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ON CYCLIC GROUPS OF MÖBIUS TRANSFORMATIONS

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1. Introduction.

For the study of Kleinian groups it is important to consider their fundamental polyhedrons. If the groups is of the second kind, i.e. discontinuous somewhere in the complex plane, then already a reasonable fundamental polygon provides some insight. A canonical polygon may then be defined as done in the book of L. R. Ford [1], by means of the isometric circles. It is useful to know how this polygon looks for a cyclic group, especially when the generator is loxodromic but close to parabolic.

In this note the latter case is investigated. The natural question is whether the polygon of a cyclic group is connected. The answer is in the affirmative. The polygons are either doubly connected and bounded by two circles or they are simply connected and bounded by two, four or six circular arcs. Hence one may distinguish between the different types of polygons. A refinement is achieved by the description of the isometric polyhedrons.

The proofs are elementary. They are based on a method of geometric continuity which generally is applicable to the deformation theory of finitely generated Kleinian groups. But this topic, being non-trivial, will be dealt with elsewhere.

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2. Preliminaries.

$\mathrm{SL}(2,\mathbf{C})$ is the group of all complex two-by-two matrices with determinant one, the composition being usual matrix multiplication. The neutral element is denoted E.

Let A belong to $\mathrm{SL}(2,\mathbf{C})$ and let τ be the trace of A. For each integer n we have by induction

$$A^n = -\beta_{n-1}E+\beta_n A , \tag{1}$$

where

$$\beta_0 = 0, \quad \beta_1 = 1 ,$$

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and

$$\beta_{n+1} = -\beta_{n-1}+\tau\beta_n . \tag{2}$$

The number $\tau_n = -\beta_{n-1}+\beta_{n+1}$ is the trace of A^n. Evidently β_n and τ_n are polynomials of degrees $|n|-1$ and $|n|$ in the variable τ. Remark that $\beta_{-n} = -\beta_n$ and $\tau_{-n} = \tau_n$. It may be convenient to change the variable: put $\tau = z+z^{-1}$, then $\tau_n = z^n+z^{-n}$ and

$$\beta_n = (z^n-z^{-n})(z-z^{-1})^{-1} .$$

Thus we have the identities

$$\begin{aligned} \tau_m\tau_n &= \tau_{m+n}+\tau_{m-n} , \\ \beta_m\tau_n &= \beta_{m+n}+\beta_{m-n} , \\ \beta_m\beta_n &= (\tau_{m+n}-\tau_{m-n})(\tau^2-4)^{-1} . \end{aligned} \tag{3}$$

Moreover, β_n and τ_n, as functions of τ, satisfy

$$\begin{aligned} \frac{d}{d\tau}\tau_n &= n\beta_n , \\ \frac{d}{d\tau}\beta_n &= (n\tau_n-\tau\beta_n)(\tau^2-4)^{-1} . \end{aligned} \tag{4}$$

$M = \mathrm{SL}(2,\mathsf{C})/\{\pm E\}$ is a representation of the group of all Möbius transformations, i.e. the group of conformal orientation preserving mappings of the Riemann sphere, $\mathsf{C}\cup\{\infty\}$, onto itself. A Kleinian group is a discrete subgroup of M. If $A \in M$ and $z \in \mathsf{C}\cup\{\infty\}$, we have explicitly

$$A = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad A(z) = \frac{az+b}{cz+d}$$

for some complex numbers a,b,c,d with $ad-bc=1$.

The isometric circle for $A \in M$ is defined, provided $A(\infty) \neq \infty$, as

$$I(A) = \{z \in \mathsf{C} \mid |dA(z)| = |dz|\} .$$

The finite closed disc bounded by $I(A)$ is denoted $D(A)$. By A the exterior of $I(A)$ is mapped onto the interior of $I(A^{-1})$. Some incidence relations between the isometric circles, to be used frequently, follow immediately from the chain rule for differentiating composite functions. For example, $D(B)$ covers an arc s of $I(A)$ if and only if $D(BA^{-1})$ covers the arc $A(s)$ of $I(A^{-1})$. Of importance is the case of three different isometric circles $I(A)$, $I(B)$ and $I(C)$ passing through a common point x. Assuming that the points x, $A(x)$, $B(x)$ and $C(x)$ are distinct, it is natural to consider three isometric circles for each point, namely through $A(x)$ the circles

