HOMOTOPY 4-SPHERES HAVE LITTLE SYMMETRY

PETER ORLIK*

Introduction.

The degree of symmetry, $N(M)$ of a topological (smooth) manifold $M$ is defined as the maximal dimension of a compact Lie group $G$ that can act continuously (smoothly) and effectively on $M$. It is well-known that if $M^n$ is smooth and $N(M^n) = n(n+1)/2$ then $M$ is $S^n$ or $\mathbb{R}P^n$. It was shown in [3] that if $\Sigma^n$, $n \geq 40$ is an exotic sphere then $N(\Sigma) < n^2/8 + 1$. A theorem of Seifert [8] implies that if $\Sigma^3$ is a counterexample to the Poincaré conjecture then $N(\Sigma^3) = 0$. The purpose of this note is to find all simply connected 4-manifolds $M^4$ with $N(M^4) > 1$ and obtain as a corollary that if $\Sigma^4$ is a counterexample to the Poincaré conjecture then $N(\Sigma^4) \leq 1$.

Groups.

Given a compact Lie group $G$ a theorem of Mann [4] computes the smallest dimension $m(G)$ of a manifold admitting an effective $G$ action. Note that for computing $N(G)$ it is sufficient to consider almost effective actions of groups $G = T^r \times G'$ where $G'$ is semi-simple.

**Proposition.** If $G$ acts almost effectively on $M^4$, then the maximal torus of $G$ is at most 2-dimensional.

**Proof.** If the maximal torus $T$ is 3-dimensional then $M^4$ has a 3-dimensional orbit and a theorem of Mostert [5] applies. The orbit space cannot be a circle, so it is a closed interval, and the action is equivalent to a smooth action. The non-principal isotropy groups of the induced $T$ action must be 1-dimensional toruses and together they do not annihilate $\pi_1(T)$.

The following is a list of all compact Lie groups $G = T^r \times G'$ where $G'$ is semi-simple with maximal torus $T^q$ so that $\dim G \leq 10$, $m(G) \leq 4$ and $r + q \leq 2$.

* Partially supported by NSF.

Received May 14, 1973.
<table>
<thead>
<tr>
<th>$G$</th>
<th>Spin 5</th>
<th>SU (3)</th>
<th>Spin 3 $\times$ Spin 3</th>
<th>$S^1 \times$ Spin 3</th>
<th>Spin 3</th>
<th>$T^2$</th>
<th>$S^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim $G$</td>
<td>10</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$m(G)$</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Note that Spin 4 = Spin 3 $\times$ Spin 3.

**Transitive actions.**

These are clearly equivalent to smooth actions and the only possibilities are:

Spin 5/Spin 4 = $S^4$, \quad SU (3)/U (2) = CP$^2$, \quad Spin 4/$T^2$ = $S^2 \times S^2$.

**Actions with 3-dimensional orbits.**

According to Mostert [5] the action is equivalent to a smooth action. The orbit space is an interval with isotropy types $\{ (H); (U_0), (U_1) \}$ and we may assume $H \subset U_i$. Now $U_i/H$ is an $r_i$-sphere so $M$ is homeomorphic to

$$G \times_{U_0} D^{r_0+1} \cup G \times_{U_1} D^{r_1+1}$$

by an equivariant homeomorphism of the common boundary $G/H$. The manifolds thus obtained are classified by the components of the double coset space $N_0 \setminus N(H)/N_1$ where $N_i = N(H) \cap N(U_i)$, see [6].

$G = $ Spin 4 admits the following isotropy structures:

$$\{ (\text{Spin 3}); \text{Spin 4}, \text{Spin 4} \} = S^4$$
$$\{ (\text{Spin 3}); (\text{Spin 3} \times S^1), \text{Spin 4} \} = \text{CP}^2$$
$$\{ (\text{Spin 3}); (\text{Spin 3} \times S^1), (\text{Spin 3} \times S^1) \} = \text{CP}^2 \# \overline{\text{CP}}^2$$

where $\overline{\text{CP}}^2$ is the reverse orientation of $\text{CP}^2$. The manifolds are determined by the isotropy structures.

$G = S^1 \times $ Spin 3 can act as a subgroup of Spin 4.

In addition we have the following possibilities:

$$\{ (S^1); (\text{Spin 3}), (T^2) \} = S^4, \quad \{ (S^1); (T^2), (T^2) \} = S^2 \times S^2,$$

and the manifolds are again determined by the isotropy structure.

$G =$ Spin 3 has finite principal orbit type ($H$). If $H = 1$ then we obtain restrictions of the above actions. If $H = \mathbb{Z}_p$ then we have

$$\{ (\mathbb{Z}_p); (S^1), (S^1) \} = Q_p$$
where $Q_p$ is the double of the $D^2$-bundle over $S^2$ with euler class $p$ and boundary the lens space $L(p,1)$.

Finally, if
$$H = D_8^* = \{ x, y \mid x^2 = (xy)^2 = y^2 \},$$
the binary dihedral group, then we have the following possible isotropy structure $\{(D_8^*); (\text{Pin}2),(\text{Pin}2)\}$. The normalizer of $D_8^*$ in Spin3 is the binary octahedral group $O^*$ and the double coset $N_0 \setminus N(H)/N_1$ has two components. The component of the identity gives a non-simply connected manifold. The other component corresponds to the irreducible 5-dimensional representation of $SO(3)$ given in [2, p. 43] so the total space is $S^4$. I am indebted to G. Bredon for explaining this example.

**Actions with 2-dimensional orbits.**

$G = \text{Spin}3$ must have principal isotropy type $(S^1)$ and principal orbit type $S^2$. The slice is a 2-dimensional cohomology manifold, hence a 2-manifold and may be taken as a disk. The only other orbits are fixed points so the orbit space, $M^*$ is a 2-manifold. Note that $M^*$ is simply connected because $M$ is. If $M^* = S^2$ then all orbits are principal and $M$ is an $S^2$ bundle over $S^2$ with structure group Spin3. Thus the associated principal bundle is classified by

$$S^3 \rightarrow S^7 \rightarrow S^4$$

and hence $M = S^2 \times S^2$. If $M^* = D^2$ then the action is easily seen to admit a cross-section and it is the action of $G$ in the first factor of the join $S^4 = S^2 \circ S^1$.

$G = T^2$ actions on simply connected 4-manifolds were classified in [7]. The only manifolds that occur are equivariant connected sums of $S^4$, $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$, $S^2 \times S^2$.

**Theorem.** The degree of symmetry of a closed simply connected 4-manifold $M$ is given as follows:

<table>
<thead>
<tr>
<th>$M$</th>
<th>$S^4$</th>
<th>$\mathbb{C}P^2$</th>
<th>$S^2 \times S^2$, $\overline{\mathbb{C}P}^2 # \mathbb{C}P^2$</th>
<th>$Q_p$, $p &gt; 1$</th>
<th># of $S^4$, $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$, $S^2 \times S^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(M)$</td>
<td>10</td>
<td>8</td>
<td>6</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

and for all other $M$, $N(M) \leq 1$.

**Corollary.** If $\Sigma^4$ is a counterexample to the Poincaré conjecture then the largest compact Lie group that can act effectively on $\Sigma^4$ is $S^1$. 
Remark. A theorem of Atiyah and Hirzebruch [1] implies that there are smooth simply connected 4-manifolds with no smooth $S^1$-action.

REFERENCES


UNIVERSITY OF WISCONSIN, MADISON