REMARKS ON RELATIONS BETWEEN MAXIMAL LATTICES AND RELATIVELY MINIMAL MODELS

J. BRZEZINSKI

In [4] we have proved that if $A$ is a Dedekind ring of characteristic $\neq 2$ with perfect residue fields (i.e. $A/p$ is a perfect field for every maximal ideal $p$ in $A$), $E/F$ a regular extension of transcendence degree 1 and genus 0 of the field of fractions of $A$ then there is a regular quadratic space $(V, Q)$ over $F$ and a lattice $L$ on $V$ such that $L$ defines a relatively minimal $\text{Spec}(A)$-model $M(L)$ of $E$. The aim of this paper is to characterize those lattices which define relatively minimal models in the way described in [4]. The main result says that every relatively minimal model can be defined by an $\alpha$-maximal lattice (see [5, § 82 H]) where $\alpha$ is an ideal in $A$ which depends only on the extension $E/F$ and $A$ (Theorem 1).

1. The discriminant of a lattice.

Let $(V, Q)$ be a regular quadratic space over the field of fractions $F$ of a discrete valuation ring $A$ of characteristic $\neq 2$ and $\dim V = 3$. Let $\pi$ be a generator of the maximal ideal of $A$. If $L$ is a lattice on $V$ then $L = Ae_0 + Ae_1 + Ae_2$ and this lattice defines a quadratic form

$$f_L = (1/\pi^r) \sum_{i,j=0}^2 B(e_i, e_j)X_iX_j = \sum_{0 \leq i \leq j \leq 2} a_{ij} X_iX_j$$

where $B$ is the bilinear form defined by $Q$, $a_{ii} = (1/\pi^r)B(e_i, e_i)$, $a_{ij} = 2 \cdot (1/\pi^r)B(e_i, e_j)$ for $i \neq j$ and $(\pi^r) = nL$ is the norm of $L$ (see [5] for the notion of norm, in [4] the form which corresponds to $L$ is

$$(1/\pi^r) \sum_{0 \leq i \leq j \leq 2} B(e_i, e_j)X_iX_j$$

where $(\pi^r) = sL$ is the scale of $L$. But it will be clear that the present definition of $f_L$ is more convenient). Since $nL$ is generated by $B(e_i, e_i)$ and $2B(e_i, e_j)$ the coefficients of the form $f_L$ belong to $A$. Now if

$$f = \sum_{0 \leq i \leq j \leq 2} a_{ij} X_iX_j$$

Received December 7, 1973.
then the determinant
\[
\det(f) = \begin{vmatrix} 2a_{00} & a_{01} & a_{02} \\ a_{01} & 2a_{11} & a_{12} \\ a_{02} & a_{12} & 2a_{22} \end{vmatrix}
\]
is called the discriminant of \( f \) (see [6, p. 2]). It is easy to check that if \( a_{ij} \in \mathbb{A} \) then \( \det(f) \in \mathbb{A} \).

If \( f_L \) is the form which corresponds to \( L \) then
\[
\det(f_L) = 4(1/\pi^3) \begin{vmatrix} B(e_0,e_0) & B(e_0,e_1) & B(e_0,e_2) \\ B(e_0,e_1) & B(e_1,e_1) & B(e_1,e_2) \\ B(e_0,e_2) & B(e_1,e_2) & B(e_2,e_2) \end{vmatrix} = 4(1/\pi^3) \det(e_0,e_1,e_2)
\]
where \( \det(e_0,e_1,e_2) \) is the discriminant of the base \( e_0,e_1,e_2 \) of \( V \) over \( F \) (see [5, p. 87]). But \( \det(e_0,e_1,e_2) \) generates the volume \( vL \) of \( L \) (see [5, p. 229]) and the last equality in the global case gives the following result:

**Lemma 1.** Let \( \mathbb{A} \) be a Dedekind ring and \( L \) a lattice on a regular quadratic space \((V, Q)\) over the field of fractions \( F \) of \( A \). Then
\[
4vL = (vL)^3 bL
\]
where \( vL \) is the volume of \( L \), \( nL \) the norm of \( L \) and \( bL \) a fractional ideal which is locally defined by \( bL = (d(f_L)) \) where \( f_L \) is the quadratic form corresponding to \( L \). Since \( d(f_L) \in \mathbb{A} \) for every prime ideal \( \mathfrak{p} \) in \( \mathbb{A} \) the ideal \( bL \) is integral.

