MATH. SCAND. 33 (1973), 261–268

ATTRACTIVE FIXED POINTS AND CONTINUED FRACTIONS

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The iteration of a Möbius transformation with distinct fixed points has a very concise and descriptive representation.

Let

$$t(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0, \tag{1}$$

and

$$T_1(z) = t(z), \quad T_n(z) = T_{n-1}(t(z)), \quad n=2,3,4,\ldots . \tag{2}$$

Then

$$\frac{T_n(z)-u}{T_n(z)-v} = K^n \frac{z-u}{z-v} \tag{3}$$

where u and v are the *fixed points* of t, $u \neq v$. $K=(a-cu)/(a-cv)$ is the *indicator* of t, with u and v chosen so that $|K| \leqq 1$. If $|K|<1$, then u is the *attractive* fixed point of t and v is the *repulsive* fixed point of t. A more complete discussion can be found in [1].

A periodic continued fraction

$$\frac{b}{d}\,\frac{}{+}\,\frac{b}{d}\,\frac{}{+}\cdots \tag{4}$$

may be interpreted in this notation. In (1) let $a=0$, $c=1$. Then, in place of (3) there is the *indicator form* of (4):

$$(T_n(0)-u)/(T_n(0)-v) = (u/v)^{n+1} = K^{n+1}, \tag{5}$$

where $T_n(0)$ is the nth approximant of (4) and tends to u if $|u|<|v|$.

(5) may be written in a form which gives the exact *a priori truncation error* for (4).

$$T_n(0)-u = -uK^n(1-K)/(1-K^{n+1}) . \tag{6}$$

Received April 9, 1973.

This research was supported by United States Air Force Office of Scientific Research grant 70-1922.

If $|K| < \frac{1}{2}$, then a convenient estimate obtained from (6) is

$$|T_n(0) - u| \leqq |u||K|^{n-1} . \tag{7}$$

Much of the remainder of this paper is devoted to using the contractive power of Möbius transformations, as quantified by the indicators, to induce convergence or at least boundedness of compositions of transformations giving rise to continued fractions not necessarily having a periodic or limit-periodic structure. Specifically, a truncation error estimate is obtained, having a form somewhat similar to (7). Information on the limit-periodic case can be found in [2] and [4].

Set

$$t_n(z) = b_n/(z+d_n), \quad n=1,2,3,\ldots ,$$
$$T_1(z) = t_1(z), \quad T_n(z) = T_{n-1}(t_n(z)), \quad n=2,3,\ldots .$$

Let u_n and v_n be the distinct fixed points of t_n, and $K_n = u_n/v_n$ be the indicator of t_n. Assume $|u_n| < |v_n|$. $T_n(0)$ is the nth approximant of

$$\frac{b_1}{d_1} + \frac{b_2}{d_2} + \cdots = \frac{u_1 v_1}{u_1+v_1} - \frac{u_2 v_2}{u_2+v_2} - \cdots . \tag{8}$$

The theorem that follows provides, among other things, a truncation error for (8) in terms of the indicators, $\{K_n\}$.

Set

$$T_n^h(z) = t_h \circ t_{h+1} \circ \cdots \circ t_n(z) \quad n \geqq h ,$$

$$F_h = \left[\frac{|v_h|}{|u_{h+1}-u_h| + |u_{h+1}|} - 1\right]^{-1} \quad h=1,2,\ldots .$$

Define s_h (when it exists) by

$$|u_{h+1}-u_h| = (1+s_h)|u_{h+1}|, \quad 0 < s_h \leqq |u_h/u_{h+1}|, \ |u_h|, \ |u_{h+1}| \neq 0 .$$

