DIVISIBLE AND CODIVISIBLE MODULES

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In this paper divisible and codivisible modules are studied relative to a torsion theory on Mod\(R\). It is shown, for example, that if the torsion theory is hereditary, then the following are equivalent; Every module is codivisible. Every torsionfree module is injective. \(R/T(R)\) is semi-simple Artinian. In addition, codivisible covers are investigated and it is shown that if every module has a codivisible cover, then \(R/T(R)\) is perfect. A partial converse to this is obtained by showing that if \(R/T(R)\) is perfect, then every torsionfree module has a codivisible cover. This leaves open the question of whether or not codivisible covers universally exist when \(R/T(R)\) is perfect.

1. Preliminaries.

Throughout this paper \(R\) will denote an associative ring with identity and our attention will be confined to the category Mod\(R\) of unital right \(R\)-modules.

In [4], Dickson defined a torsion theory on Mod\(R\) to be a pair \((A, B)\) of non-empty classes of modules such that:

\begin{enumerate}
  \item[(a)] \(A \cap B = 0\)
  \item[(b)] If \(A \rightarrow A^* \rightarrow 0\) is exact with \(A \in A\), then \(A^* \in A\).
  \item[(c)] If \(0 \rightarrow B^* \rightarrow B\) is exact with \(B \in B\), then \(B^* \in B\).
  \item[(d)] For each module \(M\) in Mod\(R\), there is an exact sequence \(0 \rightarrow A \rightarrow M \rightarrow B \rightarrow 0\) with \(A \in A\) and \(B \in B\).
\end{enumerate}

Modules in \(A\) are called torsion and those in \(B\) torsionfree. If \((A, B)\) is a torsion theory, then \(A\) is closed under isomorphic images, factor modules, extensions and direct sums, while \(B\) is closed under isomorphic images, submodules, extensions and direct products [4, p. 226, Theorem 2.3]. By saying that a class \(C\) of modules is closed under extensions, we mean that \(M \in C\) whenever \(N\) is a submodule of \(M\) such that \(N\) and \(M/N\) are in \(C\). If \((A, B)\) is a torsion theory such that \(B\) is closed under injective hulls [5], then \((A, B)\) is called hereditary. It is known that

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$(A, B)$ is hereditary if and only if $A$ is closed under submodules [4, p. 227, Theorem 2.9]. Uniquely associated with each hereditary torsion theory there is a topologizing idempotent right ideal filter

$$F = \{I \subseteq R \mid I \text{ is a right ideal of } R \text{ and } R/I \text{ is torsion}\}$$

(see Gabriel [6]).

An object functor $T : \text{Mod}
R \to \text{Mod}
R$ is said to be a radical if $T(M) \subseteq M$, $f : M \to N$ implies that $f(T(M)) \subseteq T(N)$ and $T(M/T(M)) = 0$. A radical $T$ such that $T(T(M)) = T(M)$ for all modules $M$ is called idempotent. It is well-known that $(A, B)$ is a torsion theory if and only if there exists an idempotent radical $T$ on $\text{Mod}
R$ such that

$$A = \{A \in \text{Mod}
R \mid T(A) = A\}$$

and

$$B = \{B \in \text{Mod}
R \mid T(B) = 0\}$$

[8, p. 2, Proposition 0.1]. Furthermore, this correspondence is one-to-one [10, p. 6, Proposition 2.3]. Hence a module $M$ is torsion if and only if $T(M) = M$ and torsionfree if and only if $T(M) = 0$. $T(M)$ is usually referred to as the torsion submodule of $M$ and it can be described as that (necessarily unique) torsion submodule of $M$ which contains every torsion submodule of $M$. If $(A, B)$ is a torsion theory with associated idempotent radical $T$, then $(A, B)$ is hereditary if and only if $T$ is a left exact functor [10, p. 8, Proposition 2.6]. (In this context if $f : M \to N$, then $T(f)$ is the restriction of $f$ to $T(M)$). When $T$ is left exact, it is not difficult to show that $T(N) = N \cap T(M)$ for every submodule $N$ of $M$. If $(A, B)$ is a torsion theory such that $R$ is torsionfree, then $(A, B)$ is said to be faithful.

