ON EXPONENTIAL RECURRING SEQUENCES

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1.
A (polynomial) recurring sequence \( \{z_n\} \) is an integral sequence satisfying
\[
z_n = P(z_{n-1}, \ldots, z_{n-r})
\]
for all \( n \geq r \), where \( P \) is a polynomial in \( r \) variables with integral coefficients. Every such sequence is periodic from some point on modulo any integer \( m \). In this paper we look at the more general situation where \( P \) is a function containing iterated exponentials as well, and we prove that the sequences are still periodic modulo any \( m \).

2.
To make things more precise, we introduce some notations. Let \( \mathbb{N} = \{1, 2, \ldots\} \) be the set of natural numbers and \( \mathbb{N}_1 = \{2, 3, \ldots\} \). We define a set \( \mathcal{F} \) of functions recursively as follows: \( \mathcal{F} \) contains the following elementary functions:

\[
\begin{align*}
E1. & \quad f(x_1, \ldots, x_n) = a, \quad a \in \mathbb{N}; \\
E2. & \quad f(x_1, \ldots, x_n) = x_i, \quad i = 1, 2, \ldots, n; \\
E2*. & \quad f(x) = a^x, \quad a \in \mathbb{N}_1.
\end{align*}
\]

The set \( \mathcal{F} \) is formed by the following composition rules:

\[
\begin{align*}
C1. & \quad \text{If } f, g \in \mathcal{F}, \text{ then } f + g, fg \in \mathcal{F}; \\
C2. & \quad \text{If } f \in \mathcal{F}, \text{ then } x_i^f \in \mathcal{F}; \\
C2*. & \quad \text{If } a \in \mathbb{N}_1 \text{ and } g \in \mathcal{F}, \text{ then } a^g \in \mathcal{F}; \\
C3. & \quad \text{If } f(x_1, \ldots, x_n) \in \mathcal{F}, \text{ then } f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \in \mathcal{F} \quad \text{for } i = 1, 2, \ldots, n.
\end{align*}
\]

We see that every \( f \in \mathcal{F} \) may be expressed in the form

\[
f = \sum_k a_k \left( \prod_l (q_{kl})^{f_{kl}} \prod_n x_i^{g_{kl}} \right)
\]

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where the $q_{kl}$'s are primes (not necessarily distinct), $a_k \in \mathbb{N}$, $f_{kl} \in \mathscr{F}$, $g_{kl} \in \mathscr{G}$, and the $f_{kl}$'s consist of a single term which is product of non-constant functions. Further, this representation is unique.

The subset of $\mathscr{F}$ formed by choosing $E1$ and $E2$ as elementary functions and $C1$ and $C3$ as composition rules, is the set of all polynomials with positive integral coefficients. Let $\mathcal{B}$ be the subset of $\mathscr{F}$ formed by $E1$, $E2^*$, $C1$, $C2^*$ and $C3$. For $f \in \mathcal{B}$ we have $g_{kl} \equiv 0$ in (2.1), and $f_{kl}(x)$ is either $x_i$ for some $i$ or is a product of functions from $\mathcal{B}$.

An exponential recurring sequence $\{z_n\}$ is a sequence satisfying

$$(2.2) \quad z_n = F(z_{n-1}, \ldots, z_{n-r}) \quad \text{for } n \geq r,$$

where $F \in \mathcal{F}$. If $F \in \mathcal{B}$, then we call the sequence a pure exponential recurring sequence.

We prove the following theorems.

**Theorem 1.** Every exponential recurring sequence is periodic modulo any integer $m$.

**Theorem 2.** Every pure exponential recurring sequence has period 1 modulo any integer $m$.

3.

Before we go on to the proof of the theorems we define some further concepts.

Let $\varphi$ be Euler's function. We define $\varphi_k$ for $k \geq 0$ and $\Phi$ by

$$\begin{align*}
\varphi_0(m) &= m \quad \text{for } m \in \mathbb{N}, \\
\varphi_k(m) &= \varphi(\varphi_{k-1}(m)) \quad \text{for } k \geq 1, m \in \mathbb{N}, \\
\Phi(m) &= \text{lcm}_{k \geq 0}\{\varphi_k(m)\} \quad \text{for } m \in \mathbb{N},
\end{align*}$$

where lcm denotes least common multiple. We note that if $p^a | \Phi(m)$, then $p^a | \varphi_k(m)$ for some $k$. Hence

$$\varphi(p^a) | \varphi(\varphi_k(m)) = \varphi_{k+1}(m) | \Phi(m).$$

For any $F \in \mathcal{F}$ we define $\mathcal{D}(F)$ as follows:

I. $F \in \mathcal{D}(F)$.

II. If $f \in \mathcal{D}(F)$ and we express $f$ in the form (2.1), then

$$f_{kl}, (g_{kl})^{f_{kl}}, g_{kl}, x_\lambda^{g_{kl}} \in \mathcal{D}(F)$$

for all $k$, $l$, $\lambda$. 