$I(A^{-1})$, $I(BA^{-1})$ and $I(CA^{-1})$, through $B(x)$ the circles $I(B^{-1})$, $I(CB^{-1})$ and $I(AB^{-1})$ and through $C(x)$ the circles $I(C^{-1})$, $I(AC^{-1})$ and $I(BC^{-1})$. Suppose each of these four points has arbitrarily small neighbourhoods which are not covered entirely by the three appropriate discs. Then it follows, by symmetry, that if at x the sides of the union of the three discs lie on $I(A)$ and $I(B)$, then at $A(x)$ the sides lie on $I(A^{-1})$ and $I(CA^{-1})$, at $B(x)$ the sides lie on $I(B^{-1})$ and $I(CB^{-1})$, and at $C(x)$ the sides lie on $I(AC^{-1})$ and $I(BC^{-1})$.

The cyclic group generated by $A \in M$ is discrete, except when A is elliptic of infinite order. In the discrete case $\{A^n\}$ is discontinuous in $\mathbf{C} \cup \{\infty\} \setminus \{\text{fixed points of } A\}$ and it is well-known that if $A(\infty) \neq \infty$, the set of points outside all $D_n (= D(A^n))$ is a fundamental polygon, $P(A)$ (see [1]).

The configuration of isometric circles I_n $(= I(A^n))$ is determined up to similarity by $\tau = \operatorname{trace} A$. It is symmetric with respect to the mid-point of the segment joining $A(\infty)$ and $A^{-1}(\infty)$. For the study of $P(A)$ we may therefore use τ as a parameter and consider only the case where the centre of symmetry is zero, $A(\infty) - A^{-1}(\infty) = \tau$ and $0 \leqq \arg \tau < \pi$. Under these assumptions we have

$$A^n = \begin{pmatrix} \frac{1}{2}\tau_n & \frac{1}{4}(\tau_n{}^2 - 4)\beta_n{}^{-1} \\ \beta_n & \frac{1}{2}\tau_n \end{pmatrix}.$$

Then I_n has centre $C_n = -\frac{1}{2}\tau_n \beta_n{}^{-1}$ and radius $r_n = |\beta_n|^{-1}$, provided it is defined, that is when $\beta_n \neq 0$. Now remark that (3) implies

$$C_m - C_n = \beta_{m-n} \beta_m{}^{-1} \beta_n{}^{-1}$$

from which it follows that D_m and D_n intersect if and only if

$$|\beta_{m-n}| \leqq |\beta_m| + |\beta_n| \tag{5}$$

and that D_m is contained in D_n if and only if

$$|\beta_{m-n}| + |\beta_n| \leqq |\beta_m| \,. \tag{6}$$

3. The polygon.

It is fairly easy to describe the polygon $P = P(A)$ if τ is real or $|\tau| \geqq 2$.

Assume first $\tau \in [0, 2[$. Then A is elliptic and has two fixed points through which all the isometric circles pass. If $\beta_n \neq 0$ for all $n \neq 0$, then $\bigcup I_n$ is a dense subset of the complex plane, the group $\{A^n\}$ is not discrete and there is no polygon. Otherwise there exists a smallest natural number n for which $\beta_n = 0$. In this case A, as a Möbius transformation, generates a group of order n. The boundary of P consists of two circular arcs, the sides of P, and two vertices, namely the fixed points of A. In particular P is connected and simply connected.

Next assume $|\tau| \geqq 2$. Then (2) yields

$$2|\beta_m| \leqq |\beta_{m-1}| + |\beta_{m+1}|$$

showing that $|\beta_m|$ is a convex function of the index. The minimum is $|\beta_0| = 0$. The convexity implies

$$\begin{aligned} |\beta_{m+1}| - |\beta_m| &\geqq |\beta_1| \quad \text{if } m > 0\,, \\ |\beta_{m-1}| - |\beta_m| &\geqq |\beta_{-1}| \quad \text{if } m < 0\,, \end{aligned}$$

which according to (6) shows that

$$\begin{aligned} &D_{m+1} \text{ is contained in } D_m \text{ if } m > 0\,, \\ &D_{m-1} \text{ is contained in } D_m \text{ if } m < 0\,. \end{aligned}$$

Therefore, each of the sequences $I_1, I_2, I_3, \ldots$ and $I_{-1}, I_{-2}, I_{-3}, \ldots$ consists of nested circles. This implies that P is connected and bounded by I_1 and I_{-1}. If $|\tau| > 2$, then D_1 and D_{-1} are disjoint, by (5), since

$$|\beta_2| = |\tau| > 2 = |\beta_1| + |\beta_{-1}|\,,$$

while $|\tau| = 2$ is equivalent to D_1 and D_{-1} being externally tangent. If $\tau = 2$, A is parabolic. In this case the fixed point of A is the common point of D_1 and D_{-1}.