A projective module $P(M)$ together with an epimorphism $\pi : P(M) \to M$ with small kernel is said to be a projective cover of $M$. A submodule $K$ of $M$ is said to be small in $M$ if $N = M$ whenever $N$ is a submodule of $M$ such that $K + N = M$. We will call any epimorphism with small kernel minimal. A ring $R$ is (semi-) perfect if every (cyclic) module has a projective cover. (See [2] for several characterizations of perfect and semi-perfect rings). Torsion theories for which every torsion module has a projective cover have been studied in [3].

In the discussion which follows, $(A, B)$ will denote a torsion theory on $\text{Mod}
R$ with idempotent radical $T$. When $(A, B)$ is hereditary, $F$ will denote the associated idempotent right ideal filter.
2. Divisible and codivisible modules.

Following Lambek [8, p. 8] we call a module $M$ divisible provided that every diagram of the form

$$
0 \longrightarrow L \xrightarrow{f} N \\
\downarrow \\
M
$$

where $\text{coker} f$ is torsion can be completed to a commutative diagram. Dually, $M$ is said to be codivisible if every diagram

$$
M \\
\downarrow \\
N \xrightarrow{f} L \longrightarrow 0
$$

where $\text{ker} f$ is torsionfree can be completed to a commutative diagram. The usual argument now shows that a (direct sum) direct product of modules is (codivisible) divisible if and only if each factor is (codivisible) divisible. The following result is well-known. We include a proof for completeness.

**Lemma 2.1.** If $(A,B)$ is hereditary, then for any module $M$,

$$
T(M) = \{x \in M \mid (0:x) \in F\}.
$$

($(0:x)$ denotes the right annihilator of $x$ in $R$). In particular, $T(R)$ is an ideal of $R$.

**Proof.**

$x \in T(M) \iff xR \subseteq T(M) \iff xR$ is torsion

$$
\iff R/(0:x)$ is torsion $\iff (0:x) \in F$.

Thus

$$
T(M) = \{x \in M \mid (0:x) \in F\}.
$$

If $x \in T(R)$ and $r \in R$, then $(0:x) \subseteq (0:rx)$ implies that $(0:rx) \in F$. Hence $rx \in T(R)$ and so $T(R)$ is an ideal of $R$.

**Theorem 2.2.** If $(A,B)$ is hereditary, then the following are equivalent:

(a) Every module is codivisible.
(b) Every torsionfree module is injective.
(c) $R/T(R)$ is a semi-simple Artinian ring.
Proof. (c) ⇒ (b). Let $M$ be a torsionfree $R$-module. If $m \in M$ and $x \in T(R)$, then $(0:x) \subseteq (0:mx)$ and so $(0:mx) \in F$ since $(0:x) \in F$. Thus it follows that $MT(R) \subseteq T(M) = 0$ and so $M$ is an injective $R/T(R)$-module. Next let $I$ be a right ideal of $R$ and suppose that $f: I \to M$ is $R$-linear. If

$$x \in T(I) = I \cap T(R),$$

then $f(x) = 0$ since $f(T(I)) \subseteq T(M) = 0$. Hence

$$g:(I + T(R))/T(R) \to M \text{ defined by } x + T(R) \to f(x)$$

is a well-defined $R$-linear mapping which is easily seen to be $R/T(R)$-linear. Consequently, by Baer’s criterion [1], there is an $m \in M$ such that

$$g(x + T(R)) = m(x + T(R))$$

for all $x + T(R)$ in $(I + T(R))/T(R)$. But this yields $f(x) = mx$ for all $x \in I$ and so $M$ is $R$-injective.