III. If \( F = F(x_1, \ldots, x_r) \), then the elementary functions defined by E2 (the projections) belong to \( \mathcal{D}(F) \) for \( i = 1, 2, \ldots, r \).

For any \( F \in \mathcal{F} \) we define \( h(F) \), the height of \( F \), as follows:

\[
\begin{align*}
    h(a) &= h(x_i) = 0, \quad a \in \mathbb{N}; \\
    h(a') &= h(x_i') = h(f) + 1 \quad \text{for } f \in \mathcal{F} \text{ nonconstant}; \\
    h(f+g) &= h(fg) = \max \{ h(f), h(g) \}.
\end{align*}
\]

An example may clarify these concepts. If

\[
F(x,y,z,u) = 6^{2v+3^{3v}} + z^v = 2^v 2^{2^{3^v}} 3^v 3^{3^v} + z^v
\]

then \( \mathcal{D}(F) \) consists of

\[
F, x, y, z, u, 2^v, 2^{3^v}, 3^v, 3^{3^v}, 3^{3^v}, yz, z^v,
\]
of heights 2, 0, 0, 0, 0, 1, 2, 1, 2, 1, 0, and 1 respectively.

Let \( F = F(x_1, \ldots, x_r) = F(x) \in \mathcal{F} \). Let

\[
\Phi(m) = \prod_i p_i^{e_i}
\]

be the product of \( \Phi(m) \) as primepowers and put \( v = v(m) = \max_i \{ x_i \} \). In the set \( \mathbb{N}^r \) of \( r \)-dimensional vectors with elements from \( \mathbb{N} \) we define a relation \( \sim_F \), depending on \( F \) and \( m \). It is easily seen to be an equivalence relation. We define

\[
u \sim_F v
\]

if and only if

I. \( f(u) \equiv f(v) \pmod{\Phi(m)} \) for all \( f \in \mathcal{D}(F) \).

II. If \( f(u) \neq f(v) \) for some \( f \in \mathcal{D}(F) \), then \( f(u) > v \) and \( f(v) > v \) for this \( f \).

4.

To prove theorem 1 we first prove two lemmas.

**Lemma 1.** For each \( F \in \mathcal{F} \) the equivalence relation \( \sim_F \) divides \( \mathbb{N}^r \) into a finite number of equivalence classes.

**Proof.** If \( d \) is the number of different functions in \( \mathcal{D}(F) \), then clause I divides \( \mathbb{N}^r \) into at most \( \Phi(m)^d \) classes and clause II divides each of these into at most \( (v+1)^d \) classes. Hence there are at most \( \{(v+1)^d(m)\}^d \) equivalence classes.

**Lemma 2.** If \( (u_1, \ldots, u_r) \sim_F (v_1, \ldots, v_r) \), then

\[
(F(u_1, \ldots, u_r), u_1, \ldots, u_{r-1}) \sim_F (F(v_1, \ldots, v_r), v_1, \ldots, v_{r-1})
\]
PROOF. To simplify notations, we denote the vectors appearing in lemma 2 by $u, v, u', v'$ respectively, so that $u_1' = F(u)$ and $u_i' = u_{i-1}$ for $i > 1$ and similarly for $v'$. We must show that the clauses I and II are satisfied by $u'$ and $v'$.

**Clause I.** We prove this by induction on $h(f)$. Assume $h(f) = 0$. Then $f(x)$ is a polynomial in $x_1, \ldots, x_r$. Since $u \sim_F v$ we have, by clause I, that

$$u_i' = u_{i-1} \equiv v_{i-1} = v_i' \pmod{\Phi(m)}, \quad i = 2, \ldots, r,$$

$$u_1' = F(u) \equiv F(v) = v_1' \pmod{\Phi(m)}.$$

Hence

$$f(u') \equiv f(v') \pmod{\Phi(m)}.$$

Now let $h(f) = h > 0$. We divide the induction step into three cases.