If s_h exists, set

$$G_h = \left[\frac{|v_h|}{s_h|u_{h+1}|} + 1\right]^{-1} .$$

THEOREM 1. *If*

(i) $0 < m < |u_j| < M < \infty, \quad j=1,2,\ldots$

and

(ii) $|v_j| > l_j(|u_{j+1}-u_j| + |u_{j+1}|), \quad l_j \geqq 2, \ j=1,2,\ldots ,$

then

(a) $$|T_n^h(0) - u_h| \leqq |u_h| F_h, \quad h=1,2,\ldots, h \leqq n .$$

If, in addition to (i) *and* (ii), *also*

(iii) $\prod_{j=1}^{\infty}(1-l_j^{-1})^{-2}|K_j|=0$

is satisfied, then $\lim_{n\to\infty}T_n(0)$ *exists,*

$$\text{(b)} \qquad |\lim T_n(0)-u_1| \leqq |u_1|F_1$$

and the truncation error can be estimated by

$$\text{(c)} \qquad \begin{aligned} |T_n(0)-\lim T_n(0)| &\leqq 2|u_n|(l_n-1)^{-1}\prod_1^{n-1}[(1-l_j^{-1})^{-2}|K_j|] \\ &\leqq 2M\prod_1^{n-1}[(1-l_j^{-1})^{-2}|K_j|] \\ &\leqq 2M\,4^{n-1}\prod_1^{n-1}|K_j| \,. \end{aligned}$$

Finally, if (i), (ii), (iii) *and*

(iv) s_h *exists for some* h

all are satisfied, then

$$\text{(d)} \qquad |u_h|G_h \leqq |\lim T_n^h(0)-u_h| \leqq |u_h|F_h \,.$$

REMARK. The apparent complexity of (iii) is misleading. This condition is merely a relaxation of $\prod_1^{\infty}(4|K_j|)=0$.

PROOF. Consider the disk

$$\begin{aligned} C(\varepsilon_n,u_n) &= \{z:\ |t_n(z)-u_n|\leqq\varepsilon_n\} \\ &= \{z:\ |z-u_n|\leqq c_n|z-(u_n+v_n)|\}\,, \end{aligned}$$

where $c_n=\varepsilon_n/|u_n|<1$.

$C(\varepsilon_n,u_n)$ is a circle of Apollonius with respect to the points u_n and u_n+v_n (see figure). If $c_n<1$, then one may define

$$\text{(9)} \qquad N(\varepsilon_n,u_n) = t_n(C(\varepsilon_n,u_n)) = \{z:\ |z-u_n|\leqq\varepsilon_n\}\,.$$

Computation provides the following expressions for the center g_n and radius R_n of $C(\varepsilon_n,u_n)$.

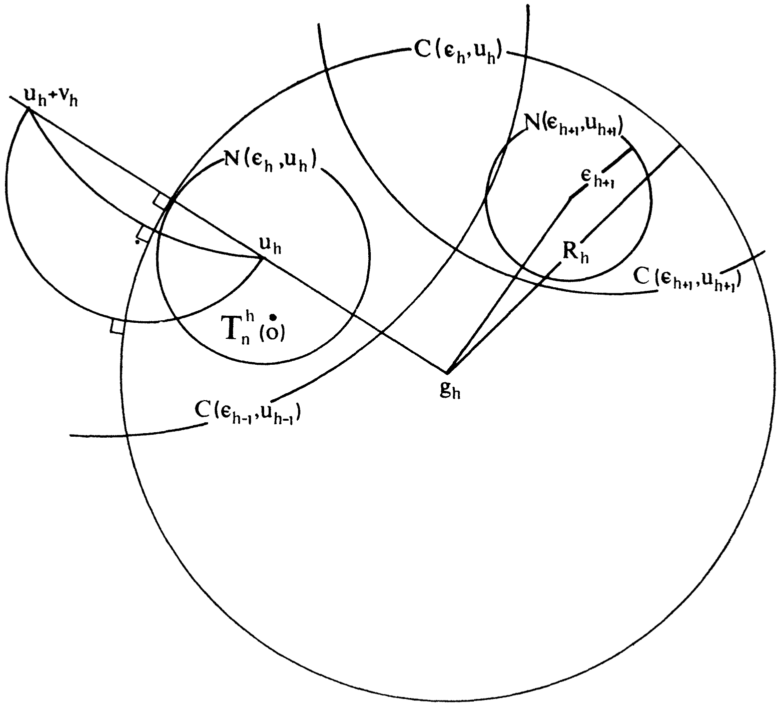
$$\text{(10)} \qquad g_n = u_n-c_n^2v_n(1-c_n^2)^{-1},$$