(b) ⇒ (a). Consider the diagram

$$
\begin{array}{ccc}
M & \to & N' \\
\downarrow g & & \downarrow \phi \\
L & \to & N \\
\end{array}
$$

where $K = \ker f$ is torsionfree. Since $K$ is injective, $L = K \oplus N'$ and $h = f|N': N' \to N$ is an isomorphism. Hence if $\varphi = h^{-1} \circ g$, then $\varphi$ completes the diagram commutatively.

(a) ⇒ (c). Let $I/T(R)$ be a right ideal of $R/T(R)$. Since $I/T(R)$ is torsionfree, the $R$-exact sequence

$$0 \to I/T(R) \to R/T(R) \to R/I \to 0$$

splits. Hence $I/T(R)$ is an $R$-direct summand and consequently an $R/T(R)$-direct summand of $R/T(R)$. Thus $R/T(R)$ is semi-simple Artinian.

Notice that from the proof of (a) ⇒ (c) we see that (a) can be replaced by: Every cyclic module is codivisible.

Corollary 2.3 If $(A,B)$ is a faithful, hereditary torsion theory and every module is codivisible, then $R$ is semi-simple Artinian.
The next theorem has a proof which is similar to that of theorem 2.2. The implications \((a) \Rightarrow (b) \Rightarrow (c)\) are straightforward while \((c) \Rightarrow (a)\) can be proven by using the generalized Baer's criterion for divisible modules [7, p. 20, Proposition 3.2].

**Theorem 2.4.** If \((A, B)\) is hereditary, then the following are equivalent:

(a) Every module is divisible.
(b) Every torsion is projective.
(c) Every \(I \in \mathcal{I}\) is a direct summand of \(R\).

If \(f: M \to N\) is \(R\)-linear, then we will call \(f\) free if \(\ker f\) is torsionfree. The following lemma will prove useful.

**Lemma 2.5.** If \(f: M \to N\) is a minimal, free epimorphism, then \(f\) is an isomorphism whenever \(N\) is codivisible.

**Proof.** Since \(K = \ker f\) is torsionfree, the exact sequence

\[
0 \to K \to M \xrightarrow{f} N \to 0
\]

splits whenever \(N\) is codivisible. Hence \(M = K \oplus N'\) where \(N'\) is a submodule of \(M\) isomorphic with \(N\). But \(K\) is small in \(M\) and so \(K = 0\).

The following lemma can be found in [9, p. 189, Hilfssatz 3.1].

**Lemma 2.6.** If \(f: M \to N\) is \(R\)-linear and \(A\) is small in \(M\), then \(f(A)\) is small in \(N\).

**Theorem 2.7.** Let \(K = \ker f\) where \(f: C \to M\) is an epimorphism and \(C\) is codivisible. Then \(M\) is codivisible if \(K\) is torsion. Conversely, if \(M\) is codivisible and \(K\) is small in \(C\), then \(K\) is torsion.

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{g} & N \\
\downarrow & & \downarrow \\
L & \xrightarrow{h} & N \\
\end{array}
\]

where \(\ker h\) is torsionfree. Since \(C\) is codivisible \(gof\) factors through \(L\). Suppose \(\varphi: C \to L\) is such that \(h \circ \varphi = gof\). Now \(\varphi: K \to \ker h\) and so if \(K\) is torsion, then

\[
\varphi(K) = \varphi(T(K) \subseteq T(\ker h)) = 0.
\]
Hence we have an induced mapping $\varphi^*: M \to L$ which completes the diagram commutatively.

Conversely, suppose that $M$ is codivisible and $K$ is small in $C$. Then $f^*: C/T(K) \to M$ defined by

$$f^*(x + T(K)) = f(x)$$

is a minimal, free epimorphism ($\ker f^* = K/T(K)$ is small in $C/T(K)$ by lemma 2.6.). Hence $f^*$ must be an isomorphism by lemma 2.5 and so it follows that $K = T(K)$.

The following theorem shows that a necessary and sufficient condition for every module to be codivisible can be found which is intrinsic to $R$ is semi-perfect.