**Case (A),** $f = a^g$ where $a \in N_1$ and $h(g) = h(f) - 1$. Let $p_i | \Phi(m)$.

Subcase (i), $p_i | a$. By the induction hypothesis

$$g(u') \equiv g(v') \pmod{\Phi(m)}.$$

In particular

$$g(u') \equiv g(v') \pmod{\phi(p_i^{\alpha_i})}.$$

Hence, by Euler’s theorem

$$f(u') = a^{\phi(u') \equiv a^{\phi(v')} = f(v') \pmod{p_i^{\alpha_i}}}.$$

Subcase (ii), $p_i | a$. If $g(u') + g(v')$, then, by clause II,

$$g(u') > v \geq \alpha_i \quad \text{and} \quad g(v') > v \geq \alpha_i.$$

Hence

$$a^{\phi(u')} \equiv a^{\phi(v')} \equiv 0 \pmod{p_i^{\alpha_i}}.$$

**Case (B),** $f(x) = x_j^{\phi(x)}$ where $h(g) = h(f) - 1$. If $p_i | u_j'$, then, since

(4.1) \hspace{1cm} u_j' \equiv v_j' \pmod{\Phi(m)}

we proceed as in case (A), subcase (i). If $p_i | u_j'$, let $p_i^\beta | u_j'$. If $\beta < \alpha_i$, then $p_i^\beta | v_j'$ by (4.1) and we may go on as in case (A), subcase (ii). If $\beta \geq \alpha_i$, then $p_i^{\alpha_i} | v_j'$ by (4.1) and hence

$$(u_j')^{\phi(u')} \equiv (v_j')^{\phi(v')} \equiv 0 \pmod{p_i^{\alpha_i}}.$$

**Case (C),** $f$ is any function of height $h$. Then $f$ is a sum of products of functions of the form considered in the cases (A) and (B). (Cf. (2.1).)

Hence

$$f(u') \equiv f(v') \pmod{p_i^{\alpha_i}}.$$
Since this congruence is true for all $p_{i}^{\varphi}|\Phi(m)$, it must hold modulo $\Phi(m)$ as well.

Clause II. This is also proved by induction on $h(f)$. $h(f)=0$. Then $f(x)$ is a polynomial. Suppose $f(u')+f(v')$. Then $u_i'+v_i'$ for at least one $i$, such that $x_i$ appears in the polynomial $f(x)$. Then, by clause I, $u_i'>v$ and $v_i'>v$. Hence $f(u') \geq u_i'>v$ and $f(v') \geq v_i'>v$.

$h(f)=h>0$. Case (A), $f=a^p$ where $a \in N_1$ and $h(g)=h(f)-1$. If $f(u')+f(v')$, then $g(u')+g(v')$. By the induction hypothesis $g(u')>v$ and $g(v')>v$. Hence

$$f(u') = a^{\varphi} > a^r > v$$

and $f(v')>v$ similarly.

Case (B), $f(x)=x_j\varphi(x)$. If $u_i'=1$, then $u_i' \leq v$. Hence, by clause II, $u_i'=v_i'=1$ and so $f(u')=f(v')$. If $u_i'>1$ and $f(u')+f(v')$, then either $u_i'+v_i'$ or $g(u')+g(v')$. Hence either $u_i'>v$ and $v_i'>v$ or $g(u')>v$ and $g(v')>v$. In either case $f(u')>v$ and $f(v')>v$.

Case (C), $f$ is any function of height $h$. Then $f$ is sum of products of functions $f_i$ of the form considered in cases (A) and (B). If $f(u')+f(v')$, then $f_i(u')+f_i(v')$ for at least one $i$. Hence

$$f(u') \geq f_i(u') > v$$

and $f(v')>v$ similarly. This completes the proof of lemma 2.

Put $z_n=(z_{n-1}, \ldots, z_{n-r})$. By lemma 1 there exist $n_1$ and $n_2$ such that $n_1 < n_2$ and $z_{n_2} \sim_F z_{n_1}$. By lemma 2 and (2.2) we have $z_{n_2+k} \sim_F z_{n_1+k}$, and applying lemma 2 repeatedly we obtain $z_{n_2+k} \sim_F z_{n_1+k}$ for all $k \geq 0$.