$$\text{(11)} \qquad R_n = c_n|v_n|(1-c_n^2)^{-1}.$$

The procedure is to determine a sequence $\{\varepsilon_n\}$ sufficient to guarantee the following three conditions

1. $t_n(C(\varepsilon_n,u_n)) = N(\varepsilon_n,u_n)$ $\quad n=1,2,\ldots,$
2. $0\in C(\varepsilon_n,u_n)$ $\quad n=1,2,\ldots,$
3. $N(\varepsilon_n,u_n)\subset C(\varepsilon_{n-1},u_{n-1})$ $\quad n=2,3,\ldots.$

Then $T_n^h(0) \in N(\varepsilon_h, u_h)$ $h = 1, 2, \ldots; n \geqq h$ (see figure).



Condition 1. holds if $0 < c_n < 1$, that is,

$$0 < \varepsilon_n < |u_n| . \tag{12}$$

Condition 2. is equivalent to

$$\varepsilon_n \geqq |u_n|^2/|u_n + v_n| \tag{13}$$

Condition 3. requires a certain overlap of circles (see figure), which is equivalent to the inequality

$$|u_{n+1} - g_n| + \varepsilon_{n+1} \leqq R_n . \tag{14}$$

Using (10) and (11) we find

$$\begin{aligned} |u_{n+1} - g_n| + \varepsilon_{n+1} &\leqq |u_{n+1} - u_n| + \varepsilon_{n+1} + |u_n - g_n| \\ &= R_n + |u_{n+1} - u_n| + \varepsilon_{n+1} - c_n|v_n|(1 + c_n)^{-1} \end{aligned}$$

The right side of the last expression is less than or equal to R_n provided

$$|u_{n+1} - u_n| \leqq c_n|v_n|(1 + c_n)^{-1} - \varepsilon_{n+1} ,$$

or, by (12), if

$$|u_{n+1}-u_n| \leqq c_n|v_n|(1+c_n)^{-1}-|u_{n+1}| ,$$

which in turn is equivalent to

$$|u_n|F_n \leqq \varepsilon_n . \tag{15}$$

If we require

$$|v_n| > l_n(|u_{n+1}-u_n|+|u_{n+1}|), \quad l_n \geqq 2 ,$$

and set $\varepsilon_n = |u_n|F_n$, then the sequence $\{\varepsilon_n\}$ satisfies (12), (13) and (15):

$$0 < |u_n|^2/|u_n+v_n| \leqq |u_n|F_n = \varepsilon_n < |u_n| ,$$

as is easily verified.

We have seen that $T_n{}^h(0) \in N(\varepsilon_h, u_h)$, so (i) and (ii) imply

$$|T_n{}^h(0)-u_h| \leqq |u_h|F_h, \quad h=1,2,\ldots ,$$

which proves the first part of (a).

The convergence of $\{T_n(0)\}$ is shown by the following argument. Consider a disk

$$D = \{z: \ |z-c|<r\} \subset C(\varepsilon_n, u_n) .$$

Then, $D'=t_n(D)=\{z: |z-c'|<r'\}\subset N(\varepsilon_n, u_n)$. The points c and ∞ are symmetric with respect to D. Therefore, $t_n(c)$ and $t_n(\infty)$ are symmetric with respect to D'. This gives