So far we have been dealing with well-known facts. The interesting case is when $|\tau| < 2$ and τ is not real.

In general, P is bounded by six sides. Following the boundary of P around with P to the left, these sides belong to

$$I_m I_{m+n} I_n I_{-m} I_{-m-n} I_{-n}$$

for some pair of natural numbers m and n. Such configurations are said to be of type $\{m, m+n, n\}$. The sides are paired, the one on I_j being mapped onto the opposite one on I_{-j} by A^j. The side-pairing transformations generate the whole group, that is, m and n are relatively prime.

In order to "change sides" by varying τ, one has to pass through a polygon with only four sides. In this case each vertex of P belongs to three isometric circles. The succession is

$$I_m(I_{m+n})I_n(I_{n-m})I_{-m}(I_{-m-n})I_{-n}(I_{m-n})$$

for some natural numbers m and n, relatively prime. Such polygons are said to be of type $\{m,n\}$. The polygons coming from the open half circle $|\tau|=2$ are considered as $\{1,1\}$-configurations.

These assertions are proved by a geometric continuity argument exhausting the deformation possibilities of P under variation of τ.

It is easy to see that P is finite sided. In fact, A has two fixed points and each D_n contains exactly one of these as interior point; but the radii r_n converge to zero as $|n|$ increases. Since the intersection of D_1 and D_{-1} is non-empty, another consequence is that each component of P is simply connected.

The main problem is to show that P is connected. Together with its boundary, P is topologically a torus if points which are equivalent with respect to the group are identified. Therefore, if "new components" of P turned up under deformation, at least one would have a side on its boundary being paired to a side of the component containing infinity. That this cannot happen follows from a general discussion below of the side-change possibilities.

First we remark that P is of type $\{1,2,1\}$ when τ belongs to the open segment joining 0 and $2i$. This is immediately verified.

Starting from a connected polygon, no new side can break out through an old side under continuous deformation. Otherwise there would be a value of τ for which some I_s touches the side on I_n from the inside. If x is the common point, then I_{-n} and I_{s-n} have $A^n(x)$ as a common point. Moreover, by (6) and (5), I_{s-n} must be externally tangent to a side on I_{-n}. It is no restriction to assume that P has a side on I_{s-n} which ends at $A^n(x)$. The symmetry of P implies that a side on I_n and a side on I_{n-s} will be tangent to each other. Therefore, by (5) and (6), I_s must also touch a side on I_{s-n} from the inside. By geometry, this is impossible unless $s-n=n=\pm 1$; but if so (6) yields $|\tau|=2$.

The corners of the isometric polyhedron (see section 5) which do not belong to the boundary of P lie strictly above the plane, because equivalent corners lie in the same height and one corner from each cycle is the common point of isometric hemispheres defined by three positive powers of A! It follows that no new component can arise. Therefore, if the side on I_n gets a new neighbour, breaking out through the vertex y, then an old side disappears through the equivalent vertex $A^n(y)$. We conclude that P must be four-sided in the critical situation. Only one pair of new sides arises and these will be opposite. The new configuration is completely determined by the old configuration when it is known which pair of sides disappears.

The previous discussion shows that P remains connected and bounded by four or six sides as long as τ lies in the domain under consideration.

Hence, by the incidence properties of the isometric circles, the sides of P are forced to succeed each other as described. Of course, this fact is also a consequence of the continuity method. That the side-pairing transformations generate the group is well-known.

4. The polygons.

Classification of the polygons according to their types results in a tesselation of the upper half plane, $\operatorname{Im}\tau > 0$. Define $f_{m,n}$ to be the set of traces τ for which P is of type $\{m,n\}$ and define $T_{m,n}$ as the set of traces τ for which P is of type $\{m,m+n,n\}$. Thus the half plane is the disjoint union of $f_{m,n}$ and $T_{m,n}$, where (m,n) runs over all ordered pairs of coprime natural numbers, and of the set outside the circle $|\tau| = 2$, where the corresponding polygons are doubly connected and bounded by the isometric circles I_1 and I_{-1}.