**Theorem 2.8.** If $(A, B)$ is hereditary and $R$ is semi-perfect, then every module is codivisible if and only if $J(R) \subseteq T(R)$. ($J(R) =$ the Jacobson radical of $R$.)

**Proof.** Due to our observation following the proof of (a) $\Rightarrow$ (c) in theorem 2.2, we can confine our attention to cyclic modules. Note also that a right ideal $I$ of $R$ is contained in $J(R)$ if and only if $I$ is small in $R$ [2, p. 472, Lemma 2.4]. Hence if $I \subseteq J(R)$ is a right ideal of $R$ and every cyclic module is codivisible, then

$$0 \to I \to R \to R/I \to 0$$

is exact with $R/I$ codivisible. Thus it follows from theorem 2.7 that $I$ is torsion.

Conversely, suppose that $J(R) \subseteq T(R)$. If $I$ is any right ideal of $R$, then $R/I$ has a projective cover

$$0 \to K \to P \to R/I \to 0.$$

Now we can assume (without loss of generality) that $P$ is a direct summand of $R$ and so since $K$ is small in $P$, it follows that $K$ is small in $R$. Hence $K \subseteq J(R)$ and so $K$ is torsion. Consequently, by theorem 2.8, $R/I$ is codivisible.

3. Codivisible covers.

A codivisible module $C(M)$ together with a minimal, free epimorphism $\mu: C(M) \to M$ is said to be a codivisible cover of $M$. As in the case of projective covers, it can be shown that if

$$\{\mu_\alpha: C(M_\alpha) \to M_\alpha\} \quad (\alpha \in \Delta)$$
is a finite family of codivisible covers, then

$$\bigoplus \mu_\alpha : \bigoplus C(M_\alpha) \to \bigoplus M_\alpha$$

is a codivisible cover. If $R/I$ is a factor ring of $R$, then an $R/I$-module $M$ will be considered to be $R/I$-codivisible if every diagram

$$\begin{array}{c}
M \\
\downarrow \\
L \overset{f}{\longrightarrow} N \longrightarrow 0
\end{array}$$

of $R/I$-modules and $R/I$-module homomorphisms can be completed to a commutative diagram where $\ker f$ is torsionfree when viewed as an $R$-module. Similar remarks hold for free $R/I$-homomorphisms and $R/I$-codivisible covers.

The following two theorems also appear in [3] though with slightly different proofs. We include them here for convenience of the reader and for the sake of continuity.

**Theorem 3.1.** A codivisible cover (when it exists) is unique up to an isomorphism.

**Proof.** Let $\mu : C(M) \to M$ and $\mu^* : C(M)^* \to M$ be codivisible covers of $M$. Then the diagram

$$\begin{array}{c}
C(M) \\
\downarrow \mu \\
C(M)^* \overset{\mu^*}{\longrightarrow} M \longrightarrow 0
\end{array}$$

can be completed to a commutative diagram by a homomorphism $\varphi$ since $\mu^*$ is free. Hence it follows that

$$C(M)^* = \text{Im}\varphi + \ker \mu^*.$$  

But $\mu^*$ is minimal and so $\varphi$ is an epimorphism. Now $\ker \varphi \subseteq \ker \mu$ and consequently $\varphi$ is minimal and free. Hence by lemma 2.5 $\varphi$ must be an isomorphism.

**Theorem 3.2.** If $\pi : P(M) \to M$ is a projective cover of $M$, then

$$\mu : P(M)/T(\ker \pi) \to M$$

is a codivisible cover of $M$ where $\mu$ is the mapping induced by $\varphi$.

**Proof.** Since $\ker \mu \subseteq \ker \pi / T(\ker \pi)$, then $\mu$ is free. Notice again by lemma 2.6 that $\ker \mu$ is small in $P(M)/T(\ker \pi)$. Hence $\mu$ is a minimal,
free epimorphism. That $P(M)/T(\ker \pi)$ is codivisible follows directly since factor modules of codivisible modules by torsion submodules are codivisible, theorem 2.7.