$$r'^2 = |c'||u_nv_n-c'(u_n+v_n-c)||u_n+v_n-c|^{-1} ,$$

and

$$r^2 = |u_n+v_n-c||c'(u_n+v_n-c)-u_nv_n||c'|^{-1} .$$

Thus,

$$r'/r = |c'|/|u_n+v_n-c| .$$

Now,

$$\begin{aligned}|c'| \leqq |u_n|+|c'-u_n| < |u_n|+\varepsilon_n &< |u_n|+|u_n|(l_n-1)^{-1} \\ &= |u_n|(1-l_n{}^{-1})^{-1} ,\end{aligned}$$

together with $c \in C(\varepsilon_n, u_n)$ and $u_n+v_n \notin C(\varepsilon_n, u_n)$ imply

$$\begin{aligned}|u_n+v_n-c| &> |g_n+R_nv_n/|v_n|-(u_n+v_n)| \\ &= |u_nv_n|/(|u_n|+\varepsilon_n) \\ &> |v_n|(1-l_n{}^{-1}) ,\end{aligned}$$

give

$$r'/r < |u_n/v_n|(1-l_n{}^{-1})^{-2} = |K_n|(1-l_n{}^{-1})^{-2} .$$

It follows that

$$\operatorname{diam} T_n(C(\varepsilon_n, u_n)) < 2|u_n|(l_n-1)^{-1}\textstyle\prod_1^{n-1}[|K_j|(1-l_j{}^{-1})^{-2}] ,$$

and $\{T_n(0)\}$ converges to a limit in $N(\varepsilon_1, u_1)$. An a priori truncation error estimate is given by

$$|T_n(0) - \lim T_n(0)| \leqq 2|u_n|(l_n - 1)^{-1} \prod_1^{n-1} [|K_j|(1 - l_j^{-1})^{-2}]$$
$$\leqq 2M \prod_1^{n-1} 4|K_j| \ .$$

This proves (a), (b) and (c).

The relationship between $|K_j|$ and l_j is

$$|K_j| < l_j^{-1} \ ,$$

since $|v_j| > l_j(|u_{j+1} - u_j| + |u_{j+1}|) \geqq l_j|u_j|$.

When $\varepsilon_{h+1} < |u_{h+1} - u_h|$ hypothesis (iv) allows a slightly sharper formulation of the region containing $T_n^h(0)$, for $n \geqq h$. $t_h(N(\varepsilon_{h+1}, u_{h+1}))$ is a disk properly contained in an annular region with center u_h, outer radius $\varepsilon_h = |u_h|F_h$, and inner radius δ_h yet to be determined. Clearly $T_n^h(0)$ lies in this ring.

Assume

$$|u_{h+1} - u_h| = (1 + s_h)|u_{h+1}|, \quad 0 < s_h \leqq |u_h/u_{h+1}| \ .$$

(A more precise criteria could be obtained from

$$|u_{h+1} - u_h| = (1 + s_h)|u_{h+1}|F_{h+1} \ ,$$

but this is difficult to apply in practice.) Let

$$\delta_h = |u_h|(|v_h|/s_h|u_{h+1}| + 1)^{-1} = |u_h|G_h \ ,$$

which is equivalent to

$$|u_h|\delta_h^{-1} = G_h^{-1} \ .$$

If x_h and r_h denote the center and radius, respectively, of $t_h^{-1}(N(\delta_h, u_h))$, then an easy computation using (11) yields

$$s_h|u_{h+1}| - |v_h|(G_h^{-2} - 1)^{-1} = G_h^{-1}|v_h|(G_h^{-2} - 1)^{-1}$$
$$= r_h \ .$$

We also find

$$\begin{aligned} |x_h - u_{h+1}| &\geqq |u_{h+1} - u_h| - |v_h|(G_h^{-2} - 1)^{-1} \\ &= |u_{h+1}| + [s_h|u_{h+1}| - |v_h|(G_h^{-2} - 1)^{-1}] \\ &\geqq |u_{h+1}|F_{h+1} + r_h \\ &= \varepsilon_{h+1} + r_h \ , \end{aligned}$$

so that

$$\operatorname{Int} C(\delta_h, u_h) \cap \operatorname{Int} N(\varepsilon_{h+1}, u_{h+1}) = \emptyset \ ,$$

and this implies

$$|u_h|G_h \leqq |T_n^h(0) - u_h| \leqq |u_h|F_h \ ,$$

which proves (d).