As one might expect, $f_{m,n}$ is a simple smooth curve with end points on the real axis. These correspond to elliptic transformations of orders m and n, elliptic of order 1 meaning parabolic. The set $T_{m,n}$ is a curvilinear triangle bounded by the curve $f_{m,n}$ from above and the curves $f_{m,m+n}$ and $f_{m+n,n}$ from below. In particular $T_{m,n}$ is connected and simply connected. The "vertices" are the common end points of $f_{m,n}$ and $f_{m,m+n}$, of $f_{m,m+n}$ and $f_{m+n,n}$ and of $f_{m+n,n}$ and $f_{m,n}$. These statements will be proved below. The tesselation induces a partial order $<$ on the set of all ordered pairs of relatively prime natural numbers, alike the one known from Euclid's algorithm and from the study of Farey series, reflecting that elliptic transformations of finite order are exactly those for which $\tau = 2\cos\pi q$, where q is a rational number. Explicitly this ordering is given by $(m,n) < (m',n')$ meaning that $f_{m',n'}$ lies below $f_{m,n}$.

Consider first a polygon of type $\{m,n\}$. Assume the sides of P on I_m and I_n meet at the vertex x through which also I_{m+n} passes. Then we have

$$\begin{aligned} &= -\tfrac{1}{2}\tau_m\beta_m{}^{-1} + \beta_m{}^{-1}\exp\theta_m \\ x \quad &= -\tfrac{1}{2}\tau_{m+n}\beta_{m+n}^{-1} + \beta_{m+n}^{-1}\exp\theta_{m+n} \\ &= -\tfrac{1}{2}\tau_n\beta_n{}^{-1} + \beta_n{}^{-1}\exp\theta_n \end{aligned}$$

for some purely imaginary numbers θ_m, θ_{m+n}, θ_n. Using that $A^{m+n}(x) = -x$, an easy calculation yields $\exp\theta_{m+n} = \pm 1$. Thus, by (3), the equations reduce to

$$\beta_{m+n}\exp\theta = \beta_m \pm \beta_n \tag{7}$$

for an imaginary number θ. Remark that $\theta = \theta_n$. The computation can be reversed. Given τ and θ (imaginary) such that (7) holds, there exists a common point of I_m, I_{m+n} and I_n, provided these circles are defined. This point need not be a vertex of P, but it is mapped onto the opposite point by A^{m+n}.

The aim is to obtain information about $f_{m,n}$ from the equation (7). The roots of β_{m+n}, as a polynomium in τ, are real. For the present purpose, therefore, it is not necessary to pay attention to the common roots of β_{m+n} and $\beta_m \pm \beta_n$ but merely to the non constant solutions of (7), $\tau = \tau(\theta)$. Dividing in (7) by β_{m+n}, we get

$$\exp \theta = (\beta_m \pm \beta_n)\beta_{m+n}^{-1} . \tag{8}$$

Differentiation with respect to τ yields

$$d\theta/d\tau = -(n\beta_m \pm m\beta_n)(\tau_m \pm \tau_n)^{-1} . \tag{9}$$

The undetermined signs in (8) and (9) are simultaneously all plus or all minus. It is a routine matter to verify that the solutions of (8) consist of a finite number of simple closed curves. Each of these has two points of intersection with the real axis, namely for θ being a multiple of $i\pi$, as seen either by a direct computation or by a geometric argument. For fixed (m,n), the solutions which correspond to the same sign do not intersect; those given by opposite signs may intersect. But it is clear that only one of the signs is correct for each given polygon of type $\{m,n\}$. Hence we conclude that $f_{m,n}$ is the union of a finite number of simple smooth curves, each of these connecting two points on the real axis.

It follows, using (9), that $f_{m,n}$ can only approach the real axis orthogonally since $d\theta$ is imaginary and $d\tau/d\theta$ becomes real and not zero. If this occurs, i.e. if $f_{m,n}$ is non-empty, the limit point corresponds to a transformation of order m or n. It is clear that the order d must be finite and divide m or n; but d cannot be a proper divisor since the radius of I_d then would become too large compared to r_m or r_n.

Each component of $f_{m,n}$ has one end for which the order is m and one for which the order is n. This fact is not true in general for the solutions of (8), but it follows by geometry for those belonging to $f_{m,n}$. Consider again a polygon of type $\{m,n\}$ and let x denote the vertex which is a common point of I_m, I_{m+n} and I_n. Then τ belongs to some curve in $f_{m,n}$. If τ approaches one end of this curve, then x approaches the midpoint of the side on I_n, i.e. that side becomes arbitrarily small. At the other end x approaches the diametrically opposite point. The first case yields ellipticity of order m; in the second case we have ellipticity of order n.