**Definition 1.** The ideal \( bL \) will be called the discriminant of \( L \).

**Remark.** We have defined \( bL \) only for lattices on three dimensional spaces. It is clear that this definition can be generalized according to the usual definition of the discriminant of a quadratic form (e.g. [6, p. 2]).

**Lemma 2.** Let \( f = \sum_{0 \leq i \leq j \leq 2} a_{ij} x_i x_j \) be a quadratic form with coefficients in a field \( F \) of an arbitrary characteristic (it can be equal 2). The form \( f \) is reducible in some extension of \( F \) if and only if \( \det(f) = 0 \).

**Remark.** If \( \text{char}(F) = 2 \) then
\[
\det(f) = 4a_{00}a_{11}a_{22} + a_{01}a_{02}a_{12} - a_{00}a_{12}^2 - a_{11}a_{02}^2 - a_{22}a_{01}^2
= a_{01}a_{02}a_{12} + a_{00}a_{12}^2 + a_{11}a_{02}^2 + a_{22}a_{01}^2.
\]

**Proof.** If \( \text{char}(F) = 2 \) then the result is well-known. If \( \text{char}(F) = 2 \) and \( f = (a_0x_0 + a_1x_1 + a_2x_2)(b_0x_0 + b_1x_1 + b_2x_2) \) then it is easy to check that
\[ d(f) = 0. \text{ Let } d(f) = 0. \text{ Then } (a_{12}, a_{02}, a_{01}) \text{ is a zero of } f. \text{ If } a_{ij} = 0 \text{ for } i \neq j \text{ then the form is reducible in some extension of } F. \text{ Let } a_{12} \neq 0. \text{ Then the transformation}
\begin{align*}
x_0 &= a_{12}y_0, \quad x_1 = a_{02}y_0 + y_1, \quad x_2 = a_{01}y_0 + y_2
\end{align*}
has determinant not equal to 0 and maps the form \( f \) on the form \( a_{11}y_1^2 + a_{22}y_2^2 + a_{12}y_1y_2 \). This form is reducible in some extension of \( F \).

2. Lattices which define models.

We shall assume that \( A \) is a Dedekind ring with perfect residue fields such that the characteristic of \( A \) is \( \neq 2 \). \( E \) is a regular extension of transcendence degree 1 and genus 0 of \( F \) where \( F \) is the field of fractions of \( A \).

**Definition 2.** We shall denote by \( a_{E/A} \) the ideal of \( A \) which is equal to the product of all maximal ideals \( \mathfrak{p} \) in \( A \) such that the fiber above \( \mathfrak{p} \) of a relatively minimal model \( M \) of \( E \) over \( \text{Spec}(A) \) is a form of two intersecting copies of \( P^1(A/\mathfrak{p}) \) (the projective line over \( A/\mathfrak{p} \)). This ideal is independent of the relatively minimal model by the Theorem 1 in [2].

**Lemma 3.** Let \( L \) be a lattice on a regular quadratic space \((V, Q)\) over \( F \).

a) If \( M(L) \) is a model of \( E \) then \( \mathfrak{d}L \) is square-free.

b) If \( M(L) \) is a relatively minimal model of \( E \) then \( \mathfrak{d}L = a_{E/A} \).