In particular (putting $\mu = n_2 - n_1$) we get

$$z_{n+\mu} \equiv z_n \pmod m$$

for all $n \geq n_1 - r$. This is theorem 1.

5.

To prove theorem 2 we need two more lemmas.

**Lemma 3.** If $F \in \mathfrak{F}$ is nonconstant and $\{z_n\}$ satisfies (2.2) then $f(z_n) \to \infty$ when $n \to \infty$ for all nonconstant $f \in \mathfrak{D}(F)$.

**Proof.** The proof is by induction on $h(f)$. First we prove that $z_n \to \infty$ when $n \to \infty$.

By (2.1)

$$F(x) \geq (g_{11})^{f_{11}(x)} > f_{11}(x)$$
where \( h(f_{11}) < h(F) \). Applying the same procedure to \( f_{11} \) we find a \( f'_{11} \) such that \( f_{11}(x) > f'_{11}(x) \) and \( h(f'_{11}) < h(f_{11}) \). Applying the procedure repeatedly a finite number of times we arrive at a function of height 0, i.e.

\[
F(x) > x_i
\]

for some fixed \( i \). By (2.2) we have

\[
z_n > z_{n-i} \quad \text{for all } n \geq r.
\]

Hence \( z_{n+kt} \geq z_n + k \), that is \( z_n \to \infty \) when \( n \to \infty \). If \( h(f) = 0 \) and \( f \) is non-constant then \( f(x) \geq x_i \) for some \( i \). Hence \( f(z_n) \geq z_{n-i} \to \infty \) when \( n \to \infty \).

If \( h(f) = h > 0 \), then

\[
f(x) \geq (q_{11})^{f_{11}(x)} > f_{11}(x)
\]

where \( h(f_{11}) < h(f) \). By the induction hypothesis \( f_{11}(z_n) \to \infty \), hence \( f(z_n) \to \infty \).

**Lemma 4.** For all primepowers \( p^\alpha \) and all \( f \in \mathcal{D}(F) \) we have

\[
(5.1) \quad f(z_{n+1}) \equiv f(z_n) \pmod{\Phi(p^\alpha)} \quad \text{for } n \geq 0.
\]

**Proof.** We prove lemma 4 by induction. Since \( \Phi(1) = 1 \), (5.1) is true when \( \alpha = 0 \). Our induction hypothesis is that (5.1) is true for all powers of all primes less than \( p \) and also for \( p^\beta \) when \( \beta < \alpha \). We prove that it is true for \( p^\alpha \). If \( f \) is nonconstant, we have

\[
f(z_n) = \sum_k a_k \prod_l (q_{kl})^{f_{kl}(z_n)}.
\]

Fix \( k \) (we look at one term at a time). If \( p = q_{kl} \) for some \( l \), then

\[
(q_{kl})^{f_{kl}(z_n)} \equiv 0 \pmod{p^\alpha} \quad \text{for } n \geq 0
\]

by lemma 3. If \( p \neq q_{kl} \), then

\[
(q_{kl})^{f_{kl}(z_{n+1})} \equiv (q_{kl})^{f_{kl}(z_n)} \pmod{p^\alpha}
\]

by Euler’s theorem since

\[
f_{kl}(z_{n+1}) \equiv f_{kl}(z_n) \pmod{\varphi(p^\alpha)} \quad \text{for } n \geq 0
\]

by the induction hypothesis. Hence

\[
f(z_{n+1}) \equiv f(z_n) \pmod{p^\alpha} \quad \text{for } n \geq 0.
\]

Further

\[
\Phi(p^\alpha) = p^\alpha \prod_{q_j < p} q_j^{\beta_j}.
\]

By the induction hypothesis

\[
f(z_{n+1}) \equiv f(z_n) \pmod{\prod q_j^{\beta_j}} \quad \text{for } n \geq 0.
\]
Hence
\[ f(z_{n+1}) \equiv f(z_n) \pmod{\Phi(p^n)} \quad \text{for } n \gg 0. \]

This completes the proof of lemma 4.

To prove theorem 2, fix \( m = \prod p_i^{r_i} \). By lemma 4 we have
\[ z_{n+1} \equiv z_n \pmod{p_i^{r_i}} \quad \text{for } n \gg 0. \]

Hence
\[ z_{n+1} \equiv z_n \pmod{m} \quad \text{for } n \gg 0. \]