1 \), that \( k_d < k_1 \). Let \( V = \bigcup_{v \in \Gamma_d(u)} \Gamma_t(v) \).

Then we find, since \( C \) is an \( e \)-error correcting code,

\[
|C \cap V| \leq |\Gamma_d(u)| = k_d < k_1 \leq |\Gamma_t(u)| ,
\]

that is, \( |C \cap V| < |\Gamma_t(u)| \). Observe that \( \Gamma_d(w) \subseteq V \) when \( w \in \Gamma_t(u) \), \( i \leq e \leq d/2 \). Hence

\[
|C \cap \bigcup_{w \in \Gamma_t(u)} \Gamma_d(w)| \leq |C \cap V| < |\Gamma_t(u)| .
\]

Since \( \Gamma_d(w_1) \cap \Gamma_d(w_2) = \emptyset \), when \( w_1 \neq w_2 \in \Gamma_t(u) \), we get

\[
\sum_{v \in \Gamma_t(u)} \left| C \cap \Gamma_d(w) \right| < |\Gamma_t(u)| ,
\]

and \( C \cap \bigcup \Gamma_d(w') = \emptyset \) for some \( w' \in \Gamma_t(u) \).

Let \( \varphi' \) be an automorphism that satisfy \( \varphi'(w') = u \) and \( \varphi'(u) = w' \). If \( (\gamma'_0, \gamma'_1, \ldots, \gamma'_d) \) is the weight enumerator of \( \varphi'(C) \) then \( \gamma'_t = 1 \) and \( \gamma'_d = 0 \).

The perfect codes \( \varphi(C) \) and \( \varphi'(C) \) have the same minimum weight, but their weight enumerators are not equal. Using theorem 2 we see that this is impossible. Consequently \( \Gamma_d(u) \setminus C = \emptyset \) if \( u \in C \) and the theorem is proved.

In the antipodal distance-transitive graph \( 2.0_4 \) (see [5]) it is easy to find a perfect code. \( 2.0_4 \) can not be represented as \( Z_2^r \) for any \( r \). So theorem 3 is in fact a generalisation of the theorem of Roos.
In [4] Smith gives an example of an antipodal distance-transitive graph \( G \) with intersection-matrix

\[
\Gamma(G) = \begin{pmatrix}
0 & 1 & 0 \\
3 & 0 & 1 \\
2 & 0 & 1 \\
2 & 0 & 2 \\
2 & 0 & 2 \\
1 & 0 & 2 \\
0 & 1 & 0
\end{pmatrix}
\]

If \( v_0(\lambda), v_1(\lambda), \ldots, v_d(\lambda) \) are defined as in section 1 and \( v_0(\lambda) = 1 \) it is easy to see that \( 1 + v_1(\lambda) + v_2(\lambda) \) divides \( 1 + v_1(\lambda) + \ldots + v_d(\lambda) \) where \( d = 8 \). This observation was made by Lindström [6].

If there exists a perfect 2-error correcting code \( C \) in \( G \) then \( |C| = 9 \). But, using theorem 3 we see that if \( u \in C \) then \( \Gamma_0(u) \cup \Gamma_8(u) \subseteq C \). The distance between any vertex of \( G \) and \( \Gamma_0(u) \cup \Gamma_8(u) \) is less or equal to 4 and there can impossibly be any more code vertices of \( G \). Consequently no perfect 2-error correcting code exists in \( G \).

In [4] Smith defines the derived graph \( G' \) of the antipodal distance-transitive graph \( G \). The vertices of \( G' \) are the sets \( \Gamma_0(u) \cup \Gamma_d(u) \), \( u \in V(G) \), and there is an edge between the vertices \( \Gamma_0(u) \cup \Gamma_d(u) \) and \( \Gamma_0(u') \cup \Gamma_d(u') \) of \( G' \) iff there are vertices \( v \in \Gamma_0(u) \cup \Gamma_d(u) \) and \( v' \in \Gamma_0(u') \cup \Gamma_d(u') \) such that \( d(v, v') = 1 \). Smith then shows that if \( d > 2 \) for the antipodal distance-transitive graph \( G \), then the derived graph \( G' \) is distance-transitive with diameter \( \lceil \frac{d}{2} \rceil \).

We show the following corollary of theorem 3.

**Corollary.** If there exists a perfect \( e \)-error correcting code in the antipodal distance-transitive graph \( G \) then there exists a perfect \( e \)-error correcting code in the derived graph \( G' \).

**Proof.** Let \( C \) be a perfect \( e \)-error correcting code of \( G \). Let \( C' \) be the vertices of the derived graph \( G' \) that satisfy

\[
\Gamma_0(u) \cup \Gamma_d(u) \in C' \quad \text{iff} \quad \Gamma_0(u) \cup \Gamma_d(u) \subseteq C.
\]

If

\[
c_1' = \Gamma_0(c_1) \cup \Gamma_d(c_1) \in C', \quad c_2' = \Gamma_0(c_2) \cup \Gamma_d(c_2) \in C'
\]

Then
and \( d(c_1', c_2') < 2e + 1 \) then it is easy to see that there exist vertices \( c_1'' \in \Gamma_0(c_1) \cup \Gamma_d(c_1), c_2'' \in \Gamma_0(c_2) \cup \Gamma_d(c_2) \) such that \( d(c_1'', c_2'') < 2e + 1 \). Since \( C \) is perfect this is impossible. Using theorem 3 we find that \( |C'| = |C|/k_o + k_d \). Now since \( |V(G')| = |V(G)|/k_o + k_d \) and

\[
|\{v \in V(G) \mid d(u, v) \leq e\}| = |\{v \in V(G') \mid d(u', v) \leq e\}|
\]

for \( u \in V(G) \) and \( u' \in V(G') \), \( C' \) must be a perfect code.

It is well-known that there exists a perfect 3-error correcting code in the antipodal distance-transitive graph \( Z_2^{23} = G \). Consequently there must exist a perfect 3-error correcting code in the derived graph \( G' \). Perhaps this is a code that Biggs [3, p. 296] question for.

**Acknowledgement.** I wish to express my sincere gratitude to Dr. B. Lindström whose kind advices have been a very fine help in writing this paper.

**References**


**University of Stockholm**

**Sweden**