GROUP RINGS OF FINITE REPRESENTATION TYPE

WILLIAM H. GUSTAFSON

1. Introduction.

In this note, we wish to discuss Artinian group rings which are of finite representation type in the sense that they have only finitely many non-isomorphic indecomposable modules. Thus let $A$ be a commutative, Artinian ring and let $G$ be a finite group; we will consider the group ring $AG$. By Maschke's Theorem, we know that if $A$ is a field whose characteristic is either zero or a prime which does not divide the order of $G$, then $AG$ is semisimple and hence of finite representation type. If, on the other hand, $A$ is a field whose characteristic $p$ divides the order of $G$, then $AG$ is of finite representation type if and only if the $p$-Sylow subgroup of $G$ is cyclic. This was shown by D. G. Higman [6]. Later, Kasch, Kupisch and Kneser [9] and Janusz [7] gave more refined information about the number of indecomposable modules in this case. Janusz [8] determined the structure of the indecomposable modules in considerable detail.

Here, we will give necessary and sufficient conditions for $AG$ to be of finite representation type, where $A$ is an arbitrary commutative, Artinian ring.

2. The theorem.

Our approach will depend on two crucial facts:

i) A commutative, Artinian ring is of finite representation type if and only if it is serial (i.e. each indecomposable projective module has a unique composition series).

This is established in Colby [1]. It can also be deduced easily from Dickson and Kelly [3].

ii) Any homomorphic image of a ring of finite representation type is also of finite representation type.

This is easy to prove.

Received November 7, 1973.
Now, each commutative, Artinian ring $A$ is a direct product $A_1 \times \ldots \times A_n$ of local, Artinian rings, and clearly we have

$$AG = A_1 G \times \ldots \times A_n G.$$  

Since the identity elements of the $A_i G$ are central idempotents of $AG$, we see that an indecomposable $AG$-module is annihilated by all but one of the $A_i G$. Thus $AG$ is of finite representation type if and only if each $A_i G$ is. Hence the problem is essentially solved once we prove

**Theorem.** Let $A$ be a local, commutative, Artinian ring, and let $G$ be a finite group. Then $AG$ is of finite representation type if and only if

a) $A$ is serial and  
b) if the characteristic of $A/\text{rad} A$ is a prime $p$ which divides the order of $G$, then $A$ is a field and the $p$-Sylow subgroup of $G$ is cyclic.

**Proof.** First, suppose that $AG$ is of finite representation type. Since the augmentation map $\varepsilon: AG \to A$ is a surjection, $A$ must be serial by i) and ii) above. Further, we have a surjection $AG \to kG$, where $k = A/\text{rad} A$ hence, $kG$ must be of finite representation type. It follows then that either the characteristic of $k$ does not divide the order of $G$, or the characteristic $p$ of $k$ divides the order of $G$ and the $p$-Sylow subgroup $H$ of $G$ is cyclic. In the latter case, we see by [4] that each $AG$-module is $(G,H)$-projective. That is, for each $AG$-module $M$, there is an $AH$-module $N$ such that $M$ is a direct summand of the induced module

$$N^G = AG \otimes_{AH} N.$$  

Now, if $AH$ is of infinite representation type, we see from [10] that for each $n \geq 1$ we can find an indecomposable $AH$-module $N$ whose length as an $A$-module is greater than $n$. But $N$ is an $AH$-summand in $N^G$ by [2, 63.6], so it follows by the Krull-Schmidt Theorem that some indecomposable $AG$-summand of $N^G$ has $A$-length greater than $n$, whence $AG$ is of infinite representation type. Thus, if $AG$ if of finite type, so is $AH$. As in [5, Theorem 1.7], it is easy to see that $AH$ is a commutative, local, Artinian ring with residue field $k$. Thus it suffices to show that if $A$ is not a field and $H \neq \{1\}$, then $AH$ is not serial. For this, we note that if $I$ denotes $(\text{rad} A) \cdot AH$ and $A$ is the kernel of the augmentation map $\varepsilon: AH \to A$, then neither of these two ideals is contained in the other. For, if $x$ is a non-zero element of $\text{rad} A$, then $x \in I$; but $x \notin A$, while if $h \in H, h \neq 1$, then $h - 1$ is in $A$, but not in $I$.

Now let us suppose that a) and b) are satisfied. If the characteristic
of \( k \) does not divide the order of \( G \), then every \( AG \)-module is \((G,1)\)-projective, by [4]. Since \( A \) is uniserial, it has only a finite number of indecomposable modules \( M_1, \ldots, M_n \), and all indecomposable \( AG \)-modules are among the direct summands of \( M_1^G, \ldots, M_n^G \). By the Krull-Schmidt Theorem, \( AG \) is of finite representation type. If the characteristic \( p \) of \( k \) divides the order of \( G \), \( A = k \), and the \( p \)-Sylow subgroup of \( G \) is cyclic, then \( AG \) is of finite type, by Higman's Theorem. This completes the proof.

Corollary. Let \( A \) be a commutative, Artinian ring and let \( G \) be a finite group. Write \( A = A_1 \times \ldots \times A_n \), where each \( A_i \) is local. Then \( AG \) is of finite representation type if and only if

a) Each \( A_i \) is serial and

b) if the characteristic of \( A_i/\text{rad}A_i \) is a prime divisor \( p \) of the order of \( G \), then \( A_i \) is a field and the \( p \)-Sylow subgroup of \( G \) is cyclic.

REFERENCES


DEPARTMENT OF MATHEMATICS,
INDIANA UNIVERSITY,
BLOOMINGTON INDIANA, U.S.A.