We also see by theorem 2.7 that if $\pi: P(M) \to M$ is a projective cover of $M$, then $M$ is codivisible if and only if $\ker \pi$ is torsion. Notice that since the class of torsionfree modules is closed under extensions, then $C(M)$ (if it exists) is torsionfree whenever $M$ is torsionfree.

The above theorem shows that every (cyclic) module has a codivisible cover when $R$ is a (semi-) perfect ring. For the sake of brevity, we will call $R$ a (t-ring) $T$-ring if every (cyclic) module has a codivisible cover.

**Theorem 3.3.** If $R$ is a (t-ring) $T$-ring, then every factor ring of $R$ is a (t-ring) $T$-ring.

**Proof.** Let $R/I$ be a factor ring of $R$ and suppose that $M$ is an $R/I$-module. If $\mu: C(M) \to M$ is an $R$-codivisible cover of $M$, then

$$C^* = C(M)/C(M)I$$

is an $R/I$-module. Since $\mu(C(M)I) = MI = 0$, we have an induced mapping $\mu^*: C^* \to M$ which is a minimal epimorphism. Now $C^*$ is $R/I$-codivisible and so it follows that $C^*/T(\ker \mu^*)$ together with the obvious induced mapping is an $R/I$-codivisible cover of $M$.

An identical proof holds for t-rings.

If $R$ is torsionfree and $M$ is codivisible then

$$0 \to K \to F \xrightarrow{f} M \to 0$$

splits where $K = \ker f$ and $f: F \to M$ is a free module on $M$. It follows then that $M$ is (projective and) torsionfree. If every codivisible module is torsionfree, then $R$ is clearly torsionfree. Hence we have the following:

**Theorem 3.4.** Every codivisible module is (projective and) torsionfree if and only if $(A,B)$ is faithful.

The above theorem shows that over faithful torsion theories the class of (t-rings) $T$-rings and the class of (semi-) perfect rings coincide. From the work completed the following theorem is now evident.

**Theorem 3.5.** If $R$ is a (t-ring) $T$-ring, then every torsionfree factor ring of $R$ is (semi-) perfect. In particular, if $(A,B)$ is hereditary, then $R/T(R)$ is (semi-) perfect.
In conclusion, we obtain a partial converse to theorem 3.5.

**Theorem 3.6.** If \((A, B)\) is hereditary and \(R/T(R)\) is (semi-)perfect, then every torsionfree (cyclic) module has a codivisible cover.

**Proof.** Suppose that \(R/T(R)\) is perfect and let \(M\) be a torsionfree \(R\)-module. Then as shown in the proof of theorem 2.2, \(M\) is an \(R/T(R)\)-module. Let \(\pi: P(M) \to M\) be an \(R/T(R)\)-projective cover of \(M\). If \(\Delta\) is a set of \(R/T(R)\)-generators for \(P(M)\), then there is a canonical \(R/T(R)\)-epimorphism

\[ \eta: \bigoplus_{\alpha \in \Delta} R_{\alpha}/T(R_{\alpha}) \to P(M) \]

which is \(R\)-linear (\(R_{\alpha} = R\) for each \(\alpha \in \Delta\)). Now \(P(M)\) is an \(R/T(R)\)-direct summand and therefore an \(R\)-direct summand of \(\bigoplus_{\alpha \in \Delta} R_{\alpha}/T(R_{\alpha})\).

Thus \(P(M)\) is \(R\)-codivisible and torsionfree since it is the direct summand of an \(R\)-codivisible, torsionfree \(R\)-module. It follows then, that \(\pi: P(M) \to M\) is an \(R\)-codivisible cover of \(M\).

The semi-perfect case has a similar proof.

**References**

1. R. Baer, *Abelian groups which are direct summands of every containing abelian group*, Bull. Amer. Math. Soc., 46 (1940), 800–806.