A NOTE ON LIFTING
OF MATRIX UNITS IN C*-ALGEBRAS

KJELD B. LAURSEN

1. Introduction.

A C*-algebra $A$ is said to be sequentially monotone closed if every norm-bounded increasing sequence of self-adjoint elements has a least upper bound. In this note we show that if $I$ is a closed, two-sided cofinite ideal in a sequentially monotone closed C*-algebra $A$ then a set of matrix units of $A/I$ lifts to a set of matrix units of $A$. As an application of this result we improve on a result concerning continuity of separable linear maps from [2] (for definitions see below).

The author wishes to thank J. Vesterstrøm for several fruitful conversations and particularly for his proof of Lemma 1 below that constitutes a marked improvement in terms of elegance on the author's own efforts.

2. Lifting projections.

We begin by proving a result on lifting one projection from a quotient algebra. A similar result may be found in [3, Theorem 3.2].

Lemma 1. Let $A$ be a sequentially monotone closed C*-algebra and $I$ a closed two-sided ideal in $A$. Then any projection in $A/I$ is the image of a projection in $A$.

Proof. Suppose $x \in A$ is an element for which $x^2 - x \in I$; we may assume $0 \leq x \leq 1$. Consider the following real-valued continuous functions on $[0, 1]$.

$$g(t) = \begin{cases} 
2t & 0 \leq t \leq \frac{1}{2}, \\
1 & \frac{1}{2} \leq t \leq 1,
\end{cases}$$

$$f_n(t) = \begin{cases} 
0 & 0 \leq t \leq \frac{1}{2}, \\
n(t - \frac{1}{2}) & \frac{1}{2} \leq t \leq \frac{1}{2} + n^{-1}, \\
1 & \frac{1}{2} + n^{-1} \leq t \leq 1,
\end{cases}$$

$$h(t) = \begin{cases} 
0 & 0 \leq t \leq \frac{1}{2}, \\
2(t - \frac{1}{2}) & \frac{1}{2} \leq t \leq 1.
\end{cases}$$

Received January 29, 1973.
We note that \( g \geq f_n \geq h \) for all \( n \) so that \( g(x) \geq f_n(x) \geq h(x) \) in \( A \). Let \( p = \sup f_n(x) \); it is easy to see that \( p \) is a projection and that \( g(x) \geq p \geq h(x) \). If \( \pi \) denotes the canonical mapping \( \pi : A \to A/I \) then

\[
\pi(g(x)) \geq \pi(p) \geq \pi(h(x)).
\]

But since \( g(0) = h(0) = 0 \) and \( g(1) = h(1) = 1 \) we get that \( \pi(g(x)) = \pi(h(x)) = \pi(x) \) and consequently \( \pi(x) = \pi(p) \).

Once we know that individual projections can be lifted it is easy to extend this fact to countable families.

**Lemma 2.** If \( A \) is a sequentially monotone closed \( C^* \)-algebra and \( I \) a closed two-sided ideal in \( A \), if \( \{E_j\} \) is a countable set of orthogonal projections in \( A/I \) then there exists a set \( \{e_j\} \) of orthogonal projections in \( A \) such that \( \pi(e_j) = E_j, \ j = 1, 2, \ldots \).

**Proof.** Once we have Lemma 1 we can quote the proof of Proposition 5.4 [2] to which we refer.

3. **Lifting the off-diagonal elements.**

To handle the lifting of the matrix units sitting off the diagonal we impose the further restriction on \( I \) that \( A/I \) be finite dimensional. To simplify notation we shall assume that \( A/I \) is a full matrix algebra. The general result will follow by a routine direct sum argument. Specifically we prove the following

**Theorem 3.** If \( A \) is a sequentially monotone closed \( C^* \)-algebra and \( I \) a cofinite closed two-sided ideal then the matrix units of \( A/I \) lift to a set of matrix units of \( A \).

**Proof.** To be precise the claim of the theorem is the following: With \( \pi \) denoting as usual the canonical mapping, if \( \{E_{ij}\} \) is a set of elements in \( A/I \) such that

\[
\sum_i E_{ii} = 1, \quad E_{ij} = E_{ji}^*,
\]

\[
E_{ij}E_{kl} = \delta_{jk}E_{il}, \quad i, j, k, l = 1, \ldots, n,
\]

then we can find \( \{e_{ij}\} \) in \( A \) such that

\[
\pi(e_{ij}) = E_{ij},
\]

\[
1 - \sum e_{ii} \in I, \quad e_{ij} = e_{ji}^*
\]

\[
e_{ij}e_{kl} = \delta_{jk}e_{il} \quad i, j, k, l = 1, \ldots, n.
\]
Suppose first (Lemma 2) we have lifted \( \{E_{ij}\} \) to \( \{e_{ij}\} \); to begin the construction of \( \{e_{ij}\} \) when \( i \neq j \) consider first \( E_{21} \) and let \( p_{21} \in E_{21} \). Using polar decomposition (see the proof of Proposition 2.3 in [1]) on \( p_{21} \) and \( |p_{21}| = (p_{21}^* p_{21})^{1/2} \) we get \( c_{21} \in A \) such that

\[
p_{21} = c_{21}|p_{21}|, \quad \text{and} \quad |p_{21}| = c_{21}^* p_{21}.
\]