Next we apply induction, taking advantage of $T_{m,n}$ being separated from the real axis except, possibly, at points corresponding to elliptic transformations of orders m, $m+n$ or n.

The case $m=n=1$ needs a special treatment because the polygons of type $\{1,2,1\}$ have two different sides on the same isometric circle. It is known that $f_{1,1}$ is the half circle $|\tau|=2$. Trivially the open segment $]-2,2[$ contains only one point corresponding to the order 2, namely zero. Hence $T_{1,1}$ is bounded from below by $f_{1,2}$ and $f_{2,1}$. In fact, $f_{1,2}$ connects -2 with 0 and $f_{2,1}$ connects 0 with 2.

Suppose $mn>1$ and let $S_{m,n}$ be a component of $T_{m,n}$, bounded from above by a curve $g_{m,n}$ which belongs to $f_{m,n}$. Thus $S_{m,n}$ has at least two boundary points on the real axis, namely the end points of $g_{m,n}$. If these are the only such points, $S_{m,n}$ must be bounded from below by another component $g'_{m,n}$ of $f_{m,n}$. But this is inconsistent with our knowledge about the solutions of (8). Both curves, $g_{m,n}$ and $g'_{m,n}$, satisfy the same equation; that the sign is the same follows by continuity, using the connectedness of $S_{m,n}$, once we recall that the sign depends on whether $C_{m+n}+\beta_{m+n}^{-1}$ or $C_{m+n}-\beta_{m+n}^{-1}$ is the mid-point of the side on I_{m+n}. Also, $S_{m,n}$ cannot have more than three boundary points on the real axis since no boundary curve connects two points corresponding to the same order, only three possible orders are in play and, by continuity, the boundary of $S_{m,n}$ does not contain points of some $f_{p,q}$ together with points of $f_{q,p}$, the corresponding types being oppositely oriented. Hence $S_{m,n}$ is bounded from below by curves $g_{m,m+n}$ and $g_{m+n,n}$ belonging to $f_{m,m+n}$ and $f_{m+n,n}$, respectively. It follows by the same arguments that $g_{m,m+n}$ bounds a component $S_{m,m+n}$ of $T_{m,m+n}$ from above and $g_{m+n,n}$ bounds a component $S_{m+n,n}$ of $T_{m+n,n}$ from above. In other words, the four-sided polygons separate six-sided polygons of different types.

In order to complete the discussion, remark that each component of $T_{m,n}$ is simply connected. Hence a covering of the upper half disc is obtained by repeated application of the above procedure, first defining $S_{1,1}=T_{1,1}$ then deducing the existence of $S_{1,2}$ and $S_{2,1}$ as neighbours of $S_{1,1}$. By induction it is verified that there is a one to one correspondence between all ordered pairs (m,n) of relatively prime natural numbers and curves $g_{m,n}$. The final conclusion is that $g_{m,n}=f_{m,n}$ and $S_{m,n}=T_{m,n}$.

It is of interest to know that P varies continuously at $\tau=2$ in all directions $d\tau$, except $d\tau$ real and negative. If τ approaches 2 in such a way that A comes sufficiently close to elliptic transformations, then P may degenerate, that is, P may have the single point ∞ as limit. On the other hand, the continuity is intuitively clear for any direction $d\tau$ with non-negative real part.

Consider a sequence $\{{}_l\tau\}$ of complex numbers with positive imaginary parts, converging to 2. The polygons corresponding to $\{{}_l\tau\}$ converge to the parabolic polygon if and only if, for each "diagonal-type sequence" of isometric circles, the radii converge to zero. This follows from the fact that the fixed points converge. Assume now that the polygons do not converge. If necessary, $\{{}_l\tau\}$ can be replaced by a subsequence, again denoted $\{{}_l\tau\}$, such that there exists an increasing sequence $\{n_l\}$ and a constant $K \in \mathsf{C}$ for which

$$\lim_{l\to\infty}\beta_{n_l}({}_l\tau) = K .$$

Put ${}_l\tau = {}_lz + {}_lz^{-1}$. Then ${}_lz \to 1$ as $l \to \infty$. Hence, using the formula

$$\beta_n = (z^n - z^{-n})(z - z^{-1})^{-1}$$

and that $K \neq \infty$, we deduce that the imaginary part of ${}_lz$ converges to zero but not faster than $\{n_l{}^{-1}\}$. It follows too that

$$\lim {}_lz^{n_l} = \pm 1 ;$$

therefore, the absolut value of ${}_lz$ converges to 1 of higher order than $\{1 + n_l{}^{-1}\}$. This implies that $\{{}_lz\}$ approaches the unit circle tangentially. Hence $\{{}_l\tau\}$ approaches the real axis tangentially.