**Proof.** If \( M(L) \) is a model then by Theorem 2 in [3] \( v_\mathfrak{p}(d(f_{L_\mathfrak{p}})) = 0 \) or 1 where \( v_\mathfrak{p} \) is the valuation corresponding to \( A_\mathfrak{p} \). Hence \( \mathfrak{d}L \) is square-free. Now if \( M(L) \) is a relatively minimal model then the last case takes place if and only if the fiber of this model above \( \mathfrak{p} \) is a form of two intersecting copies of \( P^1(A/\mathfrak{p}) \). In fact, \( v_\mathfrak{p}(d(f_{L_\mathfrak{p}})) = 1 \) if and only if \( d(\bar{f}_{L_\mathfrak{p}}) = 0 \) where \( \bar{f}_{L_\mathfrak{p}} \) is the image of \( f_{L_\mathfrak{p}} \) under the homomorphism
\[ A[x_0, x_1, x_2] \rightarrow (A/\mathfrak{p})[x_0, x_1, x_2]. \]

By the Lemma 2, \( d(\bar{f}_{L_\mathfrak{p}}) = 0 \) if and only if \( \bar{f}_{L_\mathfrak{p}} \) is reducible in some extension of \( A/\mathfrak{p} \), i.e. the fiber of \( M(L) \) above \( \mathfrak{p} \) is a form of two intersecting copies of \( P^1(A/\mathfrak{p}) \).

**Theorem 1.** Let \( M \) be a model of \( E \) over \( \text{Spec}(A) \) such that for every maximal ideal \( \mathfrak{p} \) in \( A \) the fiber \( M_\mathfrak{p} \) above \( \mathfrak{p} \) is either a form of \( P^1(A/\mathfrak{p}) \) or a form of two intersecting copies of \( P^1(A/\mathfrak{p}) \). Then there is a quadratic space \((V, Q)\) and a lattice \( L \) on \( V \) such that \( M \) is \( \text{Spec}(A) \)-isomorphic with \( M(L) \) and \( L \) is \( \mathfrak{d}L \)-maximal. Hence if \( M \) is a relatively minimal model then \( L \) is \( a_{E/A} \)-maximal.
Proof. We know that there is a quadratic space \((V, Q)\) and a lattice \(L\) on \(V\) such that \(M\) is Spec\((A)\)-isomorphic to \(M(L)\). For relatively minimal \(M\) this is proved in [4, Theorem 2]. If \(M\) is a model of \(E\) then we get \(L\) if we apply Theorem 1 in [3] and the same construction as in the proof of Theorem 2 in [4].

Since \(\nu L\) defines the neutral element in \(\text{Cl}(A)/\text{Cl}(A)^2\) where \(\text{Cl}(A)\) denotes the class group of \(A\) (by the definition of \(\nu L\) — see [5, p. 229]) hence by (1) we get that \(bL\) and \(nL\) define the same element in the group \(\text{Cl}(A)/\text{Cl}(A)^2\). We know that \(n(aL) = \alpha^2(nL)\) and \(nL^2 = \alpha(nL)\) (see [5, p. 228 and p. 238]). This means that we can choose a lattice \(L' = (aL)^*\) on a quadratic space \(V^*\) such that the model defined by \(L'\) is equal to the model defined by \(L\) and \(nL' = bL'\). We shall assume that \(L\) is such lattice and we shall prove that this lattice is \(bL\)-maximal.

Let \(K\) be a lattice such that \(nK \subseteq bL = nL\) and \(K \supseteq L\). These inclusions give \(nK = nL\) and \(\nu L = \alpha^2(\nu K)\) where \(\alpha\) is an ideal in \(A\) (see [5, § 82 E, 82:11]). Hence by (1)

\[bL = (nL)^{-3}4\nu L = (nK)^{-3}4\alpha^2\nu K = \alpha^2 bK.\]

But \(bL\) is integral, square-free (Lemma 3) and \(bK\) is integral. Hence \(\alpha = A\) and \(\nu L = \nu K\). Since \(K \supseteq L\) we get \(K = L\) (by [5, § 82 E, 82:11a]). This proves that \(L\) is \(bL\)-maximal.

REFERENCES

5. O. T. O'Meara, Introduction to quadratic forms (Grundlehren Math. Wissensc. 117), Springer-Verlag, Berlin · Göttingen · Heidelberg, 1963.

MATHEMATICAL INSTITUTE, UNIVERSITY OF GOTHENBURG

AND

CHALMER'S INSTITUTE OF TECHNOLOGY, GOTHENBURG, SWEDEN