Replacing \( p_{21} \) by \( e_{22} p_{21} e_{11} \) we see that we may assume \( p_{21} = e_{22} p_{21} e_{11} \in e_{22} A e_{11} \). Consequently, \( p_{21}^* p_{21} \in e_{11} A e_{11} \), in fact \( p_{21}^* p_{21} \in E_{11} \), and also \( |p_{21}| \in e_{11} A e_{11} \). From this it follows that

\[
p_{21} = e_{22} p_{21} e_{11} = e_{22} c_{21}|p_{21}| e_{11} = e_{22} c_{21} |p_{21}| e_{11} = (e_{22} c_{21} e_{11}) |p_{21}| e_{11} = (e_{22} c_{21} e_{11}) |p_{21}|.
\]

This means we can assume that

\[
c_{21} = e_{22} c_{21} e_{11},
\]

in fact \( c_{21} \in \lambda E_{21} \) for some \( \lambda \) so that

\[
E_{21} = \lambda E_{21} E_{11} = \lambda E_{21},
\]

i.e. \( \lambda = 1 \), and \( c_{21} \in E_{21} \).

Next we observe that if we use \([\ ]\) to denote range projections then

\[
|p_{21}| = c_{21}^* c_{21} |p_{21}|
\]

implies

\[
[|p_{21}|] = c_{21}^* c_{21} [|p_{21}|],
\]

and consequently, defining

\[
d_{21} = c_{21} [|p_{21}|],
\]

we get

\[
d_{21}^* d_{21} = [|p_{21}|] c_{21}^* c_{21} [|p_{21}|] = [|p_{21}|].
\]

This means that \( d_{21} \) is a "partial isometry". We must next check that \( d_{21} \) has the right properties,

i) \( d_{21} \in E_{21} \) and

ii) \( [|p_{21}|] \in E_{11} \)

Addressing ourselves first to ii) we see that this follows since \( |p_{21}| \in e_{11} A e_{11} \) which is a sequentially monotone closed \( C^* \)-algebra in which the range projection \([|p_{21}|]\) may be considered computed.

Concerning i):

\[
d_{21} = c_{21} [|p_{21}|] \in e_{22} A e_{11}
\]
implies
\[ \pi(d_{21}) \in E_{22}(A/I)E_{11} = \{ \lambda E_{21} \}, \]
so
\[ \pi(d_{21}) = \pi(c_{21}) \pi([p_{21}]) \]
implies
\[ \lambda E_{21} = E_{21} E_{11} = E_{21} \]
and hence \( \lambda = 1 \) which shows that \( d_{21} \in E_{21} \).

The reason it suffices to know that \([p_{21}] \in E_{11}\) is that since \([p_{21}] \leq e_{11}\) we may replace \( e_{11} \) by \([p_{21}]\).

We now repeat the above procedure for \( p_{31} \), replacing, if necessary, \([p_{21}]\) by \([p_{31}]\). After \( n \) steps we have
\[ [p_{21}] \geq [p_{31}] \geq \ldots \geq [p_{n1}] = e_{11} \]
where the last equation is to be taken as a final redefinition of \( e_{11} \). We then define
\[ e_{i1} = d_{i1}[p_{n1}], \quad i = 2, \ldots, n \]
and note the following:

a) \( e_{i1}^* e_{i1} = [p_{n1}] d_{i1}^* d_{i1}[p_{n1}] = [p_{n1}] = e_{11} \).
b) \( e_{i1} = d_{i1}[p_{n1}] = e_{i1}[p_{n1}] \), hence \( \pi(e_{i1}) = E_{i1} E_{11} = E_{i1} \).

Moreover, for the projections \( e_{i1} e_{i1}^* \) we have
\[ \pi(e_{i1} e_{i1}^*) = E_{i1} E_{i1}^* = E_{i1}, \]
so we can redefine
\[ e_{i1} = e_{i1} e_{i1}^* \]
without disturbing the orthogonality properties.

As the last step we define
\[ e_{ij} = e_{i1} e_{j1}^*, \quad i, j = 1, \ldots, n. \]
It is then a simple matter to check that \( e_{ij} \) thus defined is a set of matrix units with all the desired properties.

4. An application.

A separable linear map \( T \) from a Banach algebra \( A \) to a Banach space \( B \) is a linear map for which there exists a map \( f: A \to \mathbb{R}_+ \) such that
\[ \|T(xy)\| \leq f(x)f(y) \]
for all \( x, y \in A \). Such linear maps have been studied in [2]. The next result is an improvement of [2, Theorem 5.6].
THEOREM 4. Let $T$ be a separable linear map defined on a sequentially monotone closed $C^*$-algebra $A$. Then $T$ is continuous on a dense sub-$*$-algebra of $A$.

PROOF. Since the generalized polar decomposition [1, Proposition 2.3] is available in a sequentially monotone closed $C^*$-algebra the results of sections 3–5 of [2] carry directly over to our present setting. Using the notation from [2] and letting $M$ be the closure of $(J_T)_0$ we suppose $\{E_{ij}\}$ is a set of matrix units in $A/M$ with lifting $\{e_{ij}\} \subset A$ (Theorem 3). Define

$$B = \text{span}(e_{ij}) \oplus (J_T)_0$$

and note that $B$ is a dense sub-$*$-algebra of $A$. Since $\text{span}(e_{ij})$ is finite-dimensional the continuity of $T$ on $B$ follows.

REFERENCES


UNIVERSITY OF AARHUS, DENMARK

AND

UNIVERSITY OF COPENHAGEN, DENMARK