5. The polyhedron.

Consider the upper half space H, bounded by the extended complex plane, $\mathsf{C}\cup\{\infty\}$. That Möbius transformations operate on H in a natural way was first remarked by H. Poincarè [2]. Given $A \in M$, let S_n denote the hemisphere in H which has the same centre and radius as I_n. The set of points in H exterior to all S_n forms a fundamental domain. This set is denoted Ph. It is a non-euclidian convex polyhedron. The purpose of this section is briefly to describe its boundary.

First some notation is needed. Let φ_n denote the intersection of the closure of Ph with the closure of S_n. If φ_n is not contained in an arc of a circle, then φ_n will be called a side or a face of P according as φ_n has points in common with the complex plane or not. Obviously the boundary of Ph is made up of a finite number of sides and faces together with P. It is convenient to define φ_0 as the closure of P. An edge of Ph has one or two end points, called corners. By a vertex we mean a corner belonging to φ_0.

In a sense the boundary of Ph collects the information about types of the polygons obtained in the preceding section. It is intuitively clear that when τ is real or if $|\tau| \geqq 2$, then Ph has three sides, including φ_0, and no faces. Polygons of type $\{1,2,1\}$ correspond to polyhedrons bounded by five sides, these being φ_0, φ_1 and φ_{-1}, φ_2 and φ_{-2}.

To illustrate what can happen, we consider a polygon of type $\{1,2\}$. By a little push, changing the type of P to $\{1,3,2\}$, the four vertices get lifted into H as equivalent corners of Ph. Next time, for instance by crossing $f_{3,2}$, other four vertices become inner corners of Ph, and φ_1 and φ_{-1} become faces.

Let τ belong to some $T_{m,n}$ with $mn > 1$. Then the planar graph formed by edges and corners of Ph is cubic, i.e. every edge has two end points, no two different corners are joined by more than one edge and each corner is the end point of exactly three distinct edges. The sides of Ph are φ_0, φ_m and φ_{-m}, φ_{m+n} and φ_{-m-n}, φ_n and φ_{-n}. Faces are those φ_i and φ_{-i} for which there exists a number q such that $(|i|,q) < (m,n)$, with the omission of $|i| = m$ or $|i| = n$. The ordering is the one mentioned in section 4. Let $\lambda = \lambda(m,n)$ be the numbers of strict minorants of (m,n). Then Ph is bounded by 7 sides and $2(\lambda-1)$ faces. Moreover, Ph has 6 vertices and 4λ other corners; the number of edges is $3(2\lambda+3)$.

To see this, again the method of geometric continuity is applied. Remark the symmetry of Ph. To each "lifted vertex" corresponds a power of A by which it is mapped onto the opposite corner. It follows that edges joining lifted vertices cannot degenerate as long as A remains loxodromic. Neither can unexpected faces arise since nothing disappears. Think of a critical situation where some new S_q touches the boundary of Ph. Then φ_q and φ_{-q} are either two opposite corners or two opposite edges. In the first case it is obvious that φ_q would be fixed by some power of A and hence by A. In the second case either the two edges were already equivalent i.e. paired by another transformation, or there must be at least five isometric spheres containing φ_q. Using the symmetry of Ph, we deduce that two unexpected powers of A map φ_q onto φ_{-q}. But this is contrary to the assumption of A being loxodromic: no boundary point of Ph is a fixed point.

The conclusion is that no corner of Ph can belong to four distinct isometric spheres when A is loxodromic. Such a point would be fixed by A. No edge can degenerate under deformation without having a vertex as end point. Therefore faces are preserved or they change becoming sides. In the opposite direction, faces can only arise from sides being "pushed in" and this process does change the φ_n in question from a side to a face. In other words, any essential modification of the boundary of Ph takes place at the vertices. Since we already know how the sides change, the description is reduced to a simple combinatorial problem.

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