EXAMPLE 1. Let $|u_n| \equiv 10$, $|v_n| \equiv 100$, $s_n \equiv .9$. Then convergence occurs and

$$.825 \leqq |\lim T_n(0) - u_1| \leqq 4.085 .$$

A truncation error estimate for $n=6$ is

$$|T_6(0) - \lim T_n(0)| < 4 \times 10^{-14} .$$

EXAMPLE 2. Consider the periodic continued fraction given by $u_n \equiv 1$, $v_n \equiv 10$. Error estimates are derived from: 1. The Parabola Theorem [5], 2. Theorem 2.3 of [3], 3. Theorem 1 of this paper, and 4. the indicator form [5] giving the exact error.

$$\begin{array}{lcccc} & 1. & 2. & 3. & 4. \\ |T_6(0) - \lim T_n(0)| \leqq & 8.2 \times 10^{-3}, & 9.7 \times 10^{-4}, & 5.7 \times 10^{-5}, & 9 \times 10^{-7} . \end{array}$$

We conclude with an application of the geometrical techniques developed for theorem 1 to a certain type of limit-periodic continued fraction,

$$\left[\frac{a_n(z)}{1}\right]_1^\infty, \quad \text{where } a_n(z) = u_n z(u_n z + 1),\ z \neq 0 . \tag{16}$$

THEOREM 2. *If*

(i) $\{|u_n|\}$ *decreases monotonically to zero,*

(ii) $|u_n z - \frac{1}{15}| \leqq \frac{4}{15}$, *and*

(iii) $|u_{n+1} - u_n| < |u_n|$ *are all satisfied with regard to* (16), *then* (16) *converges, and an error estimate is given by*

$$|T_n(0) - \lim T_n(0)| \leqq 2|z| \cdot |u_n| \prod_1^n |a_j(z)|(|u_j z + 1| - |u_j z| - |u_{j+1} z|)^{-2} . \tag{17}$$

REMARKS. The fixed points of $t_n(w) = a_n(z)/(1+w)$ are $u_n z$ and $-(u_n z + 1)$. Conditions (i), (ii) and (iii) insure that $N(|u_n z|, u_n z) \subset C(|u_{n-1} z|, u_{n-1} z)$, and the proof is similar to that of theorem 1.

The following two examples provide applications of (17) of theorem 2.

EXAMPLE 3.

$$\left[\frac{1/(n+3)}{1}\right]_{n=1}^\infty$$

	actual error	error-estimate
$n = 5$	1×10^{-5}	2.5×10^{-4}
$n = 6$	1×10^{-6}	5.5×10^{-5}

EXAMPLE 4. $zf(z)$, where

$$f(z) = \left[\frac{z^n}{1}\right]_{n=0}^{\infty},$$

the continued fraction of Ramanujan. Let $z = \frac{1}{16}$.

	actual error	error estimate
$n = 2$	1.3×10^{-5}	5.9×10^{-4}
$n = 3$	3.3×10^{-9}	1.5×10^{-7}
$n = 4$	5×10^{-14}	2.3×10^{-12} .

REFERENCES

1. L. R. Ford, *Automorphic functions*, 2nd ed. Chelsea, New York, 1929.
2. J. Gill, *Infinite compositions of Möbius transformations*, Trans. Amer. Math. Soc. 176 (1973), 479–487.
3. W. B. Jones and R. I. Snell, *Truncation error bounds for continued fractions*, SIAM J. Numer. Anal. 6 (1969), 210–221.
4. M. Mandell and A. Magnus, *On convergence of sequences of linear fractional transformations*, Math. Z. 115 (1970), 11–17.
5. H. S. Wall, *Analytic theory of continued fractions*, Van Nostrand, Princeton, N.J., 1